Corrected proof of Proposition 3.3. The changes from the the published version are in equations (3.7), (3.8), and the lines -2 and -3 of page 3, and are marked in red.

Proposition 3.3 *Suppose VD, UHKP(α), NDLB(α) and UJS hold. Then PHI(α) holds.*

Proof. Let $\lambda \in (0,1]$, $R \geq 1$, $T = \lambda R^\alpha$, $x_0 \in G$, and write:

$$B_0 = B(x_0, R/2), \quad B' = B(x_0, 3R/4), \quad B = B(x_0, R),$$

and

$$Q = Q(x_0, T, R) = [0,T] \times B, \quad E = (0,T] \times B'.$$

We consider the space time process on $\mathbb{R} \times G$ given by $Z_t = (V_0 - t, X_t)$, for $t \geq 0$.

Let $u(t,x)$ be non-negative and caloric on Q. Define the réduite u_E by

$$u_E(t,x) = \mathbb{E}^x\big(u(t - T_E, X_{T_E}); T_E < \tau_Q\big),$$

where T_E is the hitting time of E by Z, and τ_Q the exit time by Z from Q. Then $u_E = u$ on E, $u_E = 0$ on Q^c, and $u_E \leq u$ on $Q - E$.

The process Z has as a dual the process $\widehat{Z}_t = (V_0 + t, X_t)$; we may therefore apply the results of Chapter VI of [BG]. The balayage formula gives

$$u_E(t,x) = \int_E p^B_{t-r}(x,y)\nu_E(dr,dy), \quad (t,x) \in Q,$$

where ν_E is a measure on $\overline{E}$. We write

$$\nu_E(dr,dy) = \sum_{z \in B'} \nu_E(dr,z)\delta_z(dy)\mu_z.$$

We can divide each of the measures $\nu_E(dr,z)$ into two parts: an atom at 0, and the remainder. Given this we can write

$$u_E(t,x) = \sum_{z \in B'} p^B_t(x,z)u(0,z)\mu_z + \sum_{z \in B'} \int_{(0,t]} p^B_{t-r}(x,z)\mu_z \nu_E(dr,z). \tag{3.4}$$

To identify $\nu_E(dr,z)$ note that if $(t,x) \in E$ then

$$\frac{\partial u_E}{\partial t} = \frac{\partial u}{\partial t} = \mathcal{L}u = \mathcal{L}(u - u_E) + \mathcal{L}u_E. \tag{3.5}$$

Differentiating (3.4) we deduce that each measure $\nu_E(dr,z)$ is absolutely continuous with respect to Lebesgue measure, and that, writing $\nu_E(dr,z) = v(r,z)\,dr$,

$$\frac{\partial u_E}{\partial t}(t,x) = v(t,x) + \mathcal{L}u_E(t,x). \tag{3.6}$$

Using (3.5) this gives

$$v(t,x) = \mathcal{L}(u - u_E)(t,x) = \mu_x^{-1} \sum_{z \in G - B'} J(x,z)(u(t,z) - u_E(t,z)). \tag{3.7}$$

Let

$$w_t(x) = u(t,x) - u_E(t,x), \qquad Jw_r(z) = \sum_{y \in G - B'} J(z,y) w_r(y). \tag{3.8}$$

Then combining (3.4) and (3.7), for $x \in B_0$, $t \in [0,T]$,

$$u(t,x) = \sum_{z \in B'} p_t^B(x,z) u(0,z) \mu_z + \sum_{z \in B'} \int_{(0,t]} p_{t-r}^B(x,z) Jw_r(z) dr. \tag{3.9}$$

Now let $(t_1, x_1) \in Q_-$ and $(t_2, x_2) \in Q_+$. To prove the parabolic Harnack inequality it is enough, using (3.9), to show that:

$$p_{t_1}^B(x_1,z) \le C p_{t_2}^B(x_2,z) \quad \text{ for } z \in B', \tag{3.10}$$

$$\sum_{z \in B'} p_{t_1-r}^B(x_1,z) Jw_r(z) \le C \sum_{z \in B'} p_{t_2-r}^B(x_2,z) Jw_r(z), \quad 0 \le r \le t_1. \tag{3.11}$$

Of these (3.10) is immediate from UHKP(α) and NDLB(α). So we consider (3.11). Since $t_2 - r \ge t_2 - t_1 \ge T/4$, using NDLB($\alpha$), and writing $V = V(x_0, R)$,

$$\sum_{z \in B'} p_{t_2-r}^B(x,z) Jw_r(z) \ge c V^{-1} \sum_{z \in B'} Jw_r(z), \quad x \in B_0. \tag{3.12}$$

Let $s = t_1 - r \in [0, T/2]$. To complete the proof of (3.11) it is enough to show that

$$\sum_{z \in B'} p_s^B(x,z) Jw_r(z) \le c V^{-1} \sum_{z \in B'} Jw_r(z). \tag{3.13}$$

If $s \ge T/8$, then using the upper bound on p^B we obtain (3.13). So suppose $s \le T/8$. Let $B_1 = B(x_0, 5R/8)$. Then

$$\sum_{z \in B'} p_s^B(x,z) Jw_r(z) = \sum_{z \in B_1} p_s^B(x,z) Jw_r(z) + \sum_{z \in B' - B_1} p_s^B(x,z) Jw_r(z). \tag{3.14}$$

If $z \in B' - B_1$ then $d(x,z) \ge R/8$ and so by UHKP(α)

$$p_s^B(x,z) \le \frac{cs}{(R/8)^\alpha V(x, R/8)} \le c' V^{-1}.$$

Hence

$$\sum_{z \in B' - B_1} p_s^B(x,z) Jw_r(z) \le c V^{-1} \sum_{z \in B' - B_1} Jw_r(z) \le c V^{-1} \sum_{z \in B'} Jw_r(z). \tag{3.15}$$

If $z \in B_1$ then using UJS

$$\begin{aligned} Jw_r(z) &= \sum_{y \in G - B'} J(z,y) w_r(y) \\ &\le \sum_{y \in G - B'} \frac{c\mu_z}{V(z, R/8)} \sum_{z' \in B(z, R/8)} J(z', y) w_r(y) \\ &= \frac{c\mu_z}{V(z, R/8)} \sum_{z' \in B(z, R/8)} Jw_r(z') \le c\mu_z V^{-1} \sum_{z' \in B'} Jw_r(z'). \end{aligned}$$

So,

$$\sum_{z\in B_1} p_s^B(x,z) J w_r(z) \le cV^{-1} \sum_{z'\in B'} J w_r(z') \sum_{z\in B_1} p_s^B(x,z)\mu_z \le cV^{-1} \sum_{z'\in B'} J w_r(z'). \tag{3.16}$$

Combining (3.15) and (3.16) proves (3.13), and hence (3.11). □

References

[BG] R.M. Blumenthal, R.K. Getoor. *Markov Processes and Potential Theory*. Academic Press, Reading, MA, 1968.