Random walk on the incipient infinite cluster on trees

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Abstract. Let $\mathcal{G}$ be the incipient infinite cluster (IIC) for percolation on a homogeneous tree of degree n_0+1. We obtain estimates for the transition density of the continuous time simple random walk Y on $\mathcal{G}$; the process satisfies anomalous diffusion and has spectral dimension $\frac{4}{3}$.

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1. Introduction

We recall the bond percolation model on the lattice $\mathbb{Z}^d$: each bond is open with probability $p \in (0,1)$, independently of all the others. Let $\mathcal{C}(x)$ be the open cluster containing x; then if $\theta(p) = P_p(|\mathcal{C}(x)| = +\infty)$ it is well known (see [Gm]) that there exists $p_c = p_c(d)$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$.

If $d = 2$ or $d \geq 19$ (or $d > 6$ for 'spread out' models) it is known (see [Gm], [HaS]) that $\theta(p_c) = 0$, and it is conjectured that this holds for $d \geq 2$. At the critical probability $p = p_c$ it is believed that in any box of side n there exist with high probability open clusters of diameter of order n – see [BCKS]. For large n the local properties of these large finite clusters can, in certain circumstances, be captured by regarding them as subsets of an infinite cluster $\widetilde{\mathcal{C}}$, called the 'incipient infinite cluster' (IIC).

This was constructed when $d = 2$ in [Ke1], by taking the limit as $N \to \infty$ of the cluster $\mathcal{C}(0)$ conditioned to intersect the boundary of a box of side N with center at the origin. See [Ja1], [Ja2] for other constructions of the IIC in two dimensions. For large d a construction of the IIC in $\mathbb{Z}^d$ is given in [HJ], using the lace expansion. It is believed that the results there will hold for any $d > 6$. [HJ] also gives the existence and some properties of the IIC for all $d > 6$ for 'spread-out' models: these include the case when there is a bond between x and y with probability pL^{-d} whenever y is in a cube side L with center x, and the parameter L is large enough. Rather more is known about the IIC for oriented percolation on $\mathbb{Z}_+ \times \mathbb{Z}^d$ (see [HHS], [HS]), but in this discussion, which mainly concerns what is conjectured rather than what is known, we specialize to the case of $\mathbb{Z}^d$. We write $\widetilde{\mathcal{C}}_d$ for the IIC in $\mathbb{Z}^d$. It is believed that the global properties of $\widetilde{\mathcal{C}}_d$ are the same for all $d > d_c$, both for nearest neighbour and spread-out models. In [HJ] it is proved for 'spread-out' models that $\widetilde{\mathcal{C}}_d$ has one end – that is that any two paths from 0 to infinity intersect infinitely often.

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For large d, it is believed that the geometry of $\widetilde{\mathcal{C}}_d$ is also similar to that of the IIC when '$d = \infty$' – that is to the IIC on a regular tree; this is supported by the results in [HHS] and [HJ]. For trees the construction of the IIC is much easier than for lattices, and there is a close connection between the IIC and a critical Bienaymé-Galton-Watson branching processes conditioned on non-extinction. In [Ke2] Kesten gave the construction of the IIC $\mathcal{G}$ for critical branching processes. This is an infinite subtree, which contains only one path from the root to infinity. This tree is quite sparse, and has polynomial volume growth: in the case when the offspring distribution has finite variance, a ball $B(x,r)$ in $\mathcal{G}$ has roughly r^2 points. (This is when distance in $\mathcal{G}$ is measured using the natural graph distance).

Let $Y = (Y_t, t \geq 0)$ be the (continuous time) simple random walk on $\widetilde{\mathcal{C}}_d$, and $q_t(x,y)$ be its transition density (see Section 3 for a precise definition). Define the *spectral dimension* of $\widetilde{\mathcal{C}}_d$ by

$$d_s(\widetilde{\mathcal{C}}_d) = -2 \lim_{t\to\infty} \frac{\log q_t(x,x)}{\log t}, \tag{1.1}$$

(if this limit exists). Alexander and Orbach [AO] conjectured that, for any $d \geq 2$, $d_s(\widetilde{\mathcal{C}}_d) = 4/3$. While it is now thought that this is unlikely to be true for small d, the results on the geometry of $\widetilde{\mathcal{C}}_d$ for spread out models in [HJ] are consistent with this holding for d above the critical dimension. For the IIC for oriented percolation on $\mathbb{Z}_+ \times \mathbb{Z}^d$, the AO conjecture is proved in [BJKS] for spread-out models for $d > 6$. Since mean field behaviour is expected to hold for oriented percolation for $d > 4$, one might initially expect this result for $d > 4$. However, an issue which arises for random walks on oriented percolation clusters is that while the percolation is oriented, the random walk is not, so that the random walk 'sees' connections in the cluster missed by the percolation process. Given this, it is not clear if one should expect the AO conjecture to hold for oriented percolation if $d = 5, 6$. See [BJKS] for a more detailed discussion of this. We also remark that an annealed version of (1.1) is obtained by [JW], using quite different methods.

Random walks on supercritical clusters in $\mathbb{Z}^d$ are studied in [B2] (transition density estimates) and [SS], [BeB], [MP] (invariance principles). In these cases the large scale behaviour of the random walk approximates that of the random walk on $\mathbb{Z}^d$, and the unique infinite cluster has spectral dimension d.

In what follows, we will specialize to the case of critical percolation on a regular rooted tree with degree $n_0 + 1$, which we denote $\mathbb{B}$. We write 0 for the root of $\mathbb{B}$. We keep n_0 fixed, but (in view of possible future applications) wish to obtain estimates which do not depend on n_0. For bond percolation with probability p on $\mathbb{B}$, it is easy to see that if X_n is the number of vertices at level n in $\mathcal{C}(0)$, then $X = (X_n)$ is a branching process with $\mathtt{Bin}(n_0, p)$ offspring distribution. Thus $p_c = 1/n_0$. For the construction of the IIC see [Ke2]: we obtain a subtree $\mathcal{G} \subset \mathbb{B}$ with law $\mathbb{P}$, on a probability space $(\Omega_1, \mathcal{F}, \mathbb{P})$. Write $\mathbb{B}_N$ for the N-th level of $\mathbb{B}$, and $\mathbb{B}_{\leq N}$ for the union of the first N levels of $\mathbb{B}$. Then the law of $\mathcal{G}$ is characterized by the fact that the law of $\mathcal{G} \cap \mathbb{B}_{\leq N}$ under $\mathbb{P}$ is the same as that of $\mathcal{C}(0)$ under P_{p_c}, conditioned on $\mathcal{C}(0)$ reaching level N.

Motivated by [AO], in [Ke2] Kesten studied the simple random walk on $\mathcal{G}(\omega)$, and also on $\widetilde{\mathcal{C}}_2$. Let $X = (X_n, n \geq 0, Q^x_\omega, x \in \mathcal{G}(\omega))$ be the simple random walk on $\mathcal{G}(\omega)$. We define the annealed law $\mathbb{P}^*$ by the semi-direct product $\mathbb{P}^* = \mathbb{P} \times Q^0_\omega$, and the rescaled height process $Z^{(n)}$ by

$$Z_t^{(n)} = n^{-1/3} d(0, X_{\lfloor nt \rfloor}), \quad t \geq 0,$$

where $d(.,.)$ is the graph distance in $\mathcal{G}(\omega)$.

The following summarizes the main results in of [Ke2] in the tree case.

Theorem 1.1. *(a) ((1.19) in [Ke2].) Let* $T_N = \min\{n : d(0, X_n) = N\}$. *Then for all* $\varepsilon > 0$ *there exist* λ_1, λ_2 *such that*

$$\mathbb{P}^*(\lambda_1 \leq N^{-3} T_N \leq \lambda_2) \geq 1 - \varepsilon, \quad \textit{for all } N \geq 1.$$

(b) ((1.16) in [Ke2], full proof in [Ke3].) Under $\mathbb{P}^*$ *the processes* $Z^{(n)}$ *converges weakly in* $C[0, \infty)$ *to a process* Z *which is not the zero process.*

The continuum limit of the IIC on regular trees is the 'continuum random tree' (CRT) of Aldous – see [A1], [A2], [A3]. A diffusion on the CRT is constructed in [Kr]. In [C1] Croydon obtains transition density estimates for this diffusion, and in [C2] proves that it is the scaling limit of random walks on the discrete trees. These papers use the work of Kigami [Ki1] on diffusions on dendrites, and the representation of the CRT as the integrated super-Brownian excursion (see [DIP]).

To understand why the $n^{-1/3}$ scaling arises in (b) it is helpful to consider the behaviour of random walks on regular deterministic graphs with a large scale fractal structure – see for example [Jo], [BB2], [HK], [GT1], [GT2] and [BCK]. Let $d_f \geq 1$ give the volume growth, so that $|B(x,r)| \sim r^{d_f}$, and suppose that the effective electrical resistance $R_{\text{eff}}(x, B(x,r)^c)$ between x and the exterior of $B(x,r)$ satisfies $R_{\text{eff}}(x, B(x,r)^c) \sim r^{\zeta}$, where $\zeta > 0$. In this 'strongly recurrent' case (see [BCK] for simple recent proofs using ideas that are also used in this paper) one finds that the mean time for X to escape from $B(x,r)$ scales as r^{d_w} where $d_w = d_f + \zeta$. While the IIC $\mathcal{G}$ is more irregular than the sets considered in these papers, it still has properties similar to regular graphs with $d_f = 2$. Further, by Proposition 2.10 below, only $O(1)$ points on $\partial B(x, r/4)$ are connected to $B(x,r)^c$ by a path outside $B(x, r/4)^c$, so one has $R_{\text{eff}}(x, B(x,r)^c) \sim r$, giving $\zeta = 1$ and $d_w = 3$.

In this paper we study the simple random walk on $\mathcal{G}$, and in particular investigate both quenched and annealed properties of its transition densities. For technical convenience we work with the continuous time simple random walk on $\mathcal{G}$, which we denote $Y = (Y_t, t \in [0, \infty), P_\omega^x, x \in \mathcal{G}(\omega))$. However, we expect similar results for the discrete time random walk X – see the note at the end of Section 5. Since we consider the law of Y with general starting points x, we need to consider the measures $\mathbb{P}_x = \mathbb{P}(\cdot | x \in \mathcal{G})$ and $\mathbb{P}_{x,y} = \mathbb{P}(\cdot | x, y \in \mathcal{G})$.

Unlike [Ke2] we restrict our attention to branching processes with a Binomial offspring distribution. Our main reason for this is to maintain good uniform control of the laws $\mathbb{P}_x$. It is clear by symmetry that $\mathbb{P}_x(|B(x,r)| > \lambda)$ is the same for any $x \in \mathbb{B}_N$, and in fact we have uniform bounds for all $x \in \mathbb{B}$. (These probabilities are not equal for all x, since a higher level x is likely to be further from the backbone of the cluster). For a general branching process, the labels of the point x may give a substantial amount of information about the size of the cluster near x.

We write $\tau(x,r) = \inf\{t : d(x, Y_t) \geq r\}$. We look at various quantities measuring the behaviour of the process Y: the transition density $q_t(x,x)$, the mean times to exit balls given by $E_\omega^x \tau(x,r)$ and the distance moved by the process $d(Y_0, Y_t)$. For each of these quantities we can discuss (i) tightness, (ii) mean values and (iii) limiting behaviour.

Theorem 1.2. *(a) There exists $\delta > 0$ such that*

$$\mathbb{P}_x(\theta^{-1} < q_t(x,x)t^{2/3} < \theta) \geq c_1 - e^{-c_2\theta^\delta}, \quad x \in \mathbb{B},\ t \geq 1,\ \theta > 1.$$

(b) There exist c_2, c_3, c_4, $S(x)$ such that for each x,

$$\mathbb{P}_x(S(x) \geq m) \leq c_2(\log m)^{-1}, \tag{1.2}$$

and on $\{\omega : x \in \mathcal{G}(\omega)\}$

$$c_3 t^{-2/3}(\log\log t)^{-17} \leq q_t^\omega(x,x) \leq c_4 t^{-2/3}(\log\log t)^3 \text{ for all } t \geq S(x). \tag{1.3}$$

(c) $d_s(\mathcal{G}) = 4/3$ $\mathbb{P}$–a.s.

The cluster $\mathcal{G}$ contains large scale fluctuations, so that $q_t(x,x)$ does have oscillations of order $(\log\log t)^c$ as $t \to \infty$ – see Lemma 5.1.

Theorem 1.3. *(a) There exists $\delta > 0$ as such that*

$$\mathbb{P}_x(\theta^{-1} \leq r^{-3}E_\omega^x(\tau(x,r)) \leq \theta) \geq 1 - e^{-c_1\theta^\delta}, \quad \text{for } x \in \mathbb{B},\ r \geq 1, \theta \geq 1.$$

(b) For $x \in \mathbb{B}$, $r \geq 1$,

$$c_2 r^3 \leq \mathbb{E}_x\left(E^x\tau(x,r)\right) \leq c_3 r^3.$$

Theorem 1.4. *(a) For $t \geq 1$, we have*

$$c_1 t^{1/3} \leq \mathbb{E}_x\left(E_\omega^x d(x, Y_t)\right) \leq \mathbb{E}_x\left(E_\omega^x \sup_{0\leq s\leq t} d(x, Y_s)\right) \leq c_2 t^{1/3}. \tag{1.4}$$

(b) There exists $T(x)$ with $\mathbb{P}_x(T(x) < \infty) = 1$ such that

$$c_3 t^{1/3}(\log\log t)^{-12} \leq E_\omega^x[d(x, Y_t)] \leq c_4 t^{1/3}\log t \quad \text{for all } t \geq T(x). \tag{1.5}$$

We also obtain (annealed) off-diagonal bounds for $q_t^\omega(x,y)$. These are of the same form as the bounds

$$ct^{-d_f/d_w}\exp(-c'(d(x,y)^{d_w}/t)^{1/(d_w-1)})$$

obtained for regular fractal graphs with $d_f = 2$, $d_w = 3$.

Theorem 1.5. *(a) Let $x, y \in \mathbb{B}$. Then, for $t \geq c_0 d(x,y)$, we have*

$$\mathbb{E}_{x,y}\big(q_t^\omega(x,y)\big) \leq c_1 t^{-2/3} \exp\big(-c_2(\frac{d(x,y)^3}{t})^{1/2}\big). \tag{1.6}$$

(b) Let $x, y \in \mathbb{B}$, with $d(x,y) = R$, and $t \geq c_3(R \vee 1)$. Then

$$\mathbb{E}_{x,y}\big(q_t^\omega(x,y)\big) \geq c_4 t^{-2/3} \exp(-c_5(R^3/t)^{1/2}). \tag{1.7}$$

Define the continuous time rescaled height process

$$\widetilde{Z}_t^{(n)} = n^{-1/3} d(0, Y_{nt}), \quad t \geq 0.$$

By Theorem 1.3(a) the processes $(\widetilde{Z}^{(n)}, n \geq 1)$ are tight with respect to the annealed law given by the semi-direct product $\mathbb{P}^* = \mathbb{P} \times P_\omega^0$. (This is much easier to prove than the full convergence given in Theorem 1.1(b).) However, the large scale fluctuations in $\mathcal{G}$ mean that we do not have quenched tightness.

Theorem 1.6. $\mathbb{P}$-a.s., *the processes $(\widetilde{Z}^{(n)}, n \geq 1)$ are not tight with respect to P_ω^0.*

In Section 2 we recall various properties of branching processes, and obtain the geometrical properties of $\mathcal{G}$ that we will require. In particular we show that, with high probability, balls $B(x,r) \subset \mathcal{G}$ have roughly r^2 points, and $O(1)$ disjoint paths between $B(x, r/4)$ and $B(x,r)^c$. Based on this, we define various types of possible 'good' behaviour of a ball $B(x,r)$, and the cluster in a neighbourhood of the path between points $x, y \in \mathcal{G}$. In Section 3 we review some general properties of random walks on graphs. Our main estimates are given in Section 4, for the random walk on a deterministic subtree $\mathcal{G}$ of $\mathbb{B}$ for which balls and paths are 'good' in the ways given in Section 2. Finally, in Section 5 we tie together the results of Sections 2 and 4, and prove Theorems 1.2–1.6.

Throughout this article, $f_n \sim g_n$ means that $\lim_{n\to\infty} f_n/g_n = 1$. We use c, c' and c'' to denote strictly positive finite constants whose values are not significant and may change from line to line. We write c_i for positive constants whose values are fixed within each theorem, lemma etc. If we cite elsewhere the constant c_1 in Lemma 2.2, we denote it as $c_{2.2.1}$. None of these constants depend on the degree n_0 of the tree.

2. The incipient infinite cluster

We begin with some estimates for the critical Bienaymé-Galton-Watson branching processes X_n, $n \geq 0$, with $X_0 = 1$ and offspring distribution $\mathtt{Bin}(n_0, 1/n_0)$ where $n_0 \geq 2$. These are quite well known, but as we did not find them anywhere in exactly the form we needed, we give the proofs (which are quite short) here.

Let f be the generator of the offspring distribution, so that

$$f(s) = E(s^{X_1}) = n_0^{-n_0}(s + n_0 - 1)^{n_0}. \tag{2.1}$$

From [Har] p. 21 we have

$$P(X_n > 0) \sim \frac{2}{nf''(1)} = \frac{2n_0}{(n_0-1)n}. \tag{2.2}$$

Let

$$Y_n = \sum_{k=0}^{n} X_k, \qquad g_n(s) = E(s^{Y_n}), \qquad f_n(s) = Es^{X_n}.$$

Then conditioning on X_1 we obtain that $f_{n+1}(s) = f(f_n(s))$, and

$$g_{n+1}(s) = sf(g_n(s)) = \frac{s}{n_0^{n_0}}(g_n(s) + n_0 - 1)^{n_0}.$$

Set

$$h_n(\theta) = \log g_n(e^\theta), \qquad k_n(\theta) = \log f_n(e^\theta).$$

Lemma 2.1. *(a) Let $1 < \alpha \le 2$. Then*

$$h_n(\theta) \le (1+\alpha n)\theta, \qquad \textit{provided } 0 \le \theta \le \frac{\alpha - 1}{(1+\alpha n)^2}. \tag{2.3}$$

(b)

$$k_n(\theta) \le \theta + 2n\theta^2, \qquad \textit{provided } 0 < \theta \le \frac{1}{6n}. \tag{2.4}$$

Proof. Note that h_n and k_n are continuous, strictly increasing and $h_n(0) = k_n(0) = 0$.

For (a) we have

$$h_{n+1}(\theta) = \log\left(\frac{e^\theta}{n_0^{n_0}}(e^{h_n(\theta)} + n_0 - 1)^{n_0}\right) = \theta + n_0 \log \frac{1}{n_0}(e^{h_n(\theta)} + n_0 - 1).$$

Let $a_n = \min\{\theta : h_n(\theta) = 1\}$. Then since $e^x \le 1 + x + x^2$ on $[0,1]$, on $[0, a_n]$,

$$h_{n+1}(\theta) \le \theta + n_0 \log(1 + \frac{1}{n_0}h_n(\theta) + \frac{1}{n_0}h_n(\theta)^2) \le \theta + h_n(\theta) + h_n(\theta)^2. \tag{2.5}$$

We verify (2.3) by induction. Since $h_0(\theta) = \theta$, (2.3) holds for $n = 0$. Writing $b_n(\alpha) = (\alpha-1)/(1+\alpha n)^2$, we have $h_n(\theta) \le 1$ for $\theta \in [0, b_n(\alpha)]$. So, using (2.5) and (2.3) for n

$$h_{n+1}(\theta) \le (1 + \alpha(n+1))\theta + (1+\alpha n)^2\theta^2 - (\alpha-1)\theta \le (1+\alpha(n+1))\theta,$$

proving (2.3) for $n+1$.

(b) Similarly, provided $k_n(\theta) \le 1$,

$$k_{n+1}(\theta) = n_0 \log\left(1 + \frac{e^{k_n(\theta)} - 1}{n_0}\right) \le k_n(\theta) + k_n(\theta)^2. \tag{2.6}$$

Using (2.4) for n we obtain, since $\theta + 2n\theta^2 \le 4\theta/3$,

$$k_{n+1}(\theta) \le (\theta + 2n\theta^2) + (\theta + 2n\theta^2)^2 \le (\theta + 2n\theta^2) + 16\theta^2/9 \le (\theta + 2(n+1)\theta^2),$$

proving (2.4) for $n+1$. $\square$

Notation. Let ξ be a random variable. We write $\lambda\xi[n]$ for a r.v. with the distribution of $\lambda\sum_1^n \xi_i$, where ξ_i are i.i.d. with $\xi_i \overset{(d)}{=} \xi$. We also write $\texttt{Ber}(p)$ and $\texttt{Bin}(n,p)$ for the Bernoulli and Binomial distributions respectively. Using this notation we have for example $(\xi[n])[m] = \xi[nm]$, and $\texttt{Bin}(n,p) \overset{(d)}{=} \texttt{Ber}(p)[n]$. We write $\succcurlyeq$ for stochastic domination.

Lemma 2.2. *For any* $\lambda > 0$

$$P(X_n[n] \geq \lambda n) \leq c_1 e^{-\lambda/6}, \tag{2.7}$$

$$P(Y_n[n] \geq \lambda n^2) \leq c_2 e^{-\lambda/5}. \tag{2.8}$$

Proof. Let $\theta = 1/6n$. Using (2.4)

$$\begin{aligned}\log P(X_n[n] \geq \lambda n) &\leq -\theta\lambda n + nk_n(\theta) \\ &\leq -n\theta(\lambda - 2) = -(\lambda-2)/6,\end{aligned}$$

proving (2.7).

If $\theta \leq b_n(\alpha)$ then

$$\begin{aligned}P(Y_n[n] \geq \lambda n^2) &= P(e^{\theta Y_n[n]} \geq e^{\theta\lambda n^2}) \leq e^{-\theta\lambda n^2} E e^{\theta Y_n[n]} \\ &= \exp(-\theta\lambda n^2 + nh_n(\theta)) \leq \exp(-\theta\lambda n^2 + (1+\alpha n)n\theta).\end{aligned}$$

So taking $\alpha = 2$ and $\theta = b_n(2) = (1+2n)^{-2}$

$$\log P(Y_n[n] \geq \lambda n^2) \leq -\frac{n^2(\lambda-2)}{(1+2n)^2} + \frac{n}{(1+2n)^2} \sim -\tfrac{1}{4}\lambda + c_3.$$

□

Lemma 2.3. *(a) There exist* $c_0 > 0$, $p_0 > 0$ *such that*

$$P(Y_n > c_0 n^2) \geq \frac{p_0}{n}.$$

(b) If $\eta_n \overset{(d)}{=} \mathtt{Bin}(n, p_0/n)$ *then* $Y_n[n] \succcurlyeq c_0 n^2 \eta_n$.

Proof. (a) This should be in the literature, but is also easy to prove directly. Let $A_n = \{X_{n/2} > 0\}$, and $a_n = P(A_n)$. Then by (2.2) $a_n \sim (4n_0/(n_0-1))n^{-1}$. We have $EY_n = n+1$ and $EY_n^2 \leq c_1 n^3$, where c_1 does not depend on n_0. On A_n^c we have $Y_{n/2} = Y_n$, so

$$n+1 = EY_n = E(Y_n; A_n) + E(Y_n; A_n^c) \leq E(Y_n|A_n)P(A_n) + EY_{n/2}.$$

It follows that

$$E(Y_n|A_n) \geq \frac{n/2}{a_n} \geq c_2 n^2.$$

Also,

$$E(Y_n^2|A_n) \leq P(A_n)^{-1} E(Y_n^2; A_n) \leq c_3 n^4.$$

Using the 'Backwards Chebyshev' inequality $P(\xi \geq \frac{1}{2}E\xi) \geq (E\xi)^2/(4E\xi^2)$ with respect to $P(\cdot|A_n)$ then gives

$$P(Y_n > \tfrac{1}{2}c_2 n^2 | A_n) \geq P(Y_n > \tfrac{1}{2}E(Y_n|A_n)|A_n) \geq \frac{c_2^2 n^4}{4c_3 n^4} = c_4.$$

So $P(Y_n > \frac{1}{2}c_2n^2) \geq P(Y_n > \frac{1}{2}c_2n^2|A_n)P(A_n) \geq c_4a_n \geq c_5n^{-1}$, and taking $c_0 = \frac{1}{2}c_2$, $p_0 = c_5$, this proves (a).
(b) Let now $Y_n^{(j)}$ be i.i.d. copies of Y_n, and $F_j = \{Y_n^{(j)} > c_0n^2\}$. Then if $\xi_j = 1_{F_j}$, by (a) we have $P(\xi_j = 1) \geq p_0/n$. So,

$$Y_n[n] = \sum_{j=1}^{n} Y_n^{(j)} \succcurlyeq \sum_{j=1}^{n} c_0n^2\xi_j \succcurlyeq c_0n^2\eta_n,$$

proving (b). □

Lemma 2.4. *For* $0 < \lambda < 1$ *and* $n \geq c_1/\lambda$,

$$\exp(-c_2/\lambda) \leq P(Y_n[n] \leq \lambda n^2) \leq \exp(-c_3/\lambda^{1/2}). \tag{2.9}$$

Proof. To prove the upper bound let $c_0 = c_{2.3.0}$, and $m = (\lambda/c_0)^{1/2}n$. Using Lemma 2.3 we have

$$Y_m[n] = \sum_{i=1}^{n} Y_m^{(i)} \succcurlyeq \sum_{i=1}^{n} c_0m^2\xi_i = \lambda n^2 \sum_{i=1}^{n} \xi_i;$$

here ξ_i are i.i.d. $\texttt{Ber}(p_0/m)$ r.v. So

$$P(Y_m[n] < \lambda n^2) \leq P(\sum_{i=1}^{n} \xi_i < 1) = (1 - p_0/m)^n \leq \exp(-p_0n/m) = \exp(-c_0^{1/2}p_0/\lambda^{1/2}).$$

For the lower bound let $k \geq 1$ and $m = n/k$. Let $G_j = \{X_m^{(j)} = 0\}$, and $G = \cap_{1\leq j\leq n} G_j$. Then $P(G) \geq (1 - c/m)^n$ so

$$\begin{aligned} P(Y_n[n] < \lambda n^2) &\geq P(Y_n[n] < \lambda n^2|G)P(G) \\ &\geq (1 - c/m)^n \Big(1 - P(Y_n[n] > \lambda n^2|G)\Big) \\ &\geq c'e^{-c''k}\Big(1 - P(Y_n[n] > \lambda n^2|G)\Big). \end{aligned}$$

On G we have $Y_n[n] = \sum_{j=1}^{n} Y_m^{(j)}$, so for $m = n/k \geq 2c$,

$$P(Y_n[n] > \lambda n^2|G) \leq \frac{E(\sum_{j=1}^{n} Y_m^{(j)}|G)}{\lambda n^2} = \frac{nE(Y_m^{(1)}|G_1)}{\lambda n^2} \leq \frac{EY_m^{(1)}}{\lambda nP(G_1)} \leq \frac{c'}{k\lambda}.$$

Taking k such that $c'/(k\lambda) = \frac{1}{2}$, we have $n \geq c_1/\lambda$, which completes the proof. □

We will need to consider the following modified branching process. Let $\widetilde{X} = (\widetilde{X}_n, n \geq 0)$ be a branching process with $\widetilde{X}_0 = 1$ and the same $\texttt{Bin}(n_0, 1/n_0)$ offspring distribution as X, except that at the first generation we have $\widetilde{X}_1 \overset{(d)}{=} \texttt{Bin}(n_0 - 1, 1/n_0)$.

Lemma 2.5. *(a) For any* $\lambda > 0$

$$P(\widetilde{X}_n[n] \geq \lambda n) \leq c_1 e^{-c_2\lambda}, \tag{2.10}$$

$$P(\widetilde{Y}_n[n] \geq \lambda n^2) \leq c_3 e^{-c_4\lambda}. \tag{2.11}$$

(b) For $0 < \lambda < 1$,

$$\exp(-c_5/\lambda) \leq P(\widetilde{Y}_n[n] \leq \lambda n^2) \leq \exp(-c_6/\lambda^{1/2}). \tag{2.12}$$

(c) There exists $p_1 > 0$ *such that* $\widetilde{Y}_n[n] \succcurlyeq c_7 n^2 \mathtt{Bin}(n, p_1/n)$.

Proof. (a) and the lower bound in (b) are immediate from Lemmas 2.2 and 2.4, since $\widetilde{X}_n \preccurlyeq X_n$ and $\widetilde{Y}_n \preccurlyeq Y_n$.

For the upper bound in (b), we can write

$$\widetilde{Y}_n[n] = n + \sum_{i=1}^{M} Y_{n-1}^{(i)},$$

where $M \overset{(d)}{=} \mathtt{Bin}(n(n_0-1), 1/n_0)$, and $Y^{(i)}$ are independent copies of Y. Similarly,

$$Y_m[m] = m + \sum_{i=1}^{M'} Y_{m-1}^{(i)},$$

where $M' \overset{(d)}{=} \mathtt{Bin}(mn_0, 1/n_0)$. So if $m = n(n_0-1)/n_0$ then

$$\widetilde{Y}_n[n] = n + \sum_{i=1}^{M} Y_{n-1}^{(i)} \geq m + \sum_{i=1}^{M'} Y_{m-1}^{(i)} = Y_m[m]. \tag{2.13}$$

(2.12) now follows from Lemma 2.4, since $\frac{1}{2}n \leq m \leq n$.
(c) We have $\mathtt{Ber}(p) \succcurlyeq \frac{1}{2}\mathtt{Ber}(p/2)[2]$. So, using (2.13), with m as in (b),

$$\begin{aligned}\widetilde{Y}_n[n] \succcurlyeq Y_m[m] &\succcurlyeq c_0 m^2 \mathtt{Bin}(m, p_0/m)\\ &\succcurlyeq \tfrac{1}{2} c_0 m^2 \mathtt{Bin}(2m, p_0/2m)\\ &\succcurlyeq \tfrac{1}{2} c_0 m^2 \mathtt{Bin}(n, p_0/2m) \succcurlyeq c_1 n^2 \mathtt{Bin}(n, p_1/n).\end{aligned}$$

□

We now define the random graph $\mathcal{G}$ we will be working with. We could regard this either as critical percolation on the n_0-ary tree $\mathbb{B}$, conditioned on the cluster containing the root 0 being infinite, or as the (critical) Bienaymé-Galton-Watson process with $\mathtt{Bin}(n_0, 1/n_0)$ offspring distribution, conditioned on non-extinction.

Let $\mathbb{B}$ be the n_0-ary tree, and let 0 be the root. A point x in the nth generation (or level) is written $x = (0, l_1, \cdots, l_n)$, where $l_i \in \{1, 2, \cdots, n_0\}$. Let $\mathbb{B}_n$ be the set of n_0^n points in the nth generation, and let $\mathbb{B}_{\leq n} = \cup_{i=0}^n \mathbb{B}_i$. If $x \in \mathbb{B}_k$ we write $|x| = k$. If $x = (0, l_1, \cdots, l_n) \in \mathbb{B}_n$, let $a(x, r) = (0, l_1, \cdots, l_{n-r})$ be the ancestor of x at level $|x| - r$.

We regard $\mathbb{B}$ as a graph (in fact a tree) with edge set $E(\mathbb{B}) = \big\{\{x, a(x,1)\}, x \in \mathbb{B} - \{0\}\big\}$. Let η_e, $e \in E(\mathbb{B})$, be i.i.d. Bernoulli $1/n_0$ r.v. defined on a probability space $(\Omega, \mathcal{F}, P)$. If $\eta_e = 1$ we say the edge e is *open*. Let

$$\mathcal{C}(0) = \{x \in \mathbb{B} : \text{ there exists an } \eta\text{–open path from 0 to } x\}$$

be the open cluster containing 0. It is clear that $Z_n = |\mathcal{C}(0) \cap \mathbb{B}_n|$ is a critical GW process with $\mathtt{Bin}(n_0, 1/n_0)$ offspring distribution. Here and in the following, $|A|$ is a cardinality of the set A. As Z has extinction probability 1, the cluster $\mathcal{C}(0)$ is P–a.s. finite.

We have

Lemma 2.6. *([Ke2, Lemma 1.14]). Let $A \subset \mathbb{B}_{\leq k}$. Then*

$$\lim_{n\to\infty} P(\mathcal{C}(0) \cap \mathbb{B}_{\leq k} = A | Z_n \neq 0) = |A \cap \mathbb{B}_k| P(\mathcal{C}(0) \cap \mathbb{B}_{\leq k} = A), \tag{2.14}$$

and writing $\mathbb{P}_0(A) = |A \cap \mathbb{B}_k| P(\mathcal{C}_{\leq k} = A)$, $\mathbb{P}_0$ has a unique extension to a probability measure $\mathbb{P}$ on the set of infinite connected subsets of $\mathbb{B}$ containing 0.

Let $\mathcal{G}'$ be a rooted labeled tree chosen with the distribution $\mathbb{P}$: we call this the *incipient infinite cluster* (IIC) on $\mathbb{B}$. For more information on $\mathcal{G}'$ see [Ke2] and [vH] but we remark that $\mathbb{P}$–a.s. $\mathcal{G}'$ has exactly one infinite descending path from 0, which we call the *backbone*, and denote H.

It will be helpful to use another construction of the IIC, obtained by modifying the cluster $\mathcal{C}(0)$ rather than its law. We can suppose the probability space $(\Omega, \mathcal{F}, P)$ carries i.i.d.r.v. ξ_i, $i \geq 1$ uniformly distributed on $\{1, 2, \cdots, n_0\}$, and independent of (η_e). For $n \geq 0$ let $\Xi_n = (0, \xi_1, \ldots, \xi_n)$, and let

$$\widetilde{\eta}_e = \begin{cases} 1 & \text{if } e = \{\Xi_n, \Xi_{n+1}\} \text{ for some } n \geq 0, \\ \eta_e & \text{otherwise.} \end{cases}$$

Then (see [vH]) if

$$\mathcal{G} = \{x \in \mathbb{B} : \text{ there exists a } \widetilde{\eta}\text{–open path from 0 to } x\},$$

$\mathcal{G}$ has law $\mathbb{P}$. It is clear that the backbone of $\mathcal{G}$ is the set $H = \{\Xi_n, n \geq 0\}$.

For $x, y \in \mathbb{B}$ let

$$\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | x \in \mathcal{G}), \qquad \mathbb{P}_{xy}(\cdot) = \mathbb{P}(\cdot | x, y \in \mathcal{G}),$$

and let $\mathbb{E}_x$ and $\mathbb{E}_{xy}$ denote expectation with respect to $\mathbb{P}_x$ and $\mathbb{P}_{xy}$ respectively. Given a descending path $b = \{0, b_1, b_2, \ldots\}$, (which we call a *possible backbone*) let

$$\mathbb{P}_{x,b}(\cdot) = \mathbb{P}(\cdot | x \in \mathcal{G}, H = b),$$

and define $\mathbb{P}_{x,y,b}$ analogously.

For each x, $y \in \mathbb{B}$, let $\gamma(x,y)$ be the unique geodesic path connecting x and y. We say that z is a *middle point* of $\gamma(x,y)$ if $z \in \gamma(x,y)$ and $|d(x,z) - \frac{1}{2}d(x,y)| \leq \frac{1}{2}$. We remark that the second construction of $\mathcal{G}$ makes it clear that $\mathbb{P}_{x,y,b}(\eta_e = 1) = 1$ if the edge e lies in any of the paths b, $\gamma(0,x)$ and $\gamma(0,y)$, and that under $\mathbb{P}_{x,y,b}$ the r.v. η_e, $e \notin b \cup \gamma(0,x) \cup \gamma(0,y)$ are i.i.d. with $\mathbb{P}_{x,y,b}(\eta_e = 1) = 1/n_0$.

Notation. We consider the tree $\mathcal{G} = \mathcal{G}(\omega)$. Let $d(x,y)$ be the graph distance between x and y, and

$$B(x,r) = \{y \in \mathcal{G} : d(x,y) < r\}.$$

We write $D(x)$ for the set of descendants of x. More precisely, $y \in D(x)$ if and only if $x \in \gamma(0,y)$. Note that $x \in D(x)$. If $y \in D(x)$ we call x an *ancestor* of y and y a *descendants* of x. We set

$$D_r(x) = \{y \in D(x) : d(x,y) = r\}, \qquad D_{\leq r}(x) = \cup_{i=0}^{r} D_i(x).$$

We also set

$$D(x;z) = \{y \in D(x) : \gamma(x,y) \cap \gamma(x,z) = \{x\}\},$$

and write $D_r(x;z) = D_r(x) \cap D(x;z)$, $D_{\leq r}(x;z) = D_{\leq r}(x) \cap D(x;z)$. Thus if $z \in D(x)$ then $y \in D(x;z)$ if and only if the lines of descent from x to y and z are disjoint, except for x. (Note that $D(x;x) = D(x)$.) For any $A \subset \mathcal{G}$ we write

$$\partial A = \{y \in \mathcal{G} - A : y \sim x \text{ for some } x \in A\}.$$

The estimates at the beginning of this Section lead to volume growth estimates for $\mathcal{G}$. For $x \in \mathcal{G}$ let μ_x be the degree of x, and for $A \subset \mathcal{G}$ set $\mu(A) = \sum_{x \in A} \mu_x$. We write

$$V(x,r) = \mu(B(x,r)).$$

Note that as $\mathcal{G}$ is a tree, we have

$$|B(x,r)| \leq V(x,r) \leq 2|B(x,r+1)|. \tag{2.15}$$

Proposition 2.7. *(a) Let $\lambda > 0$, $r \geq 1$ and $x, y \in \mathbb{B}$, and b be a possible backbone. Then*

$$\mathbb{P}_{x,y,b}(V(x,r) > \lambda r^2) \leq c_0 \exp(-c_1 \lambda), \tag{2.16}$$

and

$$\mathbb{P}_{x,y,b}(V(x,r) < \lambda r^2) \leq c_2 \exp(-c_3/\sqrt{\lambda}). \tag{2.17}$$

(b) The bounds (2.16) and (2.17) also hold for the laws $\mathbb{P}_{x,b}$, $\mathbb{P}_{x,y}$, and $\mathbb{P}_x$.

Proof. It is enough to prove (a), since the bounds for $\mathbb{P}_{x,b}$ follow by taking $y = 0$, and those for $\mathbb{P}_{x,y}$ and $\mathbb{P}_x$ then follow on integrating over b. Also, using (2.15), it is enough to bound $|B(x,r)|$.

We will assume that $|x| > r$; if not we can use the same arguments with minor modifications. Let $x_i = a(x,i)$ for $0 \le i \le r$. If the backbone intersects $B(x,r)$ then let s be the smallest i such that $x_i \in H$, and let $v_0 = x_s$ and v_i, $i \ge 1$ be the backbone descending from the point v_0. Similarly if $\gamma(0,y)$ intersects $B(x,r)$ then let t be the smallest j such that $y_j \in B(x,r)$, and let $w_0 = y_t$ and w_i, $1 \le i \le t$ be the path $\gamma(w_0,y)$.

Then we have

$$B(x,r) \subset \big(\cup_{i=0}^r D_{\le r}(x_i;x)\big) \cup \big(\cup_{i=1}^r D_{\le r}(v_i;v_{3r})\big) \cup \big(\cup_{i=1}^{r\wedge t} D_{\le r}(w_i;y)\big).$$

Under $\mathbb{P}_{x,y,b}$ the r.v. $|D_{\le r}(\cdot;\cdot)|$ above are i.i.d., with the same law as $\widetilde{Y}_r$. Thus $|B(x,r)| \preccurlyeq \widetilde{Y}_r[r][3]$, and by Lemma 2.5(a),

$$\mathbb{P}_{x,y,b}(|B(x,r)| > \lambda r^2) \le c\exp(-c'\lambda).$$

The proof of (2.17) is very similar. We have $\cup_{i=0}^{r/2} D_{\le r/2}(x_i;x) \subset B(x,r)$, so that $|B(x,r)| \succcurlyeq \widetilde{Y}_{r/2}[r/2]$, and using Lemma 2.5(b) leads to (2.17). $\square$

We also wish to show that oscillations in $n^{-2}V(0,n)$ exist. If $W \overset{(d)}{=} \texttt{Bin}(n,p/n)$ then straightforward calculations give that

$$P(W=k) \ge c_0 e^{-k\log(k/p)}, \quad 0 \le k \le n^{1/2}. \tag{2.18}$$

Proposition 2.8. *(a) For any $\varepsilon > 0$*

$$\limsup_{n\to\infty} \frac{V(0,n)}{n^2(\log\log n)^{1-\varepsilon}} = \infty, \quad \mathbb{P}-a.s.$$

(b) There exists $c_0 < \infty$ such that

$$\liminf_{n\to\infty} \frac{(\log\log n)V(0,n)}{n^2} \le c_0. \quad \mathbb{P}-a.s.$$

Proof. It is enough to prove these for the law $\mathbb{P}_b$, for any fixed possible backbone $b = \{0, y_1, y_2, \ldots\}$.
(a) Let

$$Z_n = |\{x : x \in D(y_i;y_{i+1}),\ d(x,y_i) \le 2^{n-2},\ 2^{n-1} \le i \le 2^{n-1}+2^{n-2}\}|.$$

Thus Z_n is the number of descendants off the backbone, to level 2^{n-2}, of points y on the backbone between levels 2^{n-1} and $2^{n-1}+2^{n-2}$. So $|B(0,2^n)| \ge Z_n$, the r.v. Z_n are independent, and $Z_n \overset{(d)}{=} \widetilde{Y}_{2^{n-2}}[2^{n-2}]$. Using Lemma 2.5(c) we have, if $a_n = (\log n)^{1-\varepsilon}$, and $\eta_n \overset{(d)}{=} \texttt{Bin}(n,p_1/n)$,

$$\begin{aligned}\mathbb{P}_b(|B(0,2^n)| \ge a_n 4^n) &\ge \mathbb{P}_b(Z_n \ge a_n 4^n)\\ &\ge P(\widetilde{Y}_{2^{n-2}}[2^{n-2}] \ge a_n 4^n)\\ &\ge P(\eta_{2^{n-2}} \ge a_n) \ge ce^{-a_n\log a_n}.\end{aligned}$$

As Z_n are independent, (a) follows by the second Borel-Cantelli Lemma.

(b) Let $n_k = \exp(2k \log k)$, so that $k^2 n_{k-1} \le n_k$, and let

$$W_k = \bigcup_{i=0}^{n_k - 1} D(y_i; y_{n_k}), \quad V_k = D_{\le n_k - n_{k-1}}(y_{n_{k-1}}).$$

Then the r.v. $|V_k|$ are independent and $B(0, n_k) \subset W_{k-1} \cup V_k$.

Fix $0 < \varepsilon < 1/3$ and let

$$F(i,k) = \{D_{k^{1+\varepsilon} n_k}(y_i; y_{i+1}) = \emptyset\}.$$

Then since $X_n \succcurlyeq \widetilde{X}_n$

$$\mathbb{P}(F(i,k)) = P(\widetilde{X}_{k^{1+\varepsilon} n_k} = 0) \ge P(X_{k^{1+\varepsilon} n_k} = 0) \ge 1 - \frac{c}{k^{1+\varepsilon} n_k}.$$

Let $G_k = \cap_{i=0}^{n_k - 1} F(i,k)$; we have

$$\mathbb{P}(G_k^c) \le c/k^{1+\varepsilon}.$$

On the event G_k we have that $|W_k|$ is stochastically dominated by $\sum_{i=1}^{n_k} Y^{(i)}_{k^{1+\varepsilon} n_k}$, so

$$\begin{aligned}\mathbb{P}(|W_k| \ge k^3 n_k^2) &\le \mathbb{P}(G_k^c) + P(Y_{k^{1+\varepsilon} n_k}[k^{1+\varepsilon} n_k] \ge k^{1-2\varepsilon}(k^{1+\varepsilon} n_k)^2) \\ &\le ck^{-(1+\varepsilon)} + e^{-c' k^{1-2\varepsilon}} \le c'' k^{-(1+\varepsilon)}.\end{aligned}$$

Thus $|W_k| \le k^3 n_k^2$ for all large k. Now $|V_k| \preccurlyeq Y_{n_k}[n_k]$, so

$$\mathbb{P}(|V_k| < c_1 (\log k)^{-1} n_k^2) \ge P(Y_{n_k}[n_k] < c_1 (\log k)^{-1} n_k^2) \ge e^{-c \log k} \ge k^{-1}$$

if c_1 is chosen large enough. As the r.v. $|V_k|$ are independent, we deduce that $|V_k| < c_1 (\log k)^{-1} n_k^2$ for all k in an infinite set J. For all large $k \in J$,

$$|B(0, n_k)| \le |V_k| + (k-1)^3 n_{k-1}^2 \le (c_1 (\log k)^{-1} + k^{-1}) n_k^2 \le \frac{2 c_1 n_k^2}{\log \log n_k}.$$

$\square$

Remark. Let $\mathcal{C}_\infty$ denote the unique infinite cluster for supercritical bond percolation (i.e. $p > p_c$) in $\mathbb{Z}^d$. Then writing $Q(x, N)$ for the box side N and center x

$$\frac{|\mathcal{C}_\infty \cap Q(x,N)|}{|Q(x,N)|} \to \theta(p).$$

Propositions 2.7 and 2.8 show that while the law of $R^{-2} V(0, R)$ is tight on $(0, \infty)$, $\lim_{R \to \infty} R^{-2} V(0, R)$ does not exist. Thus $\mathcal{G}$ is at large length scales a much more irregular set than the clusters considered in [B2].

Definition 2.9. Let $x \in \mathcal{G}$, $r \ge 1$. Let $M(x, r)$ be the smallest number m such that there exists a set $A = \{z_1, \dots, z_m\}$ with $d(x, z_i) \in [r/4, 3r/4]$ for each i, such that any path γ from x to $B(x, r)^c$ must pass through the set A. (Since $\mathcal{G}$ is a tree, the best choice of such a set A will in fact have the points at a distance $r/4$ from x, but we will not need this.)

Proposition 2.10. *There exist $c_1, c_2 > 0$ such that for each $r \geq 1$ and each $x, y \in \mathbb{B}$, and possible backbone b*

$$\mathbb{P}_{x,y,b}(M(x,r) \geq m) \leq c_1 e^{-c_2 m}.$$

Similar bounds hold for $\mathbb{P}_{x,y}$, $\mathbb{P}_{x,b}$ and $\mathbb{P}_x$.

Proof. We just consider the case $y = 0$; the general case is similar but a little more complicated since we would also need to consider offspring on the branch $\gamma(0,y)$. Let $w_0 = a(x, r/3)$. If $w_0 \in b$ then let w_1 be the point in the backbone at level $|x| + r/3$, otherwise let $w_1 = w_0$. Let

$$A_1 = \cup_{z \in \gamma(w_0,x), z \notin b} D_{r/4}(z;x), \qquad A_2 = \cup_{z \in \gamma(w_0,w_1), z \neq w_1} D_{r/4}(z;w_1).$$

Let $N_i = |A_i|$; we have $N_1 \preccurlyeq X_{r/4}[1 + r/4]$ and $N_2 \preccurlyeq X_{r/4}[r/2]$. Now let

$$A_i^* = \{z \in A_i : D_{r/4}(z) \neq \emptyset\}.$$

Then any path from x to $B(x,r)^c$ must pass through $A_1^* \cup A_2^* \cup \{w_0, w_1\}$, so $M = M(x,r) \leq 2 + |A_1^*| + |A_2^*|$.

Let $p_r = P(z \in A_i^* | z \in A_i) = P(X_{r/4} > 0)$, so that $p_r \leq c/r$. So, if κ_i are i.i.d. $\mathtt{Ber}(p_r)$ r.v. independent of N_i, we have

$$|A_i^*| \overset{(d)}{=} \sum_{j=1}^{N_i} \kappa_j.$$

Let

$$W_n = \sum_{i=1}^{n} (\kappa_i - p_r);$$

then $W = \{W_n\}$ is a martingale, $W_n - W_{n-1} \leq 1$, $\langle W \rangle_n = np_r(1 - p_r)$, and $|A_i^*| \overset{(d)}{=} W_{N_i} + N_i p_r$. Choose r large enough so that $p_r < \frac{1}{2}$. Then

$$\mathbb{P}_{x,b}(|A_i^*| \geq m) \leq \mathbb{P}_{x,b}(W_{N_i} + N_i p \geq m, N_i p \leq m/2) + \mathbb{P}_{x,b}(N_i p > m/2). \tag{2.19}$$

For the first term in (2.19) we have

$$\begin{aligned}\mathbb{P}_{x,b}(W_{N_i} + N_i p \geq m, N_i p \leq m/2) &\leq \mathbb{P}_{x,b}(W_{N_i} \geq m/2, \langle W \rangle_{N_i} \leq m(1-p)/2) \\ &\leq \exp\Big(-\frac{(m/2)^2}{2((m/2) + m(1-p)/2)}\Big) \leq e^{-cm},\end{aligned}$$

where we used an exponential martingale inequality – see (1.6) in [F]. For the second term, note that $N_i \preccurlyeq (X_{r/4}[r/4])[2]$ and so using Lemma 2.2 we deduce that

$$\mathbb{P}_{x,b}(N_i p > m/2) \leq c e^{-c_3 m}.$$

Combining these bounds completes the proof. $\square$

Definition 2.11. Let $x \in \mathbb{B}$, $r \geq 1$, $\lambda \geq 64$. We say that $B(x,r)$ is λ*–good* if:
(a) $x \in \mathcal{G}$
(b) $r^2 \lambda^{-2} \leq V(x,r) \leq r^2 \lambda$.
(c) $M(x,r) \leq \frac{1}{64}\lambda$.
(d) $V(x, r/\lambda) \geq r^2 \lambda^{-4}$.
(e) $V(x, r/\lambda^2) \geq r^2 \lambda^{-6}$.

Corollary 2.12. *For $x \in \mathbb{B}$ and any possible backbone b*

$$\mathbb{P}_{x,b}(B(x,r) \text{ is not } \lambda\text{–good}) \le c_1 e^{-c_2\lambda}. \tag{2.20}$$

Proof. By Propositions 2.7 and 2.10 the probability of each of conditions (a)–(d) above failing is bounded by $\exp(-c\lambda)$. □

We now need to introduce some more complicated conditions on the tree $\mathcal{G}$, and will prove that these hold with high probability. These conditions describe various kinds of 'good' behaviour of balls with centers on a path $\gamma(x,y)$, and will be used when we consider off-diagonal bounds on the transition probabilities of the random walk in Sections 4 and 5.

Fix $\lambda_1 \ge 64$ large enough so that the right hand side of (2.20) is less than $\frac{1}{4}$. For $x, y \in \mathbb{B}$ and $k \in \mathbb{N}$, define the event

$$\begin{aligned} F_1(x,y,r,k) = \{&x, y \in \mathcal{G} \text{ and there exist at least } k \text{ disjoint balls} \\ &B(z,r/2) \text{ with } z \in \gamma(x,y) \text{ and which are } \lambda_1\text{–good.}\} \end{aligned}$$

For $x, y \in \mathbb{B}$, let z_0 be a middle point of $\gamma(x,y)$. Define the events

$$\begin{aligned} A_*(z,r,N) &= \{z \in \mathcal{G} \text{ and } B(z,r) \text{ is } N\text{–good.}\}, \\ F_*(x,y,R,k;r,N) &= F_1(x,z_0,R,k/2) \cap F_1(z_0,y,R,k/2) \\ &\qquad \cap A_*(x,r,N) \cap A_*(z_0,r,N) \cap A_*(y,r,N). \end{aligned}$$

Definition 2.13. The vertex $x \in \mathbb{B}$ satisfies the condition $G_2(N,R)$ if:
(a) $x \in \mathcal{G}$,
(b) For every $z \in \partial B(x,NR)$ the event $F_1(x,z,R,\frac{1}{8}N)$ holds.

Proposition 2.14. *Let $x_0, y_0 \in \mathbb{B}$, and b be a possible backbone.*

(a) For $R \ge 1$, $N \ge 8$,

$$\mathbb{P}_{x_0,y_0,b}\big(x_0 \text{ satisfies the condition } G_2(N,R)\big) \ge 1 - c_1 \exp(-c_2 N).$$

(b) The same bounds as in (a) hold for the laws $\mathbb{P}_{x_0,b}$, $\mathbb{P}_{x_0,y_0}$, and $\mathbb{P}_{x_0}$.
(c) For $x_0, y_0 \in \mathbb{B}$, $8 \le N < d(x_0,y_0)/8$, $r \ge 1$,

$$\mathbb{P}_{x_0,y_0,b}\Big(F_*(x_0,y_0,\frac{d(x_0,y_0)}{N},\tfrac{1}{8}N;r,N)\Big) \ge 1 - c_3 \exp(-c_4 N).$$

Proof. (a) We prove this for $y_0 = 0$; as in Proposition 2.10 the general case is handled by a similar argument.

Let

$$F_0(y,s) = \{y \in \mathcal{G} \text{ and } B(y,s) \text{ is } \lambda_1\text{–good.}\},$$

and write $v_i = a(x,i)$, $R' = RN/4$. We assume that $|x| \ge NR$ and $v_{R'}$ is on the backbone b: the other cases can be handled by minor modifications to the arguments below. Let w_0

be the highest level point in both b and $\gamma(0,x)$, and w_i, $i \ge 1$ be the backbone b from w_0 on.

Under $\mathbb{P}_{x,b}$ the events $F_0(v_{Rj}, \frac{R}{2})$, $1 \le j \le N$ are independent, and $\mathbb{P}_{x,b}(F_0(v_{Rj}, \frac{R}{2})^c) \le \frac{1}{4}$. So standard exponential bounds give

$$\mathbb{P}_{x,b}(F_1(x, v_{R'}, R, N/8)^c) \le c \exp(-c'N). \tag{2.21}$$

Similarly

$$\mathbb{P}_{x,b}(F_1(w_0, w_{R'}, R, N/8)^c) \le c \exp(-c'N).$$

Now let $A_1 = \{v_i, 0 \le i \le R'\} \cup \{w_i, 0 \le i \le R'\}$; note that under $\mathbb{P}_{x,b}$ this set is non-random. Let

$$A_2 = \big\{ y \in \mathbb{B} : a(y, R') \in A_1, \gamma(y, a(y,R')) \cap A_1 = \{a(y,R')\} \big\}.$$

For $y \in A_2$ let

$$\begin{aligned} H_1(y) &= F_1(a(y,R), a(y,R'), R, N/8)^c, \\ H_2(y) &= \{y \in \mathcal{G},\ D_{R'}(y) \ne \emptyset\}. \end{aligned}$$

Then

$$\mathbb{P}_{x,b}\big(\bigcup_{y \in A_2} H_1(y) \cap H_2(y)\big) \le \sum_{y \in A_2} \mathbb{P}_{x,y,b}(H_1(y) \cap H_2(y)) \mathbb{P}_{x,b}(y \in \mathcal{G}).$$

Under $\mathbb{P}_{x,y,b}$ the events $H_1(y)$ and $H_2(y)$ are independent, and as in (2.21) we obtain $\mathbb{P}_{x,y,b}(H_1(y)) \le c \exp(-c'N)$. So,

$$\begin{aligned} \mathbb{P}_{x,b}\big(\bigcup_{y \in A_2} H_1(y) \cap H_2(y)\big) &\le ce^{-c'N} \sum_{y \in A_2} \mathbb{P}_{x,y,b}(H_2(y)) \mathbb{P}_{x,b}(y \in \mathcal{G}) \\ &= ce^{-c'N} \sum_{y \in A_2} \mathbb{P}_{x,b}(H_2(y)) \\ &= ce^{-c'N} \mathbb{E}_{x,b} \sum_{y \in A_2} 1_{H_2(y)}. \end{aligned}$$

The final sum above is bounded by a constant c' by the same argument as in Proposition 2.10.

Finally, we have

$$\begin{aligned} &\{\ G_2(N,R) \text{ fails for } x\} \subset \\ &F_1(x, v_{R'}, R, N/8)^c \cup F_1(w_0, w_{R'}, R, N/8)^c \cup \bigcup_{y \in A_2} (H_1(y) \cap H_2(y)), \end{aligned}$$

so combining the bounds above completes the proof. (b) follows on integrating the bounds in (a).

For (c), we first note that, by the argument for (2.21),

$$\mathbb{P}_{x,y,b}\big(F_1(x, y, \frac{d(x,y)}{N}, \frac{1}{16}N)^c\big) \le c' \exp(-cN).$$

So, using Corollary 2.12, we have

$$\begin{aligned}\mathbb{P}_{x,y,b}\big(F_*^c\big) \le &\mathbb{P}_{x,y,b}\big(F_1(x,z_0,\frac{d(x,y)}{N},\frac{1}{16}N)^c\big)+\mathbb{P}_{x,y,b}\big(F_1(z_0,y,\frac{d(x,y)}{N},\frac{1}{16}N)^c\big)\\ &+\sum_{w=x,z_0,y}\mathbb{P}_{x,y,b}\big(A_*(w,r,N)^c\big)\\ \le &2c'\exp(-cN)+3c'\exp(-cN)=5c'\exp(-cN).\end{aligned}$$

□

Definition 2.15. Let $x,y\in\mathbb{B}$ (with $x\neq y$), $m,\kappa\in\mathbb{N}$, and $c_1\ge 1$. Define the condition $G_3(x,y,m,\kappa)$ as follows. Let $r=d(x,y)/m$, and let $z_0=x,z_1,\dots,z_m=y$ be points on the path $\gamma(x,y)$ with $|d(z_{i-1},z_i)-r|\le 1$. (We choose these points in some fixed way – for example so that $d(z_{i-1},z_i)$ are non-decreasing.) For each $i=1,\dots m$ let Θ_i be the smallest integer $\lambda\ge\max(64,c_1)$ such that $B(z_i,\lambda^{20}r)$ is λ–good, and $|B(z_i,r)|\ge r^2/\lambda^2$. Then $G_3(x,y,m,\kappa)$ holds if:
(a) $x,y\in\mathcal{G}$,
(b) $\sum_{i=1}^m\Theta_i^{54}\le\kappa m$.

Proposition 2.16. *For each backbone b and $x,y\in\mathbb{B}$ with $x\neq y$*

$$\mathbb{P}_{x,y,b}\big(\ G_3(x,y,m,\kappa)\ \textit{holds}\ \big)\ge 1-c_1\kappa^{-1}.$$

Proof. By Proposition 2.7 and Corollary 2.12, $\mathbb{P}_{x,y,b}(\Theta_i=k)\le e^{-ck}$. Thus $\mathbb{E}_{x,y,b}\Theta_i^{54}\le c'$, and so

$$\mathbb{P}_{x,y,b}\big(\ G_3(x,y,m,\kappa)\ \text{fails}\ \big)=\mathbb{P}_{x,y,b}\big(\sum_{i=1}^m\Theta_i^{54}>\kappa m\big)\le c'/\kappa.$$

□

3. Markov chains on weighted graphs and trees

Let Γ be a infinite connected locally finite graph. Assume that the graph Γ is endowed by a weight (conductance) μ_{xy}, which is a symmetric nonnegative function on $\Gamma\times\Gamma$ such that $\mu_{xy}>0$ if and only if x and y are connected by a bond (in which case we write $x\sim y$). We call the pair (Γ,μ) a *weighted graph*. We can also regard it as an electrical network, in which the bond $\{x,y\}$ has conductance μ_{xy}. We will be mainly concerned with the case when $\mu_{xy}=1$ if and only if $\{x,y\}$ is an edge: we call these the *natural weights* on Γ. Let $\mu_x=\sum_{y\in\Gamma}\mu_{xy}$ for each $x\in\Gamma$, and set $\mu(A)=\sum_{x\in A}\mu_x$ for each $A\subset\Gamma$, so that μ is then a measure on Γ.

We next define a quadratic form $\mathcal{E}$ on Γ by

$$\mathcal{E}(f,g)=\frac{1}{2}\sum_{\substack{x,y\in\Gamma\\ x\sim y}}(f(x)-f(y))(g(x)-g(y))\mu_{xy},$$

and set

$$H^2=H^2(\Gamma,\mu)=\{f\in\mathbb{R}^\Gamma:\mathcal{E}(f,f)<\infty\}.$$

We sometimes abbreviate $\mathcal{E}(f,f)$ as $\mathcal{E}(f)$. Note that if $f=\max_{1\le i\le n} g_i$ then since

$$|f(x)-f(y)|^2 \le \max_i |g_i(x)-g_i(y)|^2 \le \sum_i |g_i(x)-g_i(y)|^2,$$

it follows that

$$\mathcal{E}(f,f) \le \sum_{i=1}^n \mathcal{E}(g_i,g_i). \tag{3.1}$$

Let $Y=\{Y_t\}_{t\ge 0}$ be the continuous time random walk on Γ associated with $\mathcal{E}$ and the measure μ. When the natural weights are given on Γ, Y is called the simple random walk on Γ. Y is the Markov process with generator

$$\mathcal{L}f(x)=\frac{1}{\mu_x}\sum_y \mu_{xy}(f(y)-f(x));$$

Y waits at x for an exponential mean 1 random time and then moves to a neighbour y of x with probability proportional to μ_{xy}. We define the transition density (heat kernel density) of Y with respect to μ by

$$q_t(x,y)=\mathbb{P}^x(Y_t=y)/\mu_y. \tag{3.2}$$

If $A\subset\Gamma$ we write

$$T_A=\inf\{t\ge 0: Y_t\in A\}, \qquad \tau_A=T_{A^c}.$$

The natural metric on the graph, obtained by counting the number of steps in the shortest path between points, is written $d(x,y)$ for $x,y\in\Gamma$. As before, we write

$$B(x,r)=\{y: d(x,y)\le r\}, \quad V(x,r)=\mu(B(x,r)).$$

Let A,B be disjoint subsets of Γ. The effective resistance between A and B is defined by:

$$R_{\text{eff}}(A,B)^{-1}=\inf\{\mathcal{E}(f,f): f\in H^2, f|_A=1, f|_B=0\}. \tag{3.3}$$

Let $R_{\text{eff}}(x,y)=R_{\text{eff}}(\{x\},\{y\})$, and $R_{\text{eff}}(x,x)=0$. In general R is a metric on Γ – see [Ki2] Section 2.3. If (Γ,μ) has natural weights then $R_{\text{eff}}(x,y)\le d(x,y)$, and if in addition Γ is a tree then $R_{\text{eff}}(x,y)=d(x,y)$.

The following is an easy consequence of (3.3).

Lemma 3.1. *For all $f\in\mathbb{R}^\Gamma$ and $x,y\in\Gamma$,*

$$|f(x)-f(y)|^2 \le R_{\text{eff}}(x,y)\mathcal{E}(f,f). \tag{3.4}$$

Further, for each $x,y\in\Gamma$, there exists f so that the equality holds in (3.4).

We recall some basic properties of Green kernels. Let Y_t^B be the continuous time random walk on (Γ,μ) killed outside $B:=B(x_0,r)$, and $q_t^B(x,y)$ be the transition density of Y_t^B. The Green kernel $g_B(x,y)$ of Y_t^B is defined by $g_B(x,y) = \int_0^\infty q_t^B(x,y)dt$. Then $g_B(\cdot,\cdot)$ has the reproducing property that

$$\mathcal{E}(g_B(x,\cdot), f) = f(x)$$

for all $f \in H^2$ such that $f|_{B^c} = 0$.

Using this and the fact that $p_B^x(y) := g_B(x,y)/g_B(x,x)$ is the equilibrium potential for $R_{\text{eff}}(x, B^c)$, we have

$$R_{\text{eff}}(x, B^c)^{-1} = \mathcal{E}(p_B^x, p_B^x) = g_B(x,x)^{-1}, \tag{3.5}$$

so that

$$R_{\text{eff}}(x, B^c) = g_B(x,x) = \int_0^\infty q_t^B(x,x)dt \qquad \forall x \in \Gamma, B \subset \Gamma. \tag{3.6}$$

4. Heat kernel estimates on graphs and trees

Recall that for $x \in \Gamma$ and $r \geq 0$, we denote $V(x,r) = \mu(B(x,r))$.

Theorem 4.1. *Let (Γ, μ) be a weighted graph and suppose that the edge weights satisfy $\mu_{xy} \geq 1$ for all x and y. Then*

$$q_{2rV(x,r)}(x,x) \leq \frac{2}{V(x,r)}, \quad x \in \Gamma,\ r \geq 1.$$

Remark. This is similar to the bound in Proposition 3.3 of [BCK].

Proof. Fix $x_0 \in \Gamma$, write $B(r) = B(x_0, r)$ and $V(r) = V(x_0, r)$. Set $f_t(y) = q_t(x_0, y)$ and

$$\psi(t) = ||f_t||_2^2 = q_{2t}(x_0, x_0) = f_{2t}(x_0);$$

note that ψ is decreasing. Let $r \geq 1$; since

$$\sum_{y \in B(r)} f_t(y)\mu_y \leq 1,$$

there exists $y = y(t,r) \in B(r)$ with $f_t(y) \leq V(r)^{-1}$. Note that, since $\mu_e \geq 1$ for every edge e, it follows that $R_{\text{eff}}(x,y) \leq d(x,y)$ for all x, y. Then by (3.4)

$$\begin{aligned} \tfrac{1}{2} f_t(x_0)^2 &\leq f_t(y)^2 + |f_t(x_0) - f_t(y)|^2 \\ &\leq \frac{1}{V(r)^2} + R_{\text{eff}}(x_0, y)\mathcal{E}(f_t, f_t) \leq \frac{1}{V(r)^2} + r\mathcal{E}(f_t, f_t). \end{aligned}$$

Hence

$$\psi'(t) = -2\mathcal{E}(f_t, f_t) \leq \frac{2V(r)^{-2} - \psi(t/2)^2}{r}.$$

Since $-\psi(s/2) \leq -\psi(t)$ for $t \leq s \leq 2t$, integrating the above inequality from t to $2t$ we obtain

$$\psi(2t) - \psi(t) \leq 2tr^{-1}V(r)^{-2} - tr^{-1}\psi(t)^2.$$

So as $\psi(2t) > 0$,

$$tV(r)^2\psi(t)^2 \leq 2t + rV(r)^2\psi(t) \leq (4t) \vee (2rV(r)^2\psi(t)).$$

Hence

$$\psi(t) \leq \frac{2}{V(r)} \vee \frac{2r}{t}. \tag{4.1}$$

Taking r such that $t = rV(r)$ completes the proof. $\square$

Corollary 4.2. *Let $V(x,r) \geq r^2/A$, $r \geq 1$, and $t = r^3$. Then*

$$q_{2t}(x,x) \leq \frac{2(A \vee 1)}{r^2} = \frac{2(A \vee 1)}{t^{2/3}}. \tag{4.2}$$

Proof. Let $t = r^3$ and $V(x,r) = \lambda r^2$ in (4.1). This gives

$$\psi(t) \leq \frac{2}{\lambda r^2} \vee \frac{2r}{r^3} = 2r^{-2}(1 \vee \lambda^{-1}) \leq \frac{2(A \vee 1)}{t^{2/3}},$$

since $\lambda \geq A^{-1}$. $\square$

Lemma 4.3. *Let $f_t(y) = q_t(x_0, y)$. Then*

$$\left| \frac{f_t(y)}{f_t(x_0)} - 1 \right|^2 \leq \frac{d(x_0,y)}{t f_t(x_0)}. \tag{4.3}$$

Proof. Let $e(t) = \mathcal{E}(f_t, f_t)$. Then $\psi'(t) = -2e(t)$, and e is decreasing, we have

$$te(t) = 2e(t) \cdot t/2 \leq 2 \int_{t/2}^{t} e(s)ds = \psi(t/2) - \psi(t) \leq \psi(t/2) = f_t(x_0).$$

So, by (3.4),

$$|f_t(x_0) - f_t(y)|^2 \leq d(x_0,y)e(t) \leq \frac{d(x_0,y)f_t(x_0)}{t},$$

and dividing by $f_t(x_0)^2$ completes the proof. $\square$

From now on we assume that Γ is a graph with natural weights.

Proposition 4.4. *Let $x_0 \in G$, $r \geq 1$, and let $m \geq 1$, $\varepsilon \leq 1/(2m)$. Write $B = B(x_0,r)$, $B' = B(x_0, \frac{1}{2}\varepsilon r)$, $V = V(x_0,r)$, $V' = V(x_0, \frac{1}{2}\varepsilon r)$. Then*
(a)

$$E^z \tau_B \leq 2rV(x_0,r), \qquad z \in B(x_0,r). \tag{4.5}$$

Suppose further that

$$R_{\text{eff}}(x, B^c) \geq r/m \quad \text{for } x \in B(x_0, \varepsilon r). \tag{4.4}$$

Then
(b)

$$E^x \tau_B \geq \frac{rV'}{4m} \quad \text{for } x \in B(x_0, \tfrac{1}{2}\varepsilon r). \tag{4.6}$$

(c) For $x \in B(x_0, \frac{1}{2}\varepsilon r)$,

$$P^x(\tau_B \leq t) \leq \Big(1 - \frac{V'}{8mV}\Big) + \frac{t}{2rV}. \tag{4.7}$$

(d)
$$q_{2t}(x,x) \ge \frac{c_1(V')^2}{m^2V^3} \quad \text{for } t \le \frac{rV'}{8m}, \quad x \in B(x_0, \tfrac{1}{2}\varepsilon r). \tag{4.8}$$

Proof. For any $z \in B$ we have
$$E^z \tau_B = \sum_{y\in B} g_B(z,y)\mu_y.$$

(a) Since $R_{\text{eff}}(z, B^c) \le 2r$ for any $z \in B$,
$$E^z\tau_B = \sum_{y\in B} g_B(z,y)\mu_y \le \sum_{y \in B} g_B(z,z)\mu_y = R_{\text{eff}}(z,B^c)V(x,r) \le 2rV(x,r).$$

(b) As in (3.5), $\mathcal{E}(p_B^x, p_B^x) = g_B(x,x)^{-1}$ and so if $x, y \in B'$
$$|1 - p_B^x(y)|^2 \le d(x,y) R_{\text{eff}}(x,B^c)^{-1} \le m\varepsilon \le \tfrac{1}{2}.$$
Hence $p_B^x(y) \ge 1 - 2^{-1/2} \ge \frac{1}{4}$. So,
$$E^x\tau_B \ge \sum_{y\in B'} g_B(x,x)p_B^x(y)\mu_y \ge \tfrac{1}{4}\mu(B')R_{\text{eff}}(x,B^c) \ge r\mu(B')/(4m).$$

(c) By the Markov property, (4.5) and (4.6), for $x \in B'$,
$$\frac{rV'}{4m} \le E^x[\tau_B] \le t + E^x[1_{\{\tau_B > t\}} E^{Y_t}(\tau_B)] \le t + 2rVP^x(\tau_B > t),$$
for all $t > 0$. Rearranging this gives (c).
(d) By (4.7),
$$P^x(Y_t \in B) \ge P^x(\tau_B > t) \ge \frac{(rV'/4m) - t}{2rV}.$$
So, if $t \le rV'/(8m)$ then
$$P^x(Y_t \in B) \ge \frac{c_2 V'}{mV}. \tag{4.9}$$
By Chapman-Kolmogorov and Cauchy-Schwarz
$$P^x(Y_t \in B)^2 = (\sum_{y\in B} q_t(x,y)\mu_y)^2 \le \mu(B)\sum_{y\in B} q_t(x,y)^2\mu_y \le q_{2t}(x,x)V,$$
and using (4.9) gives (4.8). □

We will now use these bounds for a subtree G of $\mathbb{B}$. From now on we take Γ to be a subgraph of $\mathbb{B}$, and define $M(x,r)$, and the conditions λ–good, $G_2(N,R)$ and $G_3(x,y,m,\kappa)$ as in Section 2.

Lemma 4.5. *Let $B = B(x_0, r)$, $r \ge 1$, and $x \in B(x_0, r/8)$. Then*

$$\frac{r}{8M(x_0,r)} \le g_B(x,x) = R_{\text{eff}}(x, B^c) \le 9r/8.$$

Proof. Since x is connected to $B(x_0, r)^c$ by a path of length $9r/8$, the upper bound is clear.

For the lower bound let $m = M(x_0, r)$ and $A = \{z_1, \dots, z_m\}$ be the set given in Definition 2.9: note that $d(x, z_i) \ge r/8$ for each i. Let h_i be the function on G such that $h_i(z_i) = 1, h_i(x) = 0$ and h_i is harmonic $G - \{x, z_i\}$. Then $h_i(y) = \mathbb{P}^y(T_{z_i} < T_x)$, and

$$\mathcal{E}(h_i, h_i) = R_{\text{eff}}(x, z_i)^{-1} = d(x, z_i)^{-1} \le \frac{8}{r}.$$

If $y \in B(x, r)^c$ then since any path from y to x passes through A, we have $h_i(y) = 1$ for at least one i. So if $h = \max_i h_i$ then $h(x) = 0$ and $h = 1$ on $B(x, r)^c$. So, using (3.1),

$$R_{\text{eff}}(x, B^c)^{-1} \le \mathcal{E}(h, h) \le m \max_i \mathcal{E}(h_i, h_i) \le \frac{8M(x_0, r)}{r},$$

proving the lower bound □

Theorem 4.6. *Let $\lambda \ge 64$, and suppose that $B = B(x_0, r)$ is λ–good. Let $I = I(\lambda, r) = [r^3\lambda^{-6}, r^3\lambda^{-5}]$.*
(a) For $x \in B(x_0, r/\lambda)$,

$$2\frac{r^3}{\lambda^5} \le E^x \tau_B \le 2\lambda r^3. \tag{4.10}$$

(b) For each $K \ge 0$

$$q_{2t}(x_0, y) \le (1 + \sqrt{K}) t^{-2/3} \lambda^3 \quad \text{for } t \in I, \quad y \in B(x_0, Kt^{1/3}). \tag{4.11}$$

(c) Let $x \in B(x_0, r/\lambda)$. Then

$$q_{2t}(x, y) \ge c_1 t^{-2/3} \lambda^{-17}, \quad \text{if } d(x, y) \le c_2 \lambda^{-19} r, \quad t \in I. \tag{4.12}$$

Proof. (a) Since B is λ–good, we have $M(x_0, r) \le \lambda/64$. Let $m = \lambda/8$ and $\varepsilon = 1/(2m) = 4/\lambda$. Then

$$R_{\text{eff}}(x, B^c) \ge \frac{r}{8M} \ge \frac{r}{m} \text{ for } x \in B(x_0, \varepsilon r).$$

Also $V(x_0, r) \le \lambda r^2$, while since $\frac{1}{2}\varepsilon = 2/\lambda$, $V(x_0, \frac{1}{2}\varepsilon r) \ge V(x_0, r/\lambda) \ge r^2\lambda^{-4}$. (4.10) now follows from (4.5) and (4.6).
(b) Let $t_1 = (r/\lambda^2)^3$. Then by Corollary 4.2 (taking $A = \lambda^2$), if $t \in I$,

$$q_{2t}(x_0, x_0) \le q_{2t_1}(x_0, x_0) \le 2\lambda^2 t_1^{-2/3} \le 2\lambda^{8/3} t^{-2/3} \le \lambda^3 t^{-2/3}. \tag{4.13}$$

Now, for $t \in I$ and $y \in B(x_0, Kt^{1/3})$, we have, using Lemma 4.3 and (4.13),

$$\begin{aligned} q_{2t}(x_0, y) &\le q_{2t}(x_0, x_0) + |q_{2t}(x_0, y) - q_{2t}(x_0, x_0)| \\ &\le q_{2t}(x_0, x_0) + \sqrt{\frac{K}{2t^{2/3}} q_{2t}(x_0, x_0)} \le (1 + \sqrt{K}) t^{-2/3} \lambda^3, \end{aligned}$$

proving (4.11).
(c) Since $B(x_0, \frac{1}{2}\varepsilon r) \subset B(x_0, 2r/\lambda)$, and $rV'/(8m) \geq r^3\lambda^{-5}$, by Proposition 4.4(d), for $t \in I$,

$$q_{2t}(x,x) \geq c_2 (V')^2/(V^3 m^2) \geq c_2 r^{-2}\lambda^{-13} \geq c_2 t^{-2/3}\lambda^{-17}.$$

Hence, by Lemma 4.3, if $d(x,y) \leq c_2\lambda^{-19} r$,

$$\Big|\frac{q_{2t}(x,y)}{q_{2t}(x,x)} - 1\Big|^2 \leq \frac{d(x,y)}{2t q_{2t}(x,x)} \leq \frac{d(x,y) r^2 \lambda^{13}}{2c_2 t} \leq \frac{d(x,y)\lambda^{19}}{2c_2 r} \leq \tfrac{1}{2},$$

from which (4.12) follows. □

Corollary 4.7. *Let $\lambda \geq 64$, and $B(x,r)$ and $B(x, \lambda^{-5} r)$ be λ–good. Then*

$$E^x d(x, Y_t) \geq c_1 \lambda^{-4} t^{1/3}, \qquad \text{for } \frac{r^3}{\lambda^6} \leq t \leq \frac{r^3}{\lambda^5}.$$

Proof. Let $I = [r^3\lambda^{-6}, r^3\lambda^{-5}]$ and $B' = B(x, r\lambda^{-5})$. Let $t \in I$, and $y \in B'$. Then since $r \leq \lambda^2 t^{1/3}$, $d(x_0, y) \leq \lambda^{-5} r \leq \lambda^{-3} t^{1/3}$, so by (4.11) (with $K = 1$) we have $q_{2t}(x_0, y) \leq 2t^{-2/3}\lambda^3$. Hence since B' is λ–good,

$$P^x(Y_{2t} \in B') = \sum_{y \in B'} q_{2t}(x_0, y)\mu_y \leq \mu(B') 2t^{-2/3}\lambda^3 \leq 2\lambda^{-2} \leq \tfrac{1}{2}.$$

Thus

$$E^x d(x, Y_{2t}) \geq \lambda^{-5} r P^x(Y_{2t} \notin B') = \lambda^{-5} r (1 - P^x(Y_{2t} \in B')) \geq \tfrac{1}{2} r\lambda^{-5} \geq \tfrac{1}{2} t^{1/3} \lambda^{-10/3}.$$

□

Lemma 4.8. *Suppose x satisfies $G_2(N, R)$. Then*

$$P^x(\tau_{B(x,NR)} \leq t) \leq e^{-c_1 N} \qquad \text{provided } N \geq c_2 t/R^3.$$

Proof. We use the argument of [BB1]. Let

$$A = \{y \in G : B(y, R/2) \text{ is } \lambda_1\text{–good}\}.$$

Define stopping times (T_i), (S_i) by taking $T_0 = \min\{t : Y_t \in A\}$, and

$$S_n = \min\{t \geq T_{n-1} : Y_t \notin B(Y_{T_{n-1}}, R/2)\},$$
$$T_n = \min\{t \geq S_n : Y_t \in A\}.$$

Since x satisfies $G_2(N, R)$ we have $T_{N/8} \leq \tau_{B(x,NR)}$ P^x -a.s. Let $\xi_i = S_{i+1} - T_i$, $i \geq 1$. Then by Proposition 4.4(c) and Lemma 4.5, there exist $p = p(\lambda_1) < 1$ and $c_3 = c_3(\lambda_1) > 0$ such that

$$P^x\big(\xi_i \leq s | \sigma(Y_u, 0 \leq u \leq T_i)\big) \leq p + c_3 R^{-3} s. \tag{4.14}$$

Lemma 1.1 of [BB1] (see also Lemma 3.14 of [B1]) gives that, writing $a = c_3/R^3$, (4.14) implies that

$$\log P^x\Big(\sum_{i=1}^{N/8} \xi_i \leq t\Big) \leq -\tfrac{1}{8} N \log(1/p) + 2\Big(\frac{aNt}{8p}\Big)^{1/2}.$$

Substituting for a we deduce that

$$\log P^x(\tau_{B(x,NR)} \leq t) \leq -N\Big(2c_4 - c_5 (t/(R^3 N))^{1/2}\Big) \leq -c_4 N,$$

provided $N \geq (c_5/c_4)^2 \cdot (t/R^3)$. □

Theorem 4.9. *Let $x, y \in \mathcal{G}$, $t \geq 64d(x,y)$ be such that $N := [\sqrt{d(x,y)^3/t}] \geq 8$ and suppose the event $F_*(x,y,d(x,y)N^{-1}, \frac{1}{8}N; d(x,y)^3t^{-2/3}, N)$ holds. Then*

$$q_t(x,y) \leq c_1 t^{-2/3} \exp(-c_2 N). \tag{4.15}$$

Proof. Define $T_{z_0} = \inf\{t : Y_t = z_0\}$ and $R = d(x,y)/N$, where z_0 is a middle point in $\gamma(x,y)$. Let G_x be the set of points w in $\mathcal{G}$ such that $\gamma(x,w)$ does not contain z_0, and let $G_y = \mathcal{G} - G_x$. Then, we have

$$\begin{aligned}
q_t(x,y)\mu_x\mu_y &= \mu_x P^x(Y_t = y) \\
&= \mu_x P^x(Y_{t/2} \in G_y, Y_t = y) + \mu_x P^x(Y_{t/2} \in G_x, Y_t = y) \\
&= \mu_x P^x(Y_{t/2} \in G_y, Y_t = y) + \mu_y P^y(Y_{t/2} \in G_x, Y_t = x),
\end{aligned} \tag{4.16}$$

where in the last line we used the μ–symmetry of Y. The two terms in (4.16) are bounded in the same way. For the first,

$$\begin{aligned}
P^x(Y_{t/2} \in G_y, Y_t = y) &\leq P^x(T_{z_0} \leq t/2, Y_t = y) \\
&= E^x\big(1_{(T_{z_0} \leq t/2)} P^{z_0}(Y_{t - T_{z_0}} = y)\big) \\
&\leq P^x(T_{z_0} \leq t/2) \sup_{t/2 \leq s \leq t} q_s(z_0, y)\mu_y. \\
&\leq \mu_y \sqrt{q_{t/2}(y,y) q_{t/2}(z_0,z_0)}\, P^x(T_{z_0} \leq t/2) \\
&\leq \mu_y N^3 t^{-2/3} P^x(T_{z_0} \leq t/2),
\end{aligned}$$

where we used (4.11) with $\lambda = N, r = N^2 t^{1/3}$ in the last inequality. Now, $t/R^3 \sim (d(x,y)^3/t)^{1/2} \sim N$, so $N \geq ct/R^3$. Thus, by Lemma 4.8 we have

$$P^x(T_{z_0} \leq t/2) \leq e^{-cN} \quad \text{and} \quad P^y(T_{z_0} \leq t/2) \leq e^{-cN}.$$

Combining these facts

$$q_t(x,y) \leq c' N^3 t^{-2/3} e^{-cN} \leq c t^{-2/3} e^{-c'' N},$$

which completes the proof. □

Theorem 4.10. *Let $x, y \in \mathcal{G}$ with $x \neq y$, $m \geq 1$, $\kappa \geq 1$ and suppose $G_3(x,y,m,\kappa)$ holds. Then if $T = d(x,y)^3\kappa/m^2$*

$$q_{2T}(x,y) \geq c_1 T^{-2/3} e^{-c_2(\kappa + c_3)m}. \tag{4.17}$$

Proof. Let $r = d(x,y)/m$, and (z_i), (Θ_i) be the points and integers given by the condition $G_3(x,y,m,\kappa)$ in Definition 2.15. Take the constant $c_{2.15.1} = 3c_{4.6.2}^{-1}$. Let $B_i = B(z_i, \Theta_i^{20} r)$, and $B_i' = B(z_i, r)$. Applying (4.12) to B_i we deduce that if $d(y,y') \leq c_{4.6.2}\Theta^{-19}(\Theta_i^{20} r)$, and

$$\Theta_i^{54} r^3 \leq t_i \leq \Theta_i^{55} r^3, \tag{4.18}$$

then

$$q_{2t_i}(y,y') \geq c_4 t_i^{-2/3}\Theta_i^{-17}. \tag{4.19}$$

If $y_i \in B_i'$ then by the choice of Θ_i

$$d(y_{i-1},y_i) \leq 3r \leq c_{4.6.2}\Theta_i^{-19}(\Theta_i^{20}r) = c_{4.6.2}\Theta_i r,$$

and so the bound in (4.19) holds for $q_{2t_i}(y,y')$. Therefore for $y_{i-1} \in B_{i-1}'$ and t_i satisfying (4.18),

$$\int_{B_i'} q_{2t_i}(y_{i-1},y_i)\mu(dy_i) \geq c_4 t_i^{-2/3}\Theta_i^{-17}\mu(B_i') \geq c_4\Theta_i^{-c_5};$$

we used here the fact that $\mu(B_i') \geq \Theta_i^{-2}r^2$. So if t_i satisfy (4.18), and $s = \sum t_i$ then since $\sum \log\Theta_i \leq \sum \Theta_i^{54} \leq m\kappa$,

$$\begin{aligned}
q_{2s}(x,y) &\geq \int_{B_1'} \cdots \int_{B_{m-1}'} q_{2t_1}(x,y_1)q_{2t_1}(y_1,y_2)\dots q_{2t_m}(y_{m-1},y)\mu(dy_1)\dots\mu(dy_{m-1}) \\
&\geq (ct_m^{-2/3}\Theta_m^{-17})c_4^{m-1}\Pi_{i=1}^{m-1}\Theta_i^{-c_5} \geq s^{-2/3}\exp(-c_6 m - c_5\sum\log\Theta_i) \\
&\geq s^{-2/3}e^{-(c_5\kappa+c_6)m}.
\end{aligned}$$

As $G_3(x,y,m,\kappa)$ holds we have $r^3\sum\Theta_i^{54} \leq m\kappa r^3 = T$. If $T \leq r^3\sum\Theta_i^{55}$ we can choose (t_i) satisfying (4.18) so that $s = T$. If not, let $s' = T - s$, so that $s' \leq m\kappa r^3$. Fix a j such that Θ_j is minimal and in the chaining argument above add m' extra steps (of time length t' satisfying (4.18) for $i = j$) between B_{j-1}' and B_j'. Since $c_7^{54} \leq \Theta_j^{54} \leq \kappa$, we have $c_8 r^3 \leq t' \leq \kappa r^3$. Then choose m', t' so that $m't' + s = T$; we have $m' \leq cm$. Each extra step gives a factor of $c_4\Theta_j^{-c_5}$ in the lower bound in the chaining argument, so the total contribution multiplies the lower bound by a number greater than $e^{-c(\kappa+c')m}$. Thus (4.17) holds. □

5. Random walk on the conditioned critical GW-branching process

In this section, we state and prove our main results on the random walk on the IIC. As in Section 2 we write $\mathcal{G}$ for the IIC on $\mathbb{B}$, and $\mathbb{P}$ for its law. Let $Y = \{Y_t\}_{t\geq 0}$ be the simple random walk on $\mathcal{G}(\omega)$ defined in Section 3; we write E_ω^x for its law of Y started at x. Let $q_t^\omega(x,y)$ be the transition density of Y.

Proof of Theorem 1.2. (a) Note that Theorem 4.6(b) and (c) give that, if $B(x,r)$ is λ–good, then if $t = \lambda^{-6}r^3$ then

$$c_6\lambda^{-17} \leq t^{2/3}q_t(x,x) \leq c_7\lambda^3. \tag{5.1}$$

Given $\theta \geq 1$ choose $\lambda \geq 64$ as small as possible such that $c_6\lambda^{-17} \leq \theta^{-1} \leq \theta \leq c_7\lambda^3$. Let $r = \lambda^2 t^{1/3}$. Then the probability that $B(x,r)$ is λ–good is at least $ce^{-c'\lambda}$, and using (5.1) completes the proof.

(b) Fix $x \in \mathbb{B}$, and let $c_3 = c_{2.12.2}$. Let $a = 2/c_3$ and $\lambda_n = e + a\log n$, and r_n satisfy $r_n^3\lambda_n^{-6} = e^n$. Let F_n be the event that $B(x,r_n)$ is λ_n–good. Then by Corollary 2.12

$$\mathbb{P}(F_n^c) \leq ce^{-c_3 a\log n} = c'n^{-2},$$

so by Borel-Cantelli F_n^c occurs for only finitely many n, $\mathbb{P}$–a.s. Let N be the largest m such that F_m^c occurs; then

$$\mathbb{P}(N > m) \le \sum_{m+1}^{\infty} \mathbb{P}(F_n^c) \le cm^{-1}.$$

Set $S(x) = e^N$. For $n \ge (\log S(x)) + 1$ we have, by (4.11) and (4.12),

$$c' t^{-2/3} \lambda_n^{-17} \le q_{2t}(x,x) \le c'' t^{-2/3} \lambda_n^3 \tag{5.2}$$

for $e^n \le t \le \lambda_n e^n$. Let $n(t)$ be the unique integer such that $\log t \in [n(t)-1, n(t))$. Hence, if $t \ge S(x)$, $n(t) > N$ and so (5.2) holds for $n = n(t)$. Since

$$\lambda_{n(t)} = e + a \log n(t) \sim a \log\log t,$$

we obtain (1.3).
(c) is immediate from (b). □

While the powers of the terms in $\log\log t$ given in Theorem 1.2 are not the best possible, we do have oscillations in $t^{-2/3} q_t^\omega(.,.)$ of that order.

Lemma 5.1.

$$\liminf_{t\to\infty} (\log\log t)^{1/6} t^{2/3} q_{2t}^\omega(0,0) \le 2, \quad \mathbb{P}-a.s. \tag{5.3}$$

Proof. Define a_n by $V(0,2^n) = a_n 2^{2n}$, and let $t_n = 2^n V(0,2^n) = a_n 2^{3n}$. Then by Theorem 4.1,

$$q_{2t_n}^\omega(0,0) \le \frac{2}{V(0,2^n)} = \frac{2t_n^{-2/3}}{a_n^{1/3}}.$$

By Proposition 2.8(a), $a_n > (\log n)^{1/2}$ for infinitely many n, a.s., giving (5.3). □

Proof of Theorem 1.3. (a) follows from Theorem 4.6(a) and Corollary 2.12 by an easy argument similar to that used for Theorem 1.2(a). (b) is then immediate from (a). □

Proof of Theorem 1.4. (a) The lower bound in (1.4) is an immediate consequence of Corollaries 2.12 and 4.7. For the upper bound, let $Z_t = \sup_{0\le s\le t} d(x, Y_s)$, $R = t^{1/3}$ and $T_M = \tau_{B(x,MR)}$. Let $K_t(x)(\omega)$ be the largest n such that x does not satisfy $G_2(n,R)$. Then by Proposition 2.14

$$\mathbb{P}_x(K_t(x) \ge k) \le \sum_{l=k}^{\infty} \mathbb{P}_x(x \text{ does not satisfy } G_2(l,R)) \le c' e^{-ck}. \tag{5.4}$$

Then $\{Z_t \ge nR\} \subset \{T_n \le t\}$, and so by Lemma 4.8,

$$\begin{aligned} E_\omega^x Z_t &\le R \sum_{n=0}^{\infty} P_\omega^x(T_n \le t) \\ &\le R\Big(1 + K_t(x) + \sum_{n=K_t(x)+1}^{\infty} P_\omega^x(T_n \le t)\Big) \\ &\le R\Big(1 + K_t(x) + \sum_{n=K_t(x)+1}^{\infty} e^{-cn}\Big) \le R(c + K_t(x)). \end{aligned} \tag{5.5}$$

Since $\mathbb{E}_x K_t(x) \leq c'$ this completes the proof.
(b) Let $m(t) = \lfloor t \rfloor$; Since

$$|E^x_\omega d(x, Y_t) - E^x_\omega d(x, Y_{m(t)})| \leq E^x_\omega d(Y_{m(t)}, Y_t) \leq c,$$

it is enough to prove (1.5) for integer t. Using (5.4) and Borel-Cantelli there exists c' such that

$$\mathbb{P}_x(K_n(x) > c' \log n \quad \text{i.o.}) = 0.$$

and so by (5.5)

$$E^x_\omega d(x, Y_n) \leq c'' n^{1/3} \log n$$

for all sufficiently large n. The lower bound in (1.5) follows from Corollary 4.7 by the same argument as in Theorem 1.2. □

Proof of Theorem 1.5. We begin with the on-diagonal case $x = y$. Let $\lambda_n = n$ and r_n be defined by $2r_n^3/\lambda_n^6 = t$. Let $F_n = \{B(x, r_n) \text{ is } \lambda_n\text{–good }\}$, and $N(\omega) = \min\{n : \omega \in F_n\}$. By Corollary 2.12 $\mathbb{P}_x(N > n) \leq \mathbb{P}_x(F_n^c) \leq e^{-cn}$. On F_n we have, by (4.11), $q^\omega_t(x, x) \leq ct^{-2/3} n^3$, so

$$\mathbb{E}_x[q^\omega_t(x, x)] \leq ct^{-2/3}\mathbb{E}_x N^3 \leq c' t^{-2/3}, \tag{5.6}$$

proving the on-diagonal upper bound in (1.6).

For the on-diagonal lower bound choose m_0 such that $\mathbb{P}_x(F_{m_0}) \geq \frac{1}{2}$ and then on F_{m_0}, by the lower bound in (4.12),

$$q^\omega_t(x, x) \geq ct^{-2/3} m_0^{-17}.$$

For the off-diagonal bounds, when $d(x, y) \leq 64t^{1/3}$, (1.6) can be proved similarly to (5.6) using Theorem 4.6(b). So we will assume $d(x, y) > 64t^{1/3}$. Now, let $N := [\sqrt{d(x, y)^3/t}] \geq 8$ and define $F_0 = F_*(x, y, d(x, y)N^{-1}, \frac{1}{8}N; d(x, y)^3 t^{-2/3}, N)$. Let $\lambda_0 = N$ and define $\lambda_n = N + n - 1$ for $n \geq 1$. For each $n \geq 1$, set $r_n = t^{1/3}\lambda_n^2$ and let $F_n = \{B(x, r_n)$ is λ_n-good $\}$. Then, $\mathbb{P}_{x,b}(F_n^c) \leq e^{-c\lambda_n}$. We now apply Theorem 4.6 (b) with $K = \lambda_n^2$ and obtain the following. (Note that we can apply the theorem because $d(x, y)/t^{1/3} \leq cN^{2/3} \leq c\lambda_n^2$.)

$$q_{2t}(x, y) \leq c(1 + \sqrt{\lambda_n^2}) t^{-2/3}\lambda_n^3 \leq c' t^{-2/3}\lambda_n^4. \tag{5.7}$$

Let $M(\omega) = \min\{n \geq 0 : \omega \in F_n\}$. Then, $\mathbb{P}_x(M = 0) = \mathbb{P}_x(F_0) \geq 1 - 4e^{-N}$ and $\mathbb{P}_x(M > n) \leq \mathbb{P}_x(F_n^c) \leq ce^{-c'\lambda_n}$. Thus, using Theorem 4.9 and (5.7), we obtain

$$\begin{aligned}\mathbb{E}_{x,y}[q^\omega_t(x, y)] =& \mathbb{E}_{x,y}[q^\omega_t(x, y) : M = 0] + \mathbb{E}_{x,y}[q^\omega_t(x, y) : M > 0] \\ \leq& ct^{-2/3}\exp(-c'N) + c'' t^{-2/3}\mathbb{E}[\lambda_M^4 : M > 0].\end{aligned}$$

Since $\mathbb{E}[\lambda_M^4 : M > 0] \leq c\sum_{k=1}^\infty (N + k - 1)^4 e^{-c'(N+k-1)} \leq ce^{-c''N}$, we obtain (1.6).

We next prove (b). Choose $\kappa = 2c_{2.16.1}$, so that $\mathbb{P}_{x,y}(G_3(x, y, m, \kappa) \text{ holds }) \geq \frac{1}{2}$. Now choose $m = (R^3\kappa/t)^{1/2}$; by Theorem 4.10, for ω such that $G_3(x, y, m, \kappa)$ holds,

$$q^\omega_{2t}(x, y) \geq ct^{-2/3}\exp(-c'(\kappa + c'')m).$$

Taking expectations gives (1.7). $\square$

Let

$$\widetilde{Z}_t^{(n)} = n^{-1/3} d(0, Y_{nt}), \quad t \geq 0.$$

By Theorem 1.3(a) the process $\widetilde{Z}^{(n)}$ is tight with respect to the annealed law given by the semi-direct product $\mathbb{P}^* = \mathbb{P} \times P_\omega^0$. (See Theorem 1.1 for the analogous result for the discrete time simple random walk.)

Proof of Theorem 1.6. Let $U_n = \sup_{0 \leq s \leq 1} Z_s^{(n)}$. Then, by (4.5),

$$\begin{aligned} P_\omega^0(U_n \leq \lambda) &= P_\omega^0(\sup_{t \leq n} d(0, Y_s) \leq \lambda n^{1/3}) \\ &= P_\omega^0(\tau_{B(0, \lambda n^{1/3})} \geq n) \leq \frac{2\lambda n^{1/3} V(0, \lambda n^{1/3})}{n}. \end{aligned}$$

So by Proposition 2.8(b), we have, for any $\lambda > 0$, that $\liminf_{n \to \infty} P_\omega^0(U_n \leq \lambda) = 0$, which shows that the r.v. U_n (and hence the processes $Z^{(n)}$) are not tight. $\square$

Remark 5.2. This result illustrates the difference in the type of results that can arise between the quenched and annealed cases. For the case of supercritical bond percolation in $\mathbb{Z}^d$, while an invariance principle was proved in the annealed case in [DFGW] in 1989, the quenched case was only proved recently in [SS], [BB], [MP].

Remark 5.3. In this paper we have for simplicity treated the continuous time random walk Y on $\mathcal{G}$. Similar proofs work for the discrete time random walk $X = (X_n, n \geq 0, Q_\omega^x, x \in \mathcal{G}(\omega))$. Some of the modifications are minor, but the arguments in Section 4 using the relation $\frac{d}{dt}||q_t||_2^2 = \mathcal{E}(q_t, q_t)$ are more complicated in discrete time. For a general treatment of discrete time walks, see [BCK].

Write $p_n(x, y) = Q^x(X_n = y)/\mu_y$, and set $g_n(x) = p_n(x_0, x) + p_{n+1}(x_0, x)$. See [BCK], Proposition 3.2 for a discrete time version of Theorem 4.1. We have (see for example (3.4) in [BCK]) $\mathcal{E}(g_n, g_n) = g_{2n}(x_0) - g_{2n+2}(x_0)$. Using this, and the bound $n\mathcal{E}(g_n, g_n) \leq c p_{2\lfloor n/2 \rfloor}(x_0, x_0)$, (see Lemma 3.10 in [BCK]), one obtains Lemma 4.3 for g_n.

Since the mean time between jumps of Y is 1, mean hitting times are the same for X and Y, so Proposition 4.4(a) and (b) also hold for X. The proofs of (c) and (d) just use the Markov property, and so these also hold for X.

The arguments for the remainder of Section 4 and 5 are based on the first four results in Section 4. No real issues arise in working with discrete rather than continuous time, but a careful proof would have to ensure that all the times considered were integers.

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