The Liouville property and a conjecture of de Giorgi

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Abstract. We consider bounded entire solutions of the non-linear PDE $\Delta u+u-u^3=0$ in $\mathbb{R}^d$, and prove that under certain monotonicity conditions these solutions must be constant on hyperplanes. The proof uses a Liouville theorem for harmonic functions associated with a non-uniformly elliptic divergence form operator.

0. Introduction.

In 1978 De Giorgi [Gi] formulated the following:

Conjecture. *Suppose that u is an entire solution of the equation*

$$\Delta u + u - u^3 = 0 \tag{0.1}$$

satisfying

$$|u(x)| \le 1, \quad \frac{\partial u}{\partial x_d}(x) > 0 \ \textit{for}\ x=(x', x_d) \in \mathbb{R}^d, \tag{0.2}$$

$$\lim_{x_d\to\infty} u(x', x_d) = 1, \quad \textit{and} \quad \lim_{x_d\to-\infty} u(x', x_d) = -1.$$

Then the level sets of u must be hyperplanes, i.e. there exists $g \in C^2(\mathbb{R})$ such that $u(x) = g(a \cdot x)$, for some fixed $a \in \mathbb{R}^d$ with $|a| = 1$.

Let $F \in C^{2+\varepsilon}(\mathbb{R})$ be a non-negative function such that $F(\pm 1) = 0$ and $F''(\pm 1) \ge \mu > 0$ for some constant μ. A more general form of (0.1) is the equation

$$\Delta u - F'(u) = 0, \ \text{for}\ x = (x', x_d) \in \mathbb{R}^d, \tag{0.3}$$

where

$$|u(x)| \le 1, \quad \frac{\partial u}{\partial x_d}(x) > 0 \ \text{for}\ x=(x', x_d) \in \mathbb{R}^d, \tag{0.4}$$

$$\lim_{x_d\to\infty} u(x', x_d) = 1, \quad \text{and} \quad \lim_{x_d\to-\infty} u(x', x_d) = -1.$$

A generalization of the De Giorgi conjecture is that any solution of (0.3)-(0.4) is constant on hyperplanes, and so of the form $u(x) = g(a \cdot x)$, for some fixed $a \in \mathbb{R}^d$ with $|a| = 1$. It is clear that the function g must be a solution of the ODE

$$g''(t) - F'(g(t)) = 0, \ t \in \mathbb{R}, \ |g(t)| \le 1, \quad \text{and} \lim_{t\to\pm\infty} g(t) = \pm 1. \tag{0.5}$$

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This ODE has a solution which is unique up to translation. Note that in (0.1) we have $F(u) = \frac{1}{4}(u^2-1)^2$, which satisfies the conditions above with $F''(u) = 3u^2 - 1$.

It is known [GT] that any bounded solution u of (0.3) is $C^{3+\varepsilon}$ in $\mathbb{R}^d$. In [MM] and [CGS], it is shown that any bounded solution of (0.3) satisfies the gradient bound

$$|\nabla u(x)|^2 \leq 2F(u(x)) \text{ for all } x \in \mathbb{R}^d. \tag{0.6}$$

It is also proved there that the (generalized) De Giorgi conjecture is true in any dimension for any solution u such that equality in (0.6) holds at some point $x_0 \in \mathbb{R}^d$. Also, in [MM] it is proved that if $d = 2$ then the de Giogi conjecture holds for any solution u for which the level sets are the graphs of an equilipschitzian family of functions. See also [M1], [M2] and [CGS] for other results, and also [DFP] for the existence of some entire solutions of (0.1) of a quite different form. [GNN] obtained some striking results on a related problem.

Motivated by a problem in cosmology, G.W. Gibbons (see [C]) made the weaker conjecture that the level sets of u are hyperplanes if u satisfies (0.1), (0.2), and the additional condition that the converegence of $u(x', x_3)$ to ± 1 is uniform as x_3 tends $\pm\infty$.

Recently, in [GG] Ghoussoub and Gui proved the De Giorgi conjecture for $d = 2$ without any extra assumptions, and proved Gibbons conjecture for $d = 3$.

Our first result is a proof of Gibbons' conjecture for $d \geq 3$.

Theorem 1. *Suppose that $u(x)$ satisfies (0.3), converges to 1 uniformly as x_d tends to ∞, and converges to -1 uniformly as x_d tends to $-\infty$. Then u is necessarily of the form $u(x', x_d) = g(x_d)$, where $g(t)$ is a solution of (0.5).*

We can relax the uniform convergence condition if make some additional assumptions on F, and assume that the level sets of u are Lipschitzian.

Theorem 2. *Assume that $F(u)$ in (0.3) has only one critical point u_0 in $(-1, 1)$ and that $F''(u_0) < 0$. Suppose that $u(x)$ satisfies (0.3), $u(x', x_d) \to 1$ as $x_d \to \infty$, $u(x', x_d) \to -1$ as $x_d \to -\infty$, and that the level sets of $u(x', x_d)$ are the graphs of Lipschitzian functions of x', i.e. there exists a continuous positive function $L(b)$ for $b \in (-1, 1)$ such that*

$$|\nabla_{x'} u(x)| \leq L(u(x)) \frac{\partial u(x)}{\partial x_d}, \quad x \in \mathbb{R}^d.$$

Then u is necessarily of the form $u(x', x_d) = g(a \cdot x)$ for some $a \in \mathbb{R}^d$ with $|a| = 1$, where $g(t)$ is a solution of (0.5).

Remarks. 1. Note that $F(u) = \frac{1}{4}(u^2-1)^2$ satisfies the conditions of Theorem 2.
2. The Lipschitzian condition on u in Theorem 2 is weaker than that in [MM], where (as well as taking $d = 2$) $L(b)$ is assumed to be bounded. Indeed, all we need is that $L(b) < \infty$ on an interval $[-1+\delta, 1-\delta]$, where the constant $\delta > 0$ depends only on F.
3. Let $e^{(d)} = (0, 1) \in \mathbb{R}^{d-1} \times \mathbb{R}$ be the unit vector in the x_d direction. It is easy to see that the Lipschitzian condition on u in the above theorem is equivalent to the following monotonicity condition of u in a small cone: for any $b \in (-1, 1)$ there exists a $\delta_0(b) > 0$ such that if $|\nu| = 1$ then

$$\nu \cdot \nabla u(x) > 0 \text{ whenever } \nu \cdot e^{(d)} > 1 - \delta_0(u(x)), \quad x \in \mathbb{R}^d.$$

We have recently learnt that Theorem 1 (but not Theorem 2) has also been proved, using different methods, in [BCM] and [F].

The proof of both Theorems 1 and 2 employs the same basic strategy, which uses ideas introduced by Ghoussoub and Gui in [GG]. Let

$$\sigma(x) = \frac{\partial u(x)}{\partial x_d}. \tag{0.7}$$

In the case of Theorem 2, $\sigma(x) > 0$ in $\mathbb{R}^d$ by hypothesis, while it is shown in [GG] by using the moving plane method that the hypotheses of Theorem 1 imply that $\sigma(x) > 0$ in $\mathbb{R}^d$. For $a \in \mathbb{R}^d$ with $|a| = 1$ define the directional derivative $\psi_a(x) = a \cdot \nabla u(x)$. Differentiating (0.3) we have that both σ and ψ_a satisfy

$$\Delta\varphi - F''(u(x))\varphi = 0, \quad x \in \mathbb{R}^d.$$

Let

$$h(x) = \frac{\psi_a(x)}{\sigma(x)},$$

and set

$$\mathcal{L} = \tfrac{1}{2}\sigma^{-2}\nabla(\sigma^2\nabla) = \tfrac{1}{2}\Delta + (\sigma^{-1}\nabla\sigma)\nabla. \tag{0.8}$$

Then h is $\mathcal{L}$-harmonic, since

$$\begin{aligned} 2\mathcal{L}h &= \sigma^{-2}\nabla(\sigma^2\nabla h) = \sigma^{-2}\nabla(\psi_a\nabla\sigma - \sigma\nabla\psi_a) \\ &= \sigma^{-2}\big(\psi_a\Delta\sigma - \sigma\Delta\psi_a\big) = 0. \end{aligned}$$

Note also that $\sigma h = \psi_a$ is bounded, by (0.6).

Suppose that the operator $\mathcal{L}$ satisfies the Liouville property in the form:

(LP) If h satisfies $\mathcal{L}h = 0$ in $\mathbb{R}^d$ and σh is bounded, then h is constant.

Then for each a there exists a constant $c(a)$ such that

$$\psi_a(x) = a \cdot \nabla u(x) = c(a)\sigma(x), \quad x \in \mathbb{R}^d. \tag{0.9}$$

It follows immediately from (0.9) that u is constant on any hyperplane orthogonal to $\nabla u(0)$.

Thus the proof of Theorems 1 and 2 reduces to establishing the Liouville property (LP). If σ is any C^2 function on $\mathbb{R}^d$ satisfying $\sigma \geq \varepsilon > 0$, and $\mathcal{L} = \mathcal{L}_\sigma$ is defined by (0.8), then (LP) is well known. However (LP) may fail for general $\sigma > 0$ – see [GG] and [Ba] for counterexamples in the cases $d \geq 7$, $d \geq 3$ respectively. The proof in [Ba] is probabilistic, and shows that the Liouville property fails for suitable $\mathcal{L}(= \mathcal{L}_\sigma)$ by proving non-trivial tail behaviour of the diffusion process $X = (X_t, t \geq 0)$ associated with $\mathcal{L}$. However for σ arising from (0.7), the bound (0.6) implies that

$$\sigma(x', x_d) \to 0 \text{ as } |x_d| \to \infty \text{ for each } x' \in \mathbb{R}^{d-1}. \tag{0.10}$$

As X tends to avoid regions where σ is small, (0.10) suggests that, in the case of Theorem 1, where the convergence is uniform, the process X largely lives on some 'slab' D of the form $D = \mathbb{R}^{d-1} \times [-c, c]$. Since (see Section 4) one can prove that $\sigma(x) > \varepsilon_1 > 0$ for $x \in D$, X is in some sense close to a uniformly elliptic divergence form diffusion, which suggests that (LP) should hold for X and $\mathcal{L}$.

Some additional smoothness conditions are needed to establish the Liouville property. In the theorem below, the simplest case, which is sufficient to prove Theorem 1, is when $\gamma = 0$.

Theorem 3. *Let $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ be C^2, with $|\nabla\gamma(x')| \le K_0$, $x' \in \mathbb{R}^{d-1}$, for some constant $K_0 < \infty$. For $-\infty \le a \le b \le \infty$ write*

$$I(a,b) = \{(x', x_d) \in \mathbb{R}^d : \gamma(x') + a \le x_d \le \gamma(x') + b\}.$$

Let $\sigma : \mathbb{R}^d \to (0,\infty)$ be a C^2 function, and let $\mathcal{L} = \frac{1}{2}\sigma^{-2}\nabla(\sigma^2\nabla)$. Suppose that there exist constants $0 < \varepsilon_0 < 1$, $1 \le K_1 < K_2 < \infty$, $K_3 < \infty$ such that σ satisfies

(S1) $\sigma^{-1}\Delta\sigma \ge 2\varepsilon_0$ *on* $I(-K_1, K_1)^c$,

(S2) $\sigma \ge \varepsilon_0/2$ *on* $I(-K_2, K_2)$,

(S3) $||\sigma||_\infty$, $||\nabla\sigma||_\infty$ *and* $||\Delta\sigma||_\infty$ *are all bounded by* K_3.

Then if $\mathcal{L}h = 0$ and σh is bounded, then h is constant.

Remark. Though we will not use this fact, these conditions on σ imply that $\sigma(x', x_d) \to 0$ as $|x_d| \to \infty$.

Write

$$H(\lambda) = I(\lambda, \lambda), \quad \lambda \in \mathbb{R}.$$

Let X be the diffusion associated with $\mathcal{L}$. One approach to Liouville theorems such as Theorem 3 is to obtain global upper and lower bounds on the transition density $k(t, x, y)$ of X, which is the solution to the heat equation

$$\mathcal{L}k = \frac{\partial k}{\partial t}.$$

There is a substantial literature on bounds of this type, but with most approaches some kind of uniform ellipticity condition on $\mathcal{L}$ is essential. We avoid this difficulty by considering instead the time-change of the process X on the submanifold $H(0)$. Write $\widetilde{X}$ for this process, and let Y be the projection of $\widetilde{X}$ onto $\mathbb{R}^{d-1}$. Y is a pure jump process with generator of the form

$$\mathcal{L}_Y f(x) = \int_{\mathbb{R}^{d-1}} \big(f(y') - f(x')\big) n(x', y') dy', \tag{0.11}$$

where n is symmetric and continuous away from the diagonal. Let $q = q(t, x', y')$ be the transition density of Y: q solves the equation

$$\mathcal{L}_Y q = \frac{\partial q}{\partial t}.$$

We obtain upper and lower bounds on q, and from these prove a Liouville theorem for $\mathcal{L}_Y$-harmonic functions. Theorem 3 then follows easily.

The contents of this paper are as follows. In Section 1 we consider jump processes Y given by (0.11), and, under suitable conditions on the function $n(x,y)$, which include exponential decay as $|x-y| \to \infty$, we obtain upper and lower bounds on q and prove a Liouville theorem for $\mathcal{L}_Y$. In Section 2 we use Girsanov's transformation to construct the diffusion X associated with $\mathcal{L}$. The main result in this section is an exponential bound on $|X_{\tau_0} - X_0|$, where τ_0 is the first hitting time of $H(0)$. Section 3 deals with the construction of the processes $\widetilde{X}$ and Y from X, and estimates on the jump measures n. The exponential bounds on $|X_{\tau_0} - X_0|$ lead to exponential decay of $n(x,y)$ as $|x-y| \to \infty$. Finally, in Section 4 we complete the proof of Theorems 1 and 2, by showing that the function $\sigma = \partial u(x)/\partial x_d$ satisfies the conditions of Theorem 3.

We write c_i for unimportant positive finite constants; these are fixed within each lemma, proposition, theorem and corollary.

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1. Heat kernel of a jump process.

Let N be a measure on $\mathbb{R}^n \times \mathbb{R}^n - D$ (where D is the diagonal) with a symmetric density $n(x,y)$. Throughout this section we will assume that there exist constants $\alpha_0 > 0$, c_i such that

$$\int_{|y-x|>r} n(x,y)dy \le c_0 e^{-\alpha_0 r}, \quad r \ge 1, \tag{1.1}$$

$$c_1|x-y|^{-(n+1)} \le n(x,y) \le c_2|x-y|^{-(n+1)}, \quad |x-y| \le 1. \tag{1.2}$$

Let $\mathcal{L}_Y$ be the generator

$$\mathcal{L}f(x) = \int_{\mathbb{R}^n} \big(f(y) - f(x)\big) n(x,y)dy, \quad f \in C_0^\infty(\mathbb{R}^n),$$

and $\mathcal{E}$ be the Dirichlet form on $L^2(\mathbb{R}^n, dx)$ with core $C_0^\infty(\mathbb{R}^n)$ given by

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(f(y) - f(x)\big)^2 n(x,y)dxdy, \quad f \in C_0^\infty(\mathbb{R}^n).$$

(An argument similar to that in [FOT, p. 100] implies that $\mathcal{E}$ is regular). Let $Y = (Y_t, t \ge 0, \mathbb{Q}^x, x \in \mathbb{R}^n)$ be the symmetric Markov process associated with $\mathcal{E}$. Set also

$$n_0(x,y) = |x-y|^{-(n+1)} 1_{(|x-y|<1)}, \quad x, y \in \mathbb{R}^n,\ x \ne y,$$

Replacing n by n_0 in the equations above, let $\mathcal{L}_0$ and $\mathcal{E}_0$ be the corresponding generator and Dirichlet form of the Markov process $Y^0 = (Y_t^0, t \ge 0, \mathbb{Q}_0^x, x \in \mathbb{R}^n)$.

From (1.2) we have

$$\mathcal{E}(f,f) \geq c_1 \mathcal{E}_0(f,f), \quad f \in C_0^\infty(\mathbb{R}^n). \tag{1.3}$$

The process Y^0 is a Lévy process, and therefore Y_t^0 has characteristic function $\psi(\lambda)$, given by

$$\mathbb{E}^0 e^{i\lambda\cdot Y_t^0} = e^{-t\psi(\lambda)}, \qquad \lambda \in \mathbb{R}^n,$$

where, since Y^0 is symmetric,

$$\psi(\lambda) = \int_{\mathbb{R}^n} \big(1 - \cos \lambda \cdot x\big) n_0(0,x)\, dx. \tag{1.4}$$

Lemma 1.1. *For each $t > 0$, under $\mathbb{Q}_0^0$, Y_t^0 has a continuous density $q_t^0(x)$, $x \in \mathbb{R}$, which satisfies*

$$q_t^0(x) \leq c_1 t^{-n/2}, \qquad t \geq 1. \tag{1.5}$$
$$q_t^0(x) \leq c_1 t^{-n}, \qquad t \leq 1. \tag{1.6}$$

Proof. By the radial symmetry of (1.4) we have

$$\psi(\lambda) = \int_{|x|<1} (1 - \cos(x_1|\lambda|))|x|^{-n-1}\, dx = |\lambda| \int_{|y|<|\lambda|} (1 - \cos y_1)|y|^{-n-1} dy.$$

Hence if $|\lambda| \geq 1$ then $\psi(\lambda) \geq c_2|\lambda|$, while if $|\lambda| < 1$ then $1 - \cos x_1|\lambda| \geq c_3 x_1^2 |\lambda|^2$, so

$$\psi(\lambda) \geq c_3|\lambda|^2 \int_{|x|<1} x_1^2 |x|^{-n-1}\, dx \geq c_4|\lambda|^2.$$

Therefore $\int |\lambda|^p e^{-t\psi(\lambda)} d\lambda < \infty$ for any $p < \infty$, and by Fourier inversion Y_t^0 has a C^∞ density $q_t^0(\cdot)$.

Also, by the Fourier inversion formula,

$$\begin{aligned} q_t(x) &= \int_{\mathbb{R}^n} e^{-i\lambda\cdot x} e^{-t\psi(\lambda)} d\lambda \\ &\leq q_t(0) = \int_{\mathbb{R}^n} e^{-t\psi(\lambda)} d\lambda \\ &\leq \int_{|\lambda|\leq 1} e^{-c_4 t|\lambda|^2} d\lambda + \int_{|\lambda|>1} e^{-c_2 t|\lambda|} d\lambda \\ &= c_5 \int_0^1 r^{n-1} e^{-c_4 t r^2} dr + c_5 \int_1^\infty r^{n-1} e^{-c_2 t r} dr. \end{aligned}$$

Estimating these integrals, the bounds (1.5), (1.6) follow easily. □

We can now use Theorem 2.9 of [CKS] to deduce similar estimates for Y.

Theorem 1.2. *Y has a transition density $q_t(x,y)$ which satisfies*

$$q_t(x,y) \le c_1 t^{-n/2}, \quad t \ge 1. \tag{1.7}$$
$$q_t(x,y) \le c_1 t^{-n}, \quad t \le 1. \tag{1.8}$$

Proof. Write $q_t^0(x,y)$ for the transition densities of Y^0. As Y^0 is a Lévy process $q_t^0(x,y) = q_t^0(y-x)$. From Lemma 1.1 we have, for a suitable $c_2 < \infty$,

$$q_t^0(x,y) \le c_2 t^{-n} e^t, \quad t > 0.$$

So, by Theorem 2.1 of [CKS], and writing $m = 2n$, $\mathcal{E}_0$ satisfies a Nash inequality

$$\|f\|_2^{2+4/m} \|f\|_1^{-4/m} \le c_3 \left[\mathcal{E}_0(f,f) + \|f\|_2^2\right].$$

Using (1.3), $\mathcal{E}$ satisfies a Nash inequality of the same form, and hence, by the converse implication in [CKS, Theorem 2.1], Y has a transition density $q_t(x,y)$ which satisfies

$$q_t(x,y) \le c_4 t^{-n} e^t, \quad t > 0.$$

The bound (1.8) is immediate.

To obtain bounds for $t \ge 1$, we use the conditional Nash inequalities discussed in [CKS, Theorem 2.9]. First, from (1.5) it follows that $\mathcal{E}_0$ also satisfies

$$\|f\|_2^{2+4/n} \|f\|_1^{-4/n} \le c_5 \mathcal{E}_0(f,f) \quad \text{whenever} \quad \mathcal{E}_0(f,f) \le \|f\|_1^2. \tag{1.9}$$

Again, by (1.3), $\mathcal{E}$ satisfies an inequality of the same form. Also, by (1.8) $q_1(x,y) \le c_1$ for all x,y, and so we can use the converse implication in [CKS, Theorem 2.9] to deduce that $q_t(x,y) \le c_6 t^{-n/2}$ for $t \ge 1$. Adjusting the constant c_1 if necessary this completes the proof of the theorem. □

We now wish to use Davies' method to obtain off-diagonal upper bounds on q_t, for $t \ge 1$. We encounter one technical obstacle, due to the different behaviour of q_t for large and small t. This means that $\mathcal{E}$ only satisfies a conditional Nash inequality of the form (1.9), rather than a full Nash inequality. Since verifying that the functions f_t, (which arise in [CKS, Section 3]), satisfy the condition $\mathcal{E}(f_t,f_t) \le \|f_t\|_1^2$ is quite awkward, we will avoid this difficulty by using a trick.

Let $Z = (Z_t, t \ge 0)$ be an "auxiliary" symmetric Markov process on a state space (M,m), independent of Y, with a transition density $r_t(x,y)$ with respect to a m which satisfies

$$\begin{aligned} &r_t(x',y') \le c_1 t^{-n/2}, \quad 0 < t \le 1,\ x',y' \in M \\ &r_t(x',y') \le c_1 t^{-n}, \quad t \ge 1,\ x',y' \in M \\ &r_t(x',x') \ge c_1 t^{-n/2}(t \vee 1)^{-n/2}, \quad t > 0. \end{aligned} \tag{1.10}$$

For example, if M is a sufficiently regular n dimensional manifold with volume growth given by $V(x,r) \asymp r^{2n}$, $r > 1$ and $V(x,r) \asymp r^n$, $r < 1$, we have (see for example [Gr]) that

r_t satisfies (1.10). Let $X_t = (Y_t, Z_t) \in \mathbb{R}^n \times M$. Then X has a transition density p_t given by

$$p_t\big((x,x'),(y,y')\big) = q_t(x,y) r_t(x',y'), \quad x,y \in \mathbb{R}^n,\ x',y' \in M,$$

which plainly satisfies

$$\|p_t\|_\infty \le c_1 t^{-3n/2}, \quad t > 0. \tag{1.11}$$

Write $\mathcal{E}_X$ for the Dirichlet form of X: $\mathcal{E}_X$ therefore satisfies the Nash inequality ($p = 3n$)

$$\|f\|_2^{2+4/p} \|f\|_1^{-4/p} \le c_2 \mathcal{E}_X(f,f). \tag{1.12}$$

(Here $\|\cdot\|.$ is of course the norm in the product space $(\mathbb{R}^n \times M, dx \times dm)$). Fix $0 \in M$. We can now use [CKS, Theorem 3.25] to deduce off-diagonal upper bounds for p_t. These yield immediately off-diagonal upper bounds for q_t, since if

$$p_t\big((x,0),(y,0)\big) \le c_3 t^{-p/2} e^{-K(t,x,y)}, \quad t \ge 1,$$

then by (1.10) $q_t(x,y) \le c_4 t^{-n/2} \exp(-K(t,x,y))$. In fact, the auxiliary process Z plays no role in the calculations, and to simplify notation in what follows we will therefore omit it.

Definition 1.3. Set for $f \in C(\mathbb{R}^n)$

$$\Gamma(f,f)(x) = \int \big(f(x) - f(y)\big)^2 n(x,y)\, dy,$$

where we allow $\Gamma(f,f) = +\infty$. Define for $\psi \in C(\mathbb{R}^n)$

$$\begin{aligned} \Lambda(\psi)^2 &= \|e^{-2\psi}\Gamma(e^\psi, e^\psi)\|_\infty \vee \big\|e^{2\psi}\Gamma\big(e^{-\psi}, e^{-\psi}\big)\big\|_\infty, \\ \mathcal{F}_\infty &= \{\psi \in C(\mathbb{R}^n) : \Lambda(\psi) < \infty\}, \\ D(t,x,y) &= \sup\big\{|\psi(y) - \psi(x)| - t\Lambda(\psi)^2 : \psi \in \mathcal{F}_\infty\big\}. \end{aligned}$$

From [CKS, Theorem 3.25] and the remarks above, we obtain

Lemma 1.4. *For $t \ge 1$, $x, y \in \mathbb{R}^n$,*

$$q_t(x,y) \le c_1 t^{-n/2} e^{-D(2t,x,y)}.$$

It remains to estimate $D(t,x,y)$.

Lemma 1.5. *(a) For $R \ge 1$,*

$$\int_{1 \le |x-y| \le R} |x-y|^2 n(x,y) dy \le c_1.$$

(b) For $R \geq 1$, $\theta < \alpha_0/2$,

$$\int_{|x-y|>R} e^{\theta|x-y|} n(x,y)dy \leq 2c_2 e^{-(\alpha_0-\theta)R}.$$

Proof. Write $F(r) = \int_{|x-y|>r} n(x,y)dy$. Then $F(r) \leq c_3 e^{-\alpha_0 r}$, by (1.1). So

$$\begin{aligned}\int_{1\leq|x-y|\leq R} |x-y|^2 n(x,y)dy &= -\int_1^R r^2 F(dr) \\ &= F(1) - R^2 F(R) + \int_1^R 2rF(r)dr \leq c_4.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{|x-y|>R} e^{\theta|x-y|} n(x,y)dy &= -\int_R^\infty e^{\theta r} F(dr) \\ &\leq e^{\theta R} F(R) + \int_R^\infty c_3 \theta e^{-(\alpha_0-\theta)r} dr \\ &\leq c_5 e^{-(\alpha_0-\theta)R}.\end{aligned}$$

□

Proposition 1.6. *There exist a constant c_0 such that if a is a unit vector in $\mathbb{R}^n$, $\alpha \in (0, 1 \wedge (\alpha_0/4))$, and $\psi_\alpha(x) = \alpha a \cdot x$, then $\Lambda(\psi_\alpha)^2 \leq c_0 \alpha^2$.*

Proof. We need to bound

$$e^{-2\psi_\alpha(x)} \Gamma(e^{\psi_\alpha}, e^{\psi_\alpha})(x) = \int \left(1 - e^{\psi_\alpha(y)-\psi_\alpha(x)}\right)^2 n(x,y)\, dy. \tag{1.13}$$

Split in the integral in (1.13) into three pieces, and write

$$J_1(x) = \int_{|x-y|<1}, \quad J_2(x) = \int_{1\leq|x-y|\leq 1/\alpha}, \quad J_3(x) = \int_{|x-y|>1/\alpha}.$$

Then since $e^x - 1 \leq 2x$ for $0 < x < 1$,

$$\begin{aligned}J_1(x) &\leq c_1 \int_{|x-y|<1} \alpha^2 |x-y|^2 n(x,y)\, dy \\ &\leq c_2 \alpha^2 \int_{|x-y|<1} |x-y|^{-n+1}\, dy \leq c_3 \alpha^2.\end{aligned}$$

Similarly

$$J_2(x) \leq c_4 \int_{1\leq|x-y|\leq 1/\alpha} \alpha^2 |x-y|^2 n(x,y)\, dy \leq c_5 \alpha^2,$$

by Lemma 1.5(a). Also, by Lemma 1.5(b)

$$J_3(x) \le \int_{|x-y|\ge 1/\alpha} e^{2\alpha|y-x|} n(x,y)\,dy \le c_6 e^{-\alpha_0/\alpha} \le c_7\alpha^2,$$

since $\alpha < \alpha_0/4$. Combining these estimates we have

$$e^{-2\psi_\alpha(x)}\Gamma(e^{\psi_\alpha}, e^{\psi_\alpha})(x) \le c_0\alpha^2, \quad x \in \mathbb{R}^n.$$

This bounds $\|e^{-2\psi_\alpha}\Gamma(e^{\psi_\alpha}, e^{\psi_\alpha})\|_\infty$, and replacing a by $-a$ gives an identical bound on $\|e^{2\psi_\alpha}\Gamma(e^{-\psi_\alpha}, e^{-\psi_\alpha})\|_\infty$. □

Theorem 1.7. *There exists $\alpha_1 > 0$ such that*

$$q_t(x,y) \le c_1 t^{-n/2}\exp(-\alpha_1|x-y|^2/t), \qquad t\ge 1, \quad |x-y|\le t, \tag{1.14}$$

$$q_t(x,y) \le c_1 t^{-n/2}\exp(-\alpha_1|x-y|), \qquad t\ge 1, \quad |x-y|\ge t. \tag{1.15}$$

Proof. We have, writing $\beta = 1\wedge(\alpha_0/4)$,

$$\begin{aligned} D(2T,x,y) &\ge \sup_{0\le\alpha\le\beta}\big(|\psi_\alpha(x)-\psi_\alpha(y)| - 2T\Lambda(\psi_\alpha)^2\big) \\ &\ge \sup_{0\le\alpha\le\beta}(\alpha|x-y| - 2c_0\alpha^2 T). \end{aligned}$$

If $|x-y|\le T$, take $\alpha = \theta_0|x-y|T^{-1}$ where $\theta_0 = \beta\wedge(1/4c_0)$, to obtain

$$D(2T,x,y) \ge \frac{1}{2}\theta_0|x-y|^2/T.$$

If $|x-y|\ge T$, let $\alpha=\theta_0$; then

$$D(2T,x,y) \ge \frac{1}{2}\theta_0|x-y|.$$

The bounds (1.14), (1.15) now follow from Lemma 1.4. □

Integrating the bounds in Theorem 1.7 we deduce

Corollary 1.8. *There exists $\lambda_0 < \infty$ such that for $t\ge 1$,*

$$\int_{|x-y|>\lambda_0 t^{1/2}} q_t(x,y)\,dy \le \tfrac{1}{2}.$$

We now turn to lower bounds. The first step is to obtain a suitable Poincaré inequality. Let $v \in C^\infty\big(\mathbb{R},(0,\infty)\big)$ be such that $v(x)=|x|$ for $|x|\ge 2$, $v(x)=v(-x)$, $|v'|\le 1$, and $\int e^{-v(t)}dt = 1$. Set $\psi(x) = \psi(x_1,\dots,x_n) = \sum_1^n v(x_i)$, and for $R\ge 1$ let

$$\varphi_R(x) = R^{-n}e^{-\psi(x/R)}.$$

Note that $|\nabla\psi|\le n$, and that $\int_{\mathbb{R}^n}\varphi_R = 1$. Write $\mathcal{C}$ for the set of cubes of side length 1 in $\mathbb{R}^n$ with corners in $\mathbb{Z}^n$ and edges parallel to the axes. If $f\in L^1(\mathbb{R}^n)$ set

$$f(C) = \int_C f\,dx, \qquad C\in\mathcal{C}.$$

Define $a(C,D)=1$ if C, D are adjacent (i.e. $C\cap D$) is a $n-1$ dimensional set) and $a(C,D)=0$ otherwise. From Lemma 1.19 of [SZ] we have:

Lemma 1.9. *Let $g : \mathcal{C} \to \mathbb{R}$, and write*

$$\widetilde{g}_R = \sum_C g(C)\varphi_R(C).$$

Then there exists c_1 (independent of R) such that

$$\sum_C \big(g(C) - \widetilde{g}_R\big)^2 \varphi_R(C) \le c_1 R^2 \sum_C \sum_D a(C,D)\big(g(C) - g(D)\big)^2 \varphi_R(C) \wedge \varphi_R(D).$$

Let $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function such that $m(x,y) \ge 1$ whenever $|x-y|^2 \le n+1$. So $m(x,y) \ge 1$ if $x, y \in C$ for some $C \in \mathcal{C}$, and $m(x,y) \ge 1$ if $x \in C$, $y \in D$ and $a(C,D) = 1$.

Proposition 1.10. *Let $f \in C(\mathbb{R}^n, \mathbb{R})$, and write $\overline{f}_R = \int f\varphi_R\,dx$. Then there exists c_1, independent of R, such that for $R \ge 1$,*

$$\int_{\mathbb{R}^n} \big(f(x) - \overline{f}_R\big)^2 \varphi_R(x)\,dx \le \\ c_1 R^2 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \big(f(x) - f(y)\big)^2 \varphi_R(x) \wedge \varphi_R(y) m(x,y)\,dx\,dy. \tag{1.16}$$

Proof. From the definition of φ_R we have that there exists $c_2 > 1$ such that

$$c_2^{-1}\varphi_R(C) \le \varphi_R(x) \le c_2 \varphi_R(C) \qquad \text{if} \quad x \in C. \tag{1.17}$$

It follows that

$$c_2^{-2}\varphi_R(D) \le \varphi_R(C) \le c_2^2 \varphi_R(D) \qquad \text{if} \quad a(C,D) = 1.$$

If $b \in \mathbb{R}$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} \big(f(x) - b\big)^2 \varphi_R(x)\,dx &= \sum_C \int_C \big(f(x) - b\big)^2 \varphi_R(x)\,dx \\
&\le c_3 \sum_C \varphi_R(C) \int_C \big(f(x) - b\big)^2\,dx \\
&= c_3 \sum_C \varphi_R(C) \int \big(f(x) - f(C)\big)^2\,dx + c_3 \sum_C \varphi_R(C)\big(f(C) - b\big)^2 \\
&= S_1 + S_2.
\end{aligned}$$

Since

$$\iint_{C \times C} \big(f(x) - f(y)\big)^2\,dx\,dy = 2\int_C \big(f(x) - f(C)\big)^2\,dx,$$

using (1.17) we have

$$
\begin{aligned}
S_1 &= c_3 \sum_C \varphi_R(C) \int_C \big(f(x) - f(C)\big)^2\, dx \\
&\le c_4 \sum_C \iint_{C\times C} \big(f(x) - f(y)\big)^2 \varphi_R(C)\, dx\, dy \\
&\le c_5 \sum_C \iint_{C\times C} \big(f(x) - f(y)\big)^2 \varphi_R(x) \wedge \varphi_R(y) m(x,y)\, dx\, dy \\
&= c_6 \iint \big(f(x) - f(y)\big)^2 \varphi_R(x) \wedge \varphi_R(y)\, m(x,y) dx\, dy.
\end{aligned} \tag{1.18}
$$

For S_2, by Lemma 1.9, if $b = \widetilde{f}_R = \sum_C f(C)\varphi_R(C)$,

$$
\sum_C \varphi_R(C)\big(f(C) - b\big)^2 \le c_6 R^2 \sum_C \sum_D a(C,D)\big(f(C) - f(D)\big)^2 \varphi_R(C) \wedge \varphi_R(D).
$$

Now if $C,\ D \in \mathcal{C}$,

$$
\begin{aligned}
\int_C \int_D \big(f(x) - f(y)\big)^2\, dx\, dy &= \int_C f^2 + \int_D f^2 - 2f(C)f(D) \\
&\ge \big(f(C) - f(D)\big)^2.
\end{aligned}
$$

So, again using (1.17),

$$
\begin{aligned}
S_2 &\le c_7 R^2 \sum_C \sum_D a(C,D) \int_C \int_D \big(f(x) - f(y)\big)^2 \varphi_R(x) \wedge \varphi_R(y)\, dx\, dy \\
&\le c_8 R^2 \iint \big(f(x) - f(y)\big)^2 \varphi_R(x) \wedge \varphi_R(y) m(x,y)\, dx\, dy.
\end{aligned} \tag{1.19}
$$

Since $R \ge 1$, and

$$
\int \big(f(x) - \overline{f}_R\big)^2 \varphi_R(x)\, dx \le \int \big(f(x) - b\big)^2 \varphi_R(x)\, dx,
$$

combining (1.18) and (1.19) completes the proof of the Proposition. □

Exactly the same argument (but with a subdivision of $\mathbb{R}^n$ into cubes of side $(n+1)^{-1/2}$) gives, using the bound (1.2) on $n(x,y)$, the following weighted Poincaré inequality.

Theorem 1.11. *Let $f \in C(\mathbb{R}^n, \mathbb{R})$. There exists c_1, independent of R, such that for $R \ge 1$,*

$$
\int \big(f(x) - \overline{f}_R\big)^2 \varphi_R(x)\, dx \le c_1 R^2 \iint \big(f(x) - f(y)\big)^2 \varphi_R(x) \wedge \varphi_R(y) n(x,y)\, dx\, dy.
$$

We now use an argument of Fabes and Stroock [FS] (see also [SZ]) to obtain lower bounds on $q_t(x,y)$. Let $x_0 \in \mathbb{R}^n$, and fix $T = R^2 \geq 1$. Set

$$u(t,x) = q_t(x_0,x), \qquad G(t) = \int \varphi_R(x) \log u(t,x)\, dx.$$

Then since $u_t = \mathcal{L}_Y u$,

$$\begin{aligned} G'(t) &= \int \frac{u_t}{u} \varphi_R \, dx \\ &= \int u^{-1} \varphi_R \mathcal{L}_Y u \, dx \\ &= -\mathcal{E}(\varphi_R/u, u) \\ &= -\iint \left(\frac{\varphi_R(x)}{u(t,x)} - \frac{\varphi_R(y)}{u(t,y)} \right) \big(u(t,x) - u(t,y)\big) n(x,y)\, dx\, dy. \end{aligned}$$

As in [SZ], using the inequality

$$\left(\frac{d}{b} - \frac{c}{a} \right)(b-a) \leq -\tfrac{1}{2}(c \wedge d)(\log b - \log a)^2 + \frac{(d-c)^2}{2(c \wedge d)}, \tag{1.20}$$

which holds for any positive a, b, c, d, we have

$$\begin{aligned} G'(t) \geq \tfrac{1}{2} &\iint \big(\log u(t,x) - \log u(t,y) \big)^2 \varphi_R(x) \wedge \varphi_R(y) n(x,y)\, dx\, dy \\ &- \tfrac{1}{2} \iint \big(\varphi_R(x) - \varphi_R(y) \big)^2 \big(\varphi_R(x) \wedge \varphi_R(y) \big)^{-1} n(x,y)\, dx\, dy. \end{aligned} \tag{1.21}$$

Writing $A(R)$ for the second term in (1.21), it follows from Theorem 1.11 that

$$G'(t) \geq c_2 R^{-2} \int \big(\log u(t,x) - G(t) \big)^2 \varphi_R(x)\, dx - A(R). \tag{1.22}$$

Lemma 1.12. *There exists a constant $A \in (0,\infty)$ such that $A(R) \leq AR^{-2}$, $R \geq 1$.*

Proof. We have

$$2A(R) = \int A_1(x)\, dx + \int A_2(x)\, dx + \int A_3^+(x)\, dx + \int A_3^-(x)\, dx,$$

where

$$A_1(x) = \int_{|x-y|\leq 1} \big(\varphi_R(x) - \varphi_R(y) \big)^2 \big(\varphi_R(x) \wedge \varphi_R(y) \big)^{-1} n(x,y)\, dy,$$

and $A_2(x)$, $A_3^+(x)$, A_3^- are defined similarly, but with the integration over the regions $\{y : 1 \leq |x-y| \leq R\}$, $\{y : |x-y| > R\} \cap \{y : \varphi_R(y) \geq \varphi R(x)\}$, $\{y : |x-y| > R\} \cap \{y : \varphi_R(y) < \varphi_R(x)\}$ respectively.

Now since $|\nabla\psi| \le n$, if $|x-y| \le R$ then $\varphi_R(y) \ge e^{-n}\varphi_R(x)$, and

$$|\varphi_R(x) - \varphi_R(y)| = R^{-n} e^{-\psi(x/R)} |1 - e^{\psi(x/R)-\psi(y/R)}|.$$

Hence

$$\begin{aligned}\big(\varphi_R(x) - \varphi_R(y)\big)^2 \big(\varphi_R(x) \wedge \varphi_R(y)\big)^{-1} &\le e^n \varphi_R(x)\big(1 - e^{\psi(x/R)-\psi(y/R)}\big)^2 \\ &\le c_1 \varphi_R(x)\big(e^{n|x-y|/R} - 1\big)^2 \\ &\le c_2 \varphi_R(x) R^{-2} |x-y|^2, \qquad \text{if} \quad |x-y| \le R.\end{aligned} \tag{1.23}$$

So,

$$\begin{aligned}A_1(x) &\le c_3 R^{-2} \int_{|x-y|\le 1} \varphi_R(x)|x-y|^2 n(x,y)\, dy \\ &\le c_4 R^{-2} \varphi_R(x) \int_{|x-y|\le 1} |x-y|^{2-(n+1)}\, dy = c_5 R^{-2} \varphi_R(x).\end{aligned} \tag{1.24}$$

Similarly, using (1.23) and Lemma 1.5(a)

$$\begin{aligned}A_2(x) &\le c_6 \varphi_R(x) R^{-2} \int_{1\le|x-y|\le R} |x-y|^2 n(x,y)\, dy \\ &\le c_7 R^{-2} \varphi_R(x).\end{aligned} \tag{1.25}$$

Now writing $B = \{|x-y| > R\} \cap \{\varphi_R(y) > \varphi_R(x)\}$,

$$\begin{aligned}A_3^+(x) &= \int_B \varphi_R(x)^{-1} \big(\varphi_R(y) - \varphi_R(x)\big)^2 n(x,y)\, dy \\ &\le \varphi_R(x)^{-1} \int_B \varphi_R(y)^2 n(x,y)\, dy.\end{aligned} \tag{1.26}$$

Since $\varphi_R(y) \le e^{n|x-y|/R}\varphi_R(x)$, if $R > 2n/\alpha_0$ then

$$\begin{aligned}A_3^+(x) &\le \varphi_R(x) \int_{|y-x|>R} e^{n|x-y|/R} n(x,y)\, dy \\ &\le c_8 \varphi_R(x) e^{-\alpha_0 R + n} = c_9' e^{-\alpha_0 R} \varphi_R(x).\end{aligned}$$

By symmetry $\int A_3^+(x)\, dx = \int A_3^-(x)\, dx$, so combining the estimates (1.24)–(1.26),

$$A(R) \le c_{10} \left(R^{-2} + e^{-\alpha_0 R}\right) \int \varphi_R(x)\, dx \le c_{11} R^{-2},$$

which proves the lemma. □

Lemma 1.13. *Let $T, R, G(t)$, x_0 be as above. Then there exists a constant c_1 such that*

$$G(T) \geq -c_1 + \log(T^{-n/2}) \quad \textit{provided} \quad |x_0| \leq R. \tag{1.27}$$

Proof. Set $u_0(s,x) = R^n u(sT,x)$, and

$$G_0(s) = \int \varphi_R(x) \log u_0(s,x)\, dx = G(sT) + \log R^n.$$

Then for $0 < s < 1$, using (1.22) and Lemma 1.12,

$$\begin{aligned} G_0'(s) = TG'(sT) &\geq -A + c_1 \int \big(\log u(Ts,x) - G(Ts)\big)^2 \varphi_R(x)\, dx \\ &= -A + c_1 \int \big(\log u_0(s,x) - G_0(s)\big)^2 \varphi_R(x)\, dx. \end{aligned}$$

We can now follow very closely the argument of [FS, Lemma 2.1]. By (1.7) we have

$$\sup_{\frac{1}{2} \leq s \leq 1} u_0(s,x) \leq K,$$

and so, since $(\log u_0 - G_0)^2 u_0^{-1} \geq (\log K - G_0)^2 K^{-1}$ when $u_0 \geq e^{2+G_0}$ we have

$$G_0'(s) \geq -A + c_2 \int_{u_0(s,x) \geq e^{2+G_0(s)}} \varphi_R(x) u_0(s,x)\, dx.$$

Let $\theta \geq 1$. Then for $\frac{1}{2} \leq s \leq 1$,

$$\begin{aligned} \int_{u_0(s,x)\geq e^{2+G_0(s)}} \varphi_R(x) u_0(s,x)\, dx &\geq \int \varphi_R(x) u_0(s,x)\, dx - e^{2+G_0(s)} \\ &\geq \int_{|x|<\theta R} \varphi_R(x) u_0(s,x)\, dx - e^{2+G_0(s)} \\ &\geq R^n \inf_{|x|<\theta R} \varphi_R(x) \Big(1 - \int_{|x|>\theta R} u(Ts,x)\, dx\Big) - e^{2+G_0(s)}. \end{aligned}$$

Now

$$R^n \inf_{|x|<\theta R} \varphi_R(x) = \inf_{|y|\leq \theta} e^{-\psi(y)} \geq e^{-n\theta},$$

while by Corollary 1.8, if θ is chosen large enough,

$$\int_{|x|>\theta R} u(Ts,x)\, dx \leq \tfrac{1}{2} \quad \text{for} \quad \tfrac{1}{2} \leq s \leq 1.$$

We can now proceed, exactly as in [FS], to deduce that $G_0'(s)$ satisfies a differential inequality which implies that $G_0'(1) \geq -c_1$. (1.27) is then immediate. $\square$

Theorem 1.14. *There exists a constant a_1 such that*

$$q_t(x,y) \geq c_1 t^{-n/2} \qquad \text{for} \quad t \geq 2, \quad |x-y| \leq a_1 t^{1/2}. \tag{1.28}$$

Proof. It is sufficient to prove this for $x = 0$. Write $T = t/2$, $R = T^{1/2}$. Since

$$\begin{aligned} q_{2T}(0,y) &= \int q_T(0,x) q_T(x,y)\, dx \\ &\geq \int q_T(0,x) q_T(x,y) R^n \varphi_R(x)\, dx, \end{aligned}$$

then by Jensen's inequality,

$$\log T^{-n/2} q_T(0,y) \geq \int \big(\log q_T(0,x)\big) \varphi_R(x)\, dx + \int \varphi_R(x) \log q_T(x,y)\, dx.$$

So if $|y| < T^{1/2}$, from Lemma 1.13,

$$\log T^{-n/2} q_{2T}(0,y) \geq -2c_1 + 2 \log T^{-n/2},$$

which establishes (1.28). □

We can now obtain lower bounds for q from (1.28) by a chaining argument. We omit the proof, as the argument is standard and the bound (1.28) is already sufficient to establish the Liouville property for Y.

Theorem 1.15. *There exist constants c_i such that*

$$q_t(x,y) \geq c_1 t^{-n/2} \exp(-c_2 |x-y|^2/t), \qquad t \geq 2, \quad |x-y| \leq c_3 t.$$

Definition 1.16. Write $(Q_t, t \geq 0)$ for the semigroup of Y: $Q_t f(x) = \mathbb{Q}^x f(Y_t)$. A bounded function h is *Y-harmonic* if $Q_t h = h$ for all $t \geq 0$, or equivalently, if $h(Y_t)$ is a martingale/$\mathbb{Q}^x$ for all x.

Theorem 1.17. *Let h be bounded and Y-harmonic. Then h is constant.*

Proof. Suppose h is non-constant. Replacing h by $ah + b$ if necessary, we can assume that $\inf h = 0$, $\sup h = 1$. So there exists $x_0 \in \mathbb{R}^n$ such that $h(x_0) \geq \frac{3}{4}$. We can assume $x_0 = 0$. Let λ_0 be as in Corollary 1.8, and write $B(t) = B(0, \lambda_0 t^{1/2})$ for $t \geq 1$. Then since $h(0) = Q_t h(0)$, and

$$\int_{B(t)^c} q_t(0,y) h(y)\, dy \leq \tfrac{1}{2}, \quad t \geq 1,$$

we must have

$$\int_{B(t)} q_t(0,y) h(y) dy \geq \tfrac{1}{4}, \quad t \geq 1.$$

Let $x \in \mathbb{R}^n$. Choose t large enough so $x \in B(t)$, and so that $t^{1/2} \geq \lambda_0$. Then for $y \in B(t)$,

$$q_t(0,y) \leq c_1 t^{-n/2} \exp(-c_2 \lambda_0).$$

But if $s = 2\lambda_0 t/a_1$, by Theorem 1.14

$$q_s(x,y) \geq c_3 s^{-n/2} = c_4 t^{-n/2} \geq c_5 q_t(0,y), \quad y \in B(R),$$

where $c_5 > 0$. Thus

$$h(x) = \int q_s(x,y)h(y)\,dy \geq c_5 \int_{B(t)} q_t(x,y)h(y)\,dy \geq \tfrac{1}{4}c_5.$$

So $\inf h \geq c_5/4$, a contradiction. □

2. Probabilistic estimates and Girsanov transformation.

For $x \in \mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}$ we write $x = (x', x_d)$ where $x' = (x_1, \ldots, x_n) \in \mathbb{R}^{d-1}$. Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be C^2, with

$$|\nabla\gamma(x')| \leq K_0, \quad x' \in \mathbb{R}^n.$$

For $-\infty \leq a \leq b \leq \infty$ set

$$\begin{aligned} I(a,b) &= \{(x', x_d) \in \mathbb{R}^d : \gamma(x') + a \leq x_d \leq \gamma(x') + b\}, \\ H(\lambda) &= I(\lambda, \lambda), \quad \lambda \in \mathbb{R}. \end{aligned}$$

Let $\sigma : \mathbb{R}^d \to (0,\infty)$ be a smooth function. We assume that there exist constants $0 < \varepsilon_0 < 1$, $1 \leq K_1 < K_2 < \infty$, $K_3 < \infty$ such that σ satisfies

(S1) $\sigma^{-1}\Delta\sigma \geq 2\varepsilon_0$ on $I(-K_1, K_1)^c$,
(S2) $\sigma \geq \varepsilon_0/2$ on $I(-K_2, K_2)$,
(S3) $||\sigma||_\infty$, $||\nabla\sigma||_\infty$ and $||\Delta\sigma||_\infty$ are all bounded by K_3.

We will require the following easy geometric property of the sets $I(a,b)$.

Lemma 2.1. *For $\delta > 0$, $x' \in \mathbb{R}^n$ set*

$$\begin{aligned} C(x',\delta) &= \{(y', y_d) : |y' - x'| < \delta\}, \\ C_0(x',\delta) &= C(x',\delta) \cap \{(y', y_d) : -K_1 - \delta \leq y_d - \gamma(x') \leq K_1 + \delta\}. \end{aligned}$$

Then there exists $\delta_0 = \delta_0(\varepsilon_0, K_0, K_1, K_2) > 0$ such that for $x' \in \mathbb{R}^n$

$$C_0(x', \delta_0) \subset I(-K_2 + \delta_0, K_2 - \delta_0),$$

and

$$H(0) \cap C_0(x', \delta_0) = H(0) \cap C(x', \delta_0).$$

Let

$$\mathcal{L} = \tfrac{1}{2}\sigma^{-2}\nabla(\sigma^2\nabla) = \tfrac{1}{2}\Delta + (\sigma^{-1}\nabla\sigma)\nabla. \tag{2.1}$$

We use Girsanov's theorem to construct a diffusion with generator $\mathcal{L}$. Let $X = (X_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d)$ be a standard Brownian motion on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F})$ with filtration $(\mathcal{F}_t)$. Set

$$U_t = \int_0^t \nabla(\log\sigma)(X_s) dX_s - \tfrac{1}{2}\int_0^t |\nabla \log\sigma(X_s)|^2 \, ds.$$

By Itô's formula, since

$$\log\sigma(X_t) = \log\sigma(X_0) + \int_0^t \nabla(\log\sigma)(X_s) dX_s + \tfrac{1}{2}\int_0^t \Delta(\log\sigma)(X_s)\, ds,$$

and $\Delta \log\sigma + |\nabla \log\sigma|^2 = \sigma^{-1}\Delta\sigma$, we have

$$U_t = \log\sigma(X_t) - \log\sigma(X_0) - \tfrac{1}{2}\int_0^t (\sigma^{-1}\Delta\sigma)(X_s)\, ds. \tag{2.2}$$

Write $V = \frac{1}{2}\sigma^{-1}\Delta\sigma$, and set

$$\begin{aligned} Z_t = \exp(U_t) &= \sigma(X_t)\sigma(X_0)^{-1} \exp\left(-\tfrac{1}{2}\int_0^t (\sigma^{-1}\Delta\sigma)(X_s)\, ds\right) \\ &= \sigma(X_t)\sigma(X_0)^{-1} \exp\Big(-\int_0^t V(X_s)\, ds\Big). \end{aligned}$$

Note that $u > \varepsilon_0$ on $I(-K_1, K_1)^c$, and $|u| \leq K_3/\varepsilon_0$ on $I(-K_2, K_2)$, so that $-u \leq K_3/\varepsilon_0$ everywhere.

Lemma 2.2. *(a) If $X_s \in I(-K_1, K_1)^c$ for $0 \leq s \leq t$, then*

$$\sup_{s\leq t} Z_s \leq \sigma(X_0)^{-1}\sigma(X_t)e^{-\varepsilon_0 t} \leq \sigma(X_0)^{-1}K_3 e^{-\varepsilon_0 t}.$$

(b) If $X_s \in I(-K_2, K_2)$ for $0 \leq s \leq t$, then

$$Z_t^{-1} \leq \tfrac{1}{2}K_3^{-1}\varepsilon_0 e^{-K_3 t/\varepsilon_0} \leq Z_t.$$

(c) Z satisfies

$$\sup_{s\leq t} Z_s \leq \sigma(X_0)^{-1}K_3 e^{K_3 t/\varepsilon_0}.$$

(d) For each $x \in \mathbb{R}^d$, Z is a martingale with respect to $\mathbb{P}^x$.

Proof. (a), (b) and (c) are immediate from the definition of Z, and the properties of σ and V.
(d) Z is a local martingale, since Z is of the form $Z_t = \exp(M_t - \frac{1}{2}\langle M\rangle_t)$, where M is a continuous local martingale. But then Z is a true martingale, since (c) implies that Z is $\mathbb{P}^x$ -a.s bounded on every interval $[0, t]$. □

We can now use Girsanov's transformation (see for example [RW, Theorem 38.9]) to define a probability measure $\widetilde{\mathbb{P}}^x$ on $(\Omega, \mathcal{F})$ such that $d\widetilde{\mathbb{P}}^x/d\mathbb{P}^x \mid_{\mathcal{F}_t} = Z_t$. Then under $\widetilde{\mathbb{P}}^x$,

$$X_t - \int_0^t \nabla \log \sigma(X_s)\, ds = W_t, \tag{2.3}$$

where W is a Brownian motion with respect to $\widetilde{\mathbb{P}}^x$ with $W_0 = x$. So, under $\widetilde{\mathbb{P}}^x$, X is a diffusion with generator $\mathcal{L}$ given by (2.1). Define

$$\tau_\lambda = \inf\{s \ge 0 : X_s \in H(\lambda)\}.$$

Lemma 2.3. *(a) If $\lambda \ge K_1$, $y \in I(\lambda, \infty)$ then $\widetilde{\mathbb{P}}^y(\tau_\lambda < \infty) = 1$.*
(b) For any $y \in \mathbb{R}^d$, $\widetilde{\mathbb{P}}^y(\tau_0 < \infty) = 1$.
(c) For $x \in H(0)$, $\widetilde{\mathbb{P}}^x(\tau_{K_2} < \infty) = 1$.

Proof. (a) Let $t \ge 0$. By the definition of $\widetilde{\mathbb{P}}^y$, and Lemma 2.2(a),

$$\begin{aligned}\widetilde{\mathbb{P}}^y(\tau_\lambda > t) &= \mathbb{E}^y 1_{(\tau_\lambda > t)} Z_t \\ &\le \sigma(y)^{-1} K_3 e^{-\varepsilon_0 t} \mathbb{P}^y(\tau_\lambda > t) \le \sigma(y)^{-1} K_3 e^{-\varepsilon_0 t}.\end{aligned} \tag{2.4}$$

Letting $t \to \infty$ (a) is immediate.
(b) Let $x = (x', x_d) \in I(-K_1, K_1)$, and set

$$F = \{X_s \in C_0(x', \delta_0), 0 \le s \le 1\} \cap \{\tau_0 < 1\}.$$

By the support theorem for Brownian motion (see [Bs1, p.25]) there exists $p_0 > 0$ (independent of x) such that $\mathbb{P}^x(F) \ge p_0$. By Lemma 2.1 the cylinder $C_0(x', \delta_0) \subset I(-K_1, K_1)$, so using Lemma 2.2(b)

$$\widetilde{\mathbb{P}}^x(F) = \mathbb{E}^x 1_F Z_1 \ge \tfrac{1}{2} K_3^{-1} \varepsilon_0 e^{-K_3/\varepsilon_0} p_0.$$

Using this and (a), if $x \in I(0, \infty)$, then a standard renewal argument implies that $\widetilde{\mathbb{P}}^x(\tau_0 < \infty) = 1$. Exactly the same argument works for $x \in I(-\infty, 0)$.
(c) This is proved, using the support theorem, by an argument similar to the above. □

The main result of this section is an exponential moment bound on $|X_{\tau_0} - x|$ for $x \in I(-K_1, K_1)$, under $\widetilde{\mathbb{P}}^x$. As in Lemma 2.3, it is enough to treat the case $x \in I(0, K_1)$. Define stopping times

$$\begin{aligned} T_0 &= 0, \\ S_n &= \min\{t \ge T_{n-1} : X_t \in H(0) \cup H(K_2)\}, \quad n \ge 1, \\ T_n &= \min\{t \ge S_n : X_t \in H(K_1)\}, \quad n \ge 1, \end{aligned}$$

and let

$$N = \min\{n \ge 1 : X_{S_n} \in H(0)\}.$$

These random variables are all $\widetilde{\mathbb{P}}^x$-a.s. finite by Lemma 2.3. Set

$$\xi_n = |X_{S_n} - X_{T_{n-1}}|, \qquad \eta_n = |X_{T_n} - X_{S_n}|, \quad n \geq 1.$$

Clearly

$$|X_{\tau_0} - x| \leq \sum_{n=1}^{N} \xi_n + \sum_{n=1}^{N-1} \eta_n. \tag{2.5}$$

Note that if $x \in H(K_2)$, then $\widetilde{\mathbb{P}}^x(T_0 = S_1 = 0) = 1$.

Lemma 2.4. *There exist c_0, c_1 such that*

$$\widetilde{\mathbb{P}}^x(\eta_n > \lambda \mid \mathcal{F}_{S_n}) 1_{(n<N)} \leq c_0 e^{-c_1 \lambda}. \tag{2.6}$$

Proof. Using the strong Markov property of X, it is enough to prove

$$\widetilde{\mathbb{P}}^y(\eta_1 > \lambda) \leq c_0 e^{-c_1\lambda}, \qquad y \in H(K_2).$$

So let $y \in H(K_2)$. Then $\eta_1 = |X_{T_1} - y|$ and

$$\widetilde{\mathbb{P}}^y(\eta_1 > \lambda) \leq \widetilde{\mathbb{P}}^y(T_1 > \lambda) + \widetilde{\mathbb{P}}^y(T_1 \leq \lambda, \eta_1 > \lambda). \tag{2.7}$$

Using (2.4), we have

$$\widetilde{\mathbb{P}}^y(T_1 > \lambda) \leq c_2 e^{-\varepsilon_0 \lambda}.$$

For the second term in (2.7), note that by Lemma 2.2(a) $Z_{\lambda \wedge T_1} \leq c_3 e^{-\lambda \varepsilon_0}$, so that

$$\begin{aligned}\widetilde{\mathbb{P}}^y(\eta > \lambda, T_1 \leq \lambda) &\leq c_4 e^{-\varepsilon_0 \lambda}\, \mathbb{P}^y\big(\sup_{0 \leq s \leq \lambda} |X_s - y| > \lambda\big) \\ &\leq c_5 \exp\left(-\varepsilon_0 \lambda - c_6 \lambda\right),\end{aligned}$$

where we used a standard bound on Brownian motion in the last line. Combining these estimates for the two terms in (2.7) proves the lemma. □

Lemma 2.5. *There exist $\delta_1 > 0$, $c_1, c_2 < \infty$ such that*

$$\widetilde{\mathbb{P}}^x(\xi_n > \lambda \mid \mathcal{F}_{T_{n-1}}) 1_{(N > n-1)} \leq c_1 e^{-c_2 \lambda}, \quad \lambda > 0, \tag{2.8}$$

$$\widetilde{\mathbb{P}}^x(X_{S_n} \in H(0) \mid \mathcal{F}_{T_{n-1}}) 1_{(N > n-1)} \geq \delta_1. \tag{2.9}$$

Proof. As in the previous lemma, it is sufficient to take $x \in H(K_1)$ and prove unconditional versions of (2.8) and (2.9) with $n = 1$.

The estimate (2.9) follows from the support theorem for Brownian motion by the same argument as in Lemma 2.3(b).

Let $\sigma_1 \in C^2(\mathbb{R}^d)$ be defined by taking $\sigma_1 = \sigma$ in $I(-K_2, K_2)$, and be such that $\frac{1}{3}\varepsilon_0 \leq \sigma_1(y) \leq 2K_3$ for $y \in \mathbb{R}^d$. Let $X^* = (X^*_t, t \geq 0, \mathbb{Q}^x, x \in \mathbb{R}^d)$ be the divergence form diffusion with generator $L^* = \frac{1}{2}\nabla \sigma_1^2 \nabla$. Then X_s, $s \in [0, T_1]$, is, under $\widetilde{\mathbb{P}}^x$, a time change of X^*, and so $\widetilde{\mathbb{P}}^x(\xi_1 > \lambda) = \mathbb{Q}^x\big(|X^*_R - x| > \lambda\big)$, where $R = \inf\{t \geq 0 : X^*_t \in H(0) \cup H(K_2)\}$. The bound (2.8) now follows from standard properties of uniformly elliptic divergence form diffusions; see [BBu], Lemma 2.2. and Section 4 and [Bs2], pp. 187–188. □

Theorem 2.6. *There exist constants c_0, c_1, such that*

$$\widetilde{\mathbb{P}}^x(|X_{\tau_0} - x| > \lambda) \le c_0 e^{-c_1\lambda}, \qquad \lambda > 0, \quad x \in I(-K_2, K_2).$$

Proof. Set $V_1 = \xi_1$, and

$$V_n = |X_{S_n} - X_{S_{n-1}}| 1_{(N>n-1)} \le (\xi_n + \eta_{n-1}) 1_{(N>n-1)}, \quad n \ge 2.$$

Combining (2.6) and (2.8) we deduce that there exists $\alpha > 0$ such that

$$\widetilde{\mathbb{P}}^x(V_n > \lambda \mid \mathcal{F}_{S_{n-1}}) \le c_1 e^{-\alpha\lambda}, \quad \lambda > 0.$$

Integrating this bound, for $\theta < \alpha$,

$$\widetilde{\mathbb{E}}^x(e^{\theta V_n} \mid \mathcal{F}_{S_{n-1}}) \le 1 + \frac{c_1\theta}{\alpha - \theta}.$$

Write $\psi(\theta) = \log(1 + \frac{c_1\theta}{\alpha-\theta})$. Then if

$$M_n = \exp\left(\theta \sum_{i=1}^{n} V_i - n\psi(\theta)\right),$$

$$\widetilde{\mathbb{E}}^x(M_n \mid \mathcal{F}_{S_{n-1}}) = M_{n-1} e^{-\psi(\theta)} \widetilde{\mathbb{E}}^x(e^{\theta V_n} \mid \mathcal{F}_{S_{n-1}}) \le M_{n-1},$$

so that M_n is a supermartingale. Since N is a stopping time with respect to $(\mathcal{F}_{S_n})$, if $k \ge 1$ then $\widetilde{\mathbb{E}}^x(M_{N\wedge k}) \le 1$. So, by Cauchy-Schwarz

$$\begin{aligned}
\widetilde{\mathbb{E}}^x \exp\big(\tfrac{1}{2}\theta \sum_1^{N\wedge k} V_i\big) &= \widetilde{\mathbb{E}}^x\big(\exp\big(\tfrac{1}{2}\theta \sum_{i=1}^{N\wedge k} V_i - \tfrac{1}{2}(N\wedge k)\psi(\theta)\big) \exp\big(\tfrac{1}{2}(N\wedge k)\psi(\theta)\big)\big) \\
&\le \left(\widetilde{\mathbb{E}}^x \exp\big(\theta \sum_{i=1}^{N\wedge k} V_i - (N\wedge k)\psi(\theta)\big)\right)^{1/2} \left(\widetilde{\mathbb{E}}^x e^{\psi(\theta)(N\wedge k)}\right)^{1/2} \\
&\le \left(\widetilde{\mathbb{E}}^x e^{\psi(\theta)N}\right)^{1/2}.
\end{aligned}$$

The bound (2.9) implies that $P(N = n \mid N > n-1) \ge \delta_1$, so $P(N \ge n) \le (1-\delta_1)^{n-1}$, and

$$\widetilde{\mathbb{E}}^x(e^{\psi(\theta)N}) < \infty \quad \text{provided} \quad e^{\psi(\theta)}(1-\delta_1) < 1.$$

So, taking θ_1 small enough so that this last condition holds, and letting $k \to \infty$, it follows that

$$\widetilde{\mathbb{E}}^x \exp\left(\tfrac{1}{2}\theta \sum_{i=1}^{N} V_i\right) < \infty \qquad \text{for} \quad \theta < \theta_1.$$

Since

$$|X_{\tau_0} - X_0| = |X_{S_N} - X_0| \le \sum_{i=1}^{N} V_i,$$

this implies that $|X_{\tau_0} - X_0|$ has exponential moments with respect to $\widetilde{\mathbb{P}}^x$, proving the theorem. □

The final result in this section will be used to weaken the hypotheses of boundedness in our Liouville Theorem.

Proposition 2.7. *Let $h \in C^2(\mathbb{R}^d)$ be a function such that $\mathcal{L}h = 0$ and σh is bounded. Then h is bounded.*

Proof. We can assume that $|\sigma h| \le 1$. Set $M_t = h(X_t)$. By Itô's formula M is a continuous local martingale with respect to $\widetilde{\mathbb{P}}^x$, and $M_t Z_t$ is a continuous local martingale with respect to $\mathbb{P}^x$. However,

$$|M_t Z_t| = \sigma(X_0)^{-1} |\sigma(X_t) h(X_t)| \exp\Big(- \int_0^t V(X_s) ds \Big) \le \sigma(X_0)^{-1} e^{tK_3/\varepsilon_0},$$

so that $|M_t Z_t|$ is bounded on each interval $[0, t]$. Therefore MZ is a martingale with respect to $\mathbb{P}^x$, and hence M is a martingale with respect to $\widetilde{\mathbb{P}}^x$.

Set $T = \inf\{t \ge 0 : X_t \in I(-K_2, K_2)\}$; by Lemma 2.3 $\widetilde{\mathbb{P}}^x(T < \infty) = 1$ for all x. Note that, as h is bounded by $2/\varepsilon_0$ on $I(-K_2, K_2)$, $|h(X_T)| \le 2/\varepsilon_0$. Since M is a martingale with respect to $\widetilde{\mathbb{P}}^x$, $\widetilde{\mathbb{E}}^x h(X_{t\wedge T}) = h(x)$. So

$$\begin{aligned} |\widetilde{\mathbb{E}}^x h(X_T) - h(x)| &= |\widetilde{\mathbb{E}}^x 1_{(T>t)} \big(h(X_T) - h(X_t)\big)| \\ &\le 2\varepsilon_0^{-1} \widetilde{\mathbb{P}}^x(T > t) + \widetilde{\mathbb{E}}^x 1_{(T>t)} |h(X_t)|. \end{aligned} \tag{2.10}$$

By Lemma 2.2(a) the second term in (2.10) is bounded by

$$\sigma(x)^{-1} \mathbb{E}^x 1_{(T>t)} \sigma(X_t) |h(X_t)| e^{-\varepsilon_0 t} \le \sigma(x)^{-1} e^{-\varepsilon_0 t}.$$

Since $\widetilde{\mathbb{P}}^x(T = \infty) = 0$, letting $t \to \infty$ in (2.10) it follows that $\widetilde{\mathbb{E}}^x h(X_T) = h(x)$. Hence $|h(x)| \le \widetilde{\mathbb{E}}^x |h(X_T)| \le 2/\varepsilon_0$, proving that h is bounded. □

3. Transformation to a Jump Process

We continue with the notation and hypotheses of the previous section. We write $X = (X_t, t \ge 0, \widetilde{\mathbb{P}}^x, x \in \mathbb{R}^d)$ for the diffusion with generator $\mathcal{L}$. When we refer to properties of X, it is with respect to the probabilities $\widetilde{\mathbb{P}}^x$. Let $\mu(dx) = \sigma^2(x)dx$, and define

$$\mathcal{E}(f, f) = \int |\nabla f(x)|^2 \sigma^2(x) dx, \quad f \in C_0^2(\mathbb{R}^d).$$

Then (see [FOT, Thm. 3.1.3]) $\mathcal{E}$ can be extended to a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^d, \mu)$ with (special standard) core $C_0^2(\mathbb{R}^d)$, and X is the Hunt process (in fact a Feller diffusion) associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Define $\pi : H(0) \to \mathbb{R}^{d-1}$ by $\pi((x', x_d) = x'$. Let ν be the measure on $\mathbb{R}^d$ with support $H(0)$ given by

$$\nu(A \cap H(0)) = |\pi(A \cap H(0))|,$$

where $|\cdot|$ denotes $d-1$–dimensional Lebesgue measure. Let A_t, $t \ge 0$ be the continuous additive functional with Revuz measure ν. Note that A_t increases only when $X_t \in H(0)$. Set $\zeta_t = \inf\{s \ge 0 : A_s > t\}$, and let

$$\widetilde{X}_t = X_{\zeta_t}.$$

It is clear from Lemma 2.3 that $\widetilde{\mathbb{P}}^x(\zeta_t < \infty) = 1$ for all t, so that $\widetilde{X}$ is a conservative Markov process.

We now use the relation between traces of Dirichlet forms and time changes of Hunt processes – see [FOT], Theorem 6.2.1, to describe the Dirichlet form of $\widetilde{X}$. Let

$$\Gamma(x, dy) = \widetilde{\mathbb{P}}^x(X_{\tau_0} \in dy), \quad x \in \mathbb{R}^d - H(0),\, y \in H(0),$$

be the harmonic measure on $H(0)$ for X. Since X is a diffusion with C^2 coefficients, Γ is absolutely continuous with respect to ν. Write $\Gamma(x, y)$ for the density of Γ:

$$\Gamma(x, dy) = \Gamma(x, y)\,\nu(dy), \qquad y \in H(0), \quad x \in \mathbb{R}^d - H(0).$$

Further $\Gamma(x, y)$ is continuous on $(\mathbb{R}^d - H(0)) \times H(0)$. For $g \in C_0^2(\mathbb{R}^d)$, set

$$\Gamma g(x) = \begin{cases} g(x), & \text{if } x \in H(0), \\ \int \Gamma(x, dy) g(y), & \text{if } x \in \mathbb{R}^d - H(0). \end{cases}$$

Then Γg is $\mathcal{L}$-harmonic on $H(0)^c$, and continuous on $\mathbb{R}^d$. By [FOT, Theorem 6.2.1] $\widetilde{X}$ is a ν-symmetric Hunt process with Dirichlet form $(\widetilde{\mathcal{E}}, \mathcal{D}(\widetilde{\mathcal{E}}))$ with core $C_0^2(H(0))$, where $\widetilde{\mathcal{E}}$ is given by

$$\widetilde{\mathcal{E}}(g, g) = \mathcal{E}(\Gamma g, \Gamma g), \quad g \in C_0^2(H(0)). \tag{3.1}$$

Remark 3.1. Let W be $BM(\mathbb{R}^d)$; W has Dirichlet form

$$\mathcal{E}_B(f, f) = \int |\nabla f(x)|^2 dx.$$

If $H(0) = \{x : x_d = 0\}$, then a similar construction gives a Dirichlet form $\widetilde{\mathcal{E}}_B$. See [FOT, Example 6.2.2] for a detailed discussion of this case: $\widetilde{\mathcal{E}}_B$ is the Dirichlet form of the $d-1$ –dimensional Cauchy process, and is given by

$$\widetilde{\mathcal{E}}_B(g, g) = c_d \iint_{H(0)\times H(0)} \big(g(x) - g(y)\big)^2 |x - y|^{-d}\, dx\, dy. \tag{3.2}$$

Since the law of X is locally absolutely continuous with respect to that of Brownian motion, one expects an analogous expression for $\widetilde{\mathcal{E}}$.

We now wish to identify more precisely the Dirichlet form $\widetilde{\mathcal{E}}$. We first show there exists a nice harmonic function to compare to.

Lemma 3.2. *There exist c_1 and c_2 and an $\mathcal{L}$-harmonic function h such that $h = 0$ on $H(0)$, $h = 1$ on $H(1)$, $|\nabla h| \le c_1$ in $I(0, 1)$, and $\partial h/\partial n \ge c_2$ on $H(0)$.*

Proof. Let $h(x) = \mathbb{P}^x(\tau_1 < \tau_0)$. Since h solves a Dirichlet problem in a C^2 domain with C^2 boundary values and C^2 strictly elliptic coefficients for the operator, the upper bound on $|\nabla h|$ follows. We need only prove the lower bound.

We flatten the boundary. That is, we look at $\widetilde{X}_t = \Phi(X_t)$, where $\Phi(x', x_d) = (x', x_d - \gamma(x'))$. A routine calculation using Ito's formula shows that on $\Phi(I(0,1))$ the process $\widetilde{X}_t$ is associated with an operator in divergence form that is strictly elliptic, that the coefficients are C^2, and that the normal derivative gets mapped into the conormal derivative; also the angle made by the conormal vector with the hyperplane is bounded away from 0. If $\widetilde{h}$ is the image of h under this map, we need to show that $\partial\widetilde{h}/\partial n > c_3$.

Since the coefficients of the operator are C^2, the process $\widetilde{X}_t^d$ is a semimartingale $M_t + A_t$, where $c_4 \le d\langle M\rangle_t/dt \le c_5$ and $|dA_t| \le c_6 dt$. By performing a nondegenerate time change, we may assume M_t is a Brownian motion. By a comparison theorem, (see, e.g., [IW], p. 352)

$$\widetilde{h}(x) \ge \mathbb{P}^x(W_t - c_7 t \text{ hits 1 before 0}),$$

where W_t is a standard one-dimensional Brownian motion. This, it is well known, is larger than $c_8 x_d$. □

Next we need some routine harmonic measure estimates. For $x \in H(0)$ we let $B_{H(0)}(x,r)$ be $\{y \in H(0) : |y - x| < r\}$, G be the Green function for the process killed on hitting $H(0)$, and $y_r(x)$ the point whose coordinates are the same as those for x, except that the x_d coordinate is r larger; thus $y_r(x)$ lies r units above x. Since $K_1 \ge 1$ $\mathcal{L}$ is strictly elliptic on $I(0,1)$.

Proposition 3.3. *Suppose $x_0 \in I(0,1)$ with* $\operatorname{dist}(x_0, H(0)) \ge 1/4$. *Let $x \in H(0)$ with $|x - x_0| \le 2$. Then there exist c_1, c_2, c_3, c_4, and A_0 not depending on x_0 or x such that*
(a) $c_1 \le \Gamma(x_0, y) \le c_2$ if $|y - x| \le 1$.
(b) If $\lambda \ge A_0$, then $\widetilde{\mathbb{P}}^{x_0}(|X_{\tau_0} - x| > \lambda) \le c_3 \exp(-c_4\lambda)$.

Proof. Suppose $x_0 \in I(1/2, 1)$. As in [Bs1], Theorem III.5.4,

$$\Gamma(x_0, B_{H(0)}(x,r)) \approx G(x_0, y_r(x)) r^{d-2}, \quad r < \tfrac{1}{8}.$$

Here '$\approx$' means that the ratio is bounded above and below by positive constants not depending on x or r as long as $r < 1$. The proof in [Bs1] is given for Brownian motion, but the identical proof works for strictly elliptic divergence form operators. We now apply the the boundary Harnack principle for divergence from operators (see [BBu]) to compare the harmonic functions $h(y)$ and $G(x_0, y)$. We conclude $G(x_0, y_r(x)) \approx h(y_r(x))$ and hence $G(x_0, y_r(x)) \approx r$. So $\Gamma(x_0, (B(x,r)) \approx r^{d-1}$, as long as $r < \frac{1}{8}$ and $|x - x_0| < 2$. Since the surface measure of $B(x,r)$ is comparable to r^{d-1}, (a) follows.

(b) follows immediately from Theorem 2.6. □

Now we estimate $\partial\Gamma/\partial n$. Let S be the surface measure on $H(0)$: note that S and ν are absolutely continuous. Since Γ is a solution to a Dirichlet problem, $\partial\Gamma/\partial n$ exists. Set

$$m(x,y) = \frac{\partial\Gamma(\cdot, y)}{\partial n}(x).$$

Proposition 3.4. *There exist c_1, c_2, c_3, c_4, and A_0 such that for $x, y \in H(0)$,*

(a) *If* $|x-y|\le 1$, *then*

$$c_1|x-y|^d \le m(x,y) \le c_2|x-y|^d.$$

(b) *If* $A > A_0$, *then*

$$\int_{B_{H(0)}(x,A)^c} m(x,y)\, S(dy) \le c_3 \exp(-c_4 A).$$

Proof. Let us first flatten the boundary as above. Pick $z \in I(0,1)$ with $z' = x'$.

First suppose $|x-y| = 1$. By the boundary Harnack principle, $\Gamma(z,y)$ is comparable to

$$\frac{h(z)}{h((z',1/2))}\Gamma((z',1/2),y).$$

This is comparable to z_d by Lemma 3.2 and Proposition 3.3. (a) follows when $|x-y| = 1$ by letting $z_d \to 0$. (Recall that the angle between the conormal vector and the hyperplane is bounded away from 0.)

We get the case where $|x-y| < 1$ by scaling. Let $r = |x-y|$ and scale by a factor $1/r$. This enlarges things, so the region on which σ is nice is larger and the coefficients are smoother. This increases the area of a surface ball by a factor r^{d-1}, and since the distance from z to the boundary becomes r times as large, the derivative increases by a factor r. So altogether we get a factor r^d, as we should.

To get (b), we use the boundary Harnack principle as above. So $\Gamma(z, (B_{H(0)}(x,A)^c)$ is comparable to

$$\frac{h(z)}{h((z',1/2))}\Gamma((z',1/2),(B_{H(0)}(x,A)^c).$$

By Proposition 3.3, this is less than $c_3 z_d e^{-c_4 A}$. (b) now follows by letting $z_d \to 0$. □

A measure $J(dx\,dy)$ on $H(0)\times H(0)$ is symmetric if it remains unchanged under the map $(x,y)\to(y,x)$.

Proposition 3.5. *There exists a symmetric measure* J *such that*

$$\widetilde{\mathcal{E}}(g,g) = \int_{H(0)} \big(g(x)-g(y)\big)^2 J(dx\,dy). \tag{3.3}$$

Proof. Since the Dirichlet form $\mathcal{E}$ for X is regular, with core $C_0^2(\mathbb{R}^d)$, [FOT], Theorem 6.2.1 implies that $\widetilde{\mathcal{E}}$ is also regular, with core

$$C' = \{f : f = g|_H \text{ for some g } \in C_0^2(\mathbb{R}^d)\} = C_0^2(H(0)).$$

Hence, by [FOT], Theorem 3.2.1 $\widetilde{\mathcal{E}}$ can be written in the form $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}^{(c)} + \widetilde{\mathcal{E}}^{(d)} + \widetilde{\mathcal{E}}^{(k)}$, where

$$\widetilde{\mathcal{E}}^{(d)}(f,g) = \int\int (f(x)-f(y))(g(x)-g(y))J(dx\,dy);$$

here J is a measure on $H(0)\times H(0)$ that is symmetric.

Since Y is conservative, the killing term $\widetilde{\mathcal{E}}^{(k)}=0$. By [JY], all martingales adapted to the filtration of X can be written as stochastic integrals with respect to d fixed martingales; the quadratic variation of each of these is absolutely continuous with respect to dt. Since X spends zero time on $H(0)$, any continuous martingale on the filtration of X which is constant except on $\{t: X_t\in H(0)\}$ is therefore constant everywhere. It follows from [FOT], Section 5.3 that $\widetilde{\mathcal{E}}^{(c)}=0$. □

When f and g are both C_0^2 with disjoint supports, we have using the symmetry of J

$$\widetilde{\mathcal{E}}^{(d)}(f,g)=2\int\int f(x)g(y)J(dx\,dy). \tag{3.4}$$

Define a metric d' on $H(0)$ by $d'(x,y)=d'((x',x_n),(y',y_n))=|x'-y'|$. Since $|\nabla\gamma|$ is bounded, d' is equivalent to the Euclidean metric.

Theorem 3.6. *There exists a symmetric function $n(x,y)$, $x,y\in H(0)$ such that*

$$\widetilde{\mathcal{E}}(g,g)=\int_{H(0)}\big(g(x)-g(y)\big)^2 n(x,y)\,\nu(dx)\nu(dy). \tag{3.5}$$

The function $n(x,y)$ satisfies

$$c_1|x-y|^{-d}\le n(x,y)\le c_2|x-y|^{-d},\quad d'(x,y)\le 1, \tag{3.6}$$

$$\int_{d'(x,y)>\lambda} n(x,y)dy\le c_2e^{-\alpha\lambda},\quad x\ge 1. \tag{3.7}$$

for constants c_1, c_2, $\alpha\in(0,\infty)$.

Proof. Let $f,g\in C_0^2(\mathbb{R}^d)$ with disjoint support. Choose a cube $D\subset\mathbb{R}^d$ which is large enough so that f and g are 0 outside D. Write $D_+=D\cap I(0,\infty)$, $D_0=D\cap H(0)$ and $D_-=D\cap I(-\infty,0)$. Since D_0 has measure zero,

$$\begin{aligned}\widetilde{\mathcal{E}}(f,g)&=\int_{\mathbb{R}^d}\nabla\Gamma f(x)\cdot\nabla\Gamma g(x)\sigma(x)^2dx \\ &=\int_{D_+}\nabla\Gamma f(x)\cdot\nabla\Gamma g(x)\sigma(x)^2dx+\int_{D_-}\nabla\Gamma f(x)\cdot\nabla\Gamma g(x)\sigma(x)^2dx.\end{aligned} \tag{3.8}$$

By the Gauss-Green theorem, since $\mathcal{L}\Gamma g=0$ on $H(0)^c$ and $f=0$ on $\partial D-D_0$,

$$\int_{D_+}\nabla\Gamma f(x)\cdot\nabla\Gamma g(x)\sigma(x)^2dx=\int_{D_0}\Gamma f(x)\sigma(x)^2\frac{\partial\Gamma g(x)}{\partial n}dS,$$

where dS is surface measure on D_0. But note that $\Gamma f=f$ on $H(0)$.

Using Proposition 3.4 and the fact that f and g have disjoint support, we may pass the limit under the integration sign to obtain

$$\int_{D_+} \nabla\Gamma f(x)\cdot\nabla\Gamma g(x)\sigma(x)^2 dx = \int_{D_0} f(x)\sigma(x)^2 \int_{D_0} g(y)\frac{\partial\Gamma(\cdot,y)}{\partial n}(x)\, dS(y)\, dS(x).$$

Summing with the analogous equality for the integral over D_- and using Proposition 3.4, we obtain

$$\widetilde{\mathcal{E}}(f,g) = \int_{D_0}\int_{D_0} f(x)g(y)m(x,y)dS(y)\, dS(x).$$

If we now compare this with (3.4), we see that $J(dx\, dy) = m(x,y)\, dS(x)\, dS(y)$. The symmetry of J shows that m is symmetric. Since $dS(x) \le c_3 d\nu(x)$, the bounds in Proposition 3.4 complete the proof. □

Note that $\pi : H(0) \to \mathbb{R}^{d-1}$ is bijective. So, setting $Y_t = \pi(\widetilde{X}_t)$, $n = d-1$, Y is a Markov process on $\mathbb{R}^n$, with Dirichlet form $\mathcal{E}_Y$ given by $\mathcal{E}_Y(f,f) = \widetilde{\mathcal{E}}(f\circ\pi, f\circ\pi)$. Writing $n'(x',y') = n(\pi^{-1}(x'), \pi^{-1}(y'))$ we have

$$\mathcal{E}_Y(f,f) = \iint (f(x') - f(y'))^2 n'(x',y')dx'dy'.$$

It is immediate from Theorem 3.6 that n' satisfies (1.1) and (1.2) so that Y satisfies the hypotheses of Theorems 1.14 and 1.17.

Recall from the introduction that h is $\mathcal{L}$-harmonic if $\mathcal{L}h = 0$.

Theorem 3.7. *Let γ, σ be as above, with σ satisfying (S1)-(S3). Suppose h is $\mathcal{L}$-harmonic, and σh is bounded. Then h is constant.*

Proof. By Proposition 2.7 h is bounded. Set $M_t = h(X_t)$. Then, as in the proof of Proposition 2.7, M is a martingale/$\widetilde{\mathbb{P}}^x$ for any $x \in \mathbb{R}^d$. As M is bounded, and $\widetilde{\mathbb{P}}^x(\zeta_t < \infty)$ for all t, it follows that $h(\widetilde{X}_{\zeta_t})$ is a martingale/$\widetilde{\mathbb{P}}^x$. So, if g is the function on $\mathbb{R}^n = \mathbb{R}^{d-1}$ defined by $g(x') = h(\pi^{-1}(x'))$, then g is bounded and $g(Y_t)$ is a bounded martingale. Thus g is Y-harmonic, and so g is equal to a constant c_0 by Theorem 1.17. But then for any $x \in \mathbb{R}^d$, since $\widetilde{\mathbb{P}}^x(\tau_0 < \infty) = 1$, and h is bounded,

$$h(x) = \widetilde{\mathbb{E}}^x h(X_{\tau_0}) = c_0,$$

which proves that h is constant. □

4. Applications to PDE.

In this section, we apply the Liouville theorem (Theorem 3) to prove Theorems 1 and 2. First we have an elementary lemma.

Lemma 4.1. *Let $\sigma = \partial u/\partial x_d$. Suppose $\sigma(0) > 0$ and for each a there exists a constant $c(a)$ such that $a \cdot \nabla u(x) = c(a)\sigma(x)$ for all $x \in \mathbb{R}^d$. Then u is of the form $u(x) = g(a \cdot x_d)$, for some fixed $a \in \mathbb{R}^d$ with $|a| = 1$.*

Proof. Since $\sigma(0) > 0$ we have $\nabla u(0) \neq 0$; let a be orthogonal to $\nabla u(0)$. Using the hypothesis shows $c(a) = 0$, so that $a \cdot \nabla u(x) = 0$ for all $x \in \mathbb{R}^d$, proving that u is constant on every hyperplane orthogonal to $\nabla u(0)$. □

Proof of Theorem 1. Let $\sigma(x) = \frac{\partial u(x)}{\partial x_d}$. It is shown in Lemma 3.2 of [GG], by using the moving plane method, that $\sigma(x) > 0$ in $\mathbb{R}^d$. Recall from the introduction that σ satisfies

$$\Delta\sigma - F''(u(x))\sigma = 0, \quad x \in \mathbb{R}^d.$$

In view of Lemma 4.1 it is enough to prove that σ satisfies the conditions (S1)-(S3) in Theorem 3.

We choose $\gamma(x') \equiv 0, x' \in \mathbb{R}^{d-1}$. Hence

$$I(a, b) = \{x = (x', x_d) \in \mathbb{R}^d : a \leq x_d \leq b, x' \in \mathbb{R}^{d-1}\}.$$

Since $F''(\pm 1) \geq \mu > 0$ and $u(x) \to \pm 1$ uniformly as $x_d \to \pm\infty$, we can choose K_1 large enough so that

$$\sigma^{-1}\Delta\sigma = F''(u(x)) > 2\varepsilon_1, \quad x \in I(-K_1, K_1)$$

for some $\varepsilon_1 > 0$. It can be shown, using (0.6), that $|u(x)| < 1, x \in \mathbb{R}^d$. The standard Schauder estimates for elliptic equations imply that $||\sigma||_\infty$, $||\nabla\sigma||_\infty$ and $||\Delta\sigma||_\infty$ are all bounded by some $K_3 > 0$. Also, by [GT], u and σ are $C^{2+\varepsilon}$ in $\mathbb{R}^d$. Set $K_2 = 1 + K_1$; it remains to show that

$$\sigma \geq \varepsilon/2 > 0, \quad x \in I(-K_2, K_2) \tag{4.1}$$

for some positive constant ε.

If (4.1) is not true, there exists a sequence $\{x^{(m)}\} = \{(x^{(m)'}, x_d^{(m)})\}$ such that $|x_d^{(m)}| \leq K_2$ and $\lim_{m\to\infty} \sigma(x^{(m)}) = 0$. Without loss of generality, we can assume $x_d^{(m)} \to x_d$ as $m \to \infty$. Now we define a sequence of solutions to (0.3)

$$u^{(m)}(x) := u(x^{(m)} + x), \quad x \in \mathbb{R}^d.$$

By the standard Schauder estimates for elliptic equations, we know that $\|u^{(m)}\|_{C^{3+\varepsilon}(\mathbb{R}^d)} \leq K_3 < \infty$. Therefore there exists a subsequence of $u^{(m)}$, which we still denote by $u^{(m)}$, and a solution of (0.3) $v(x) \in C^{3+\varepsilon}(\mathbb{R}^d)$ such that $\|u^{(m)} - v\|_{C^{2+\varepsilon}(\Omega)} \to 0$ as $m \to \infty$ for any bounded set Ω. Since $|x_d^{(m)}| \leq K_2$, $v(x)$ converges uniformly to ± 1 as x_d tends to $\pm\infty$. So, by Lemma 3.2 of [GG],

$$\frac{\partial v}{\partial x_d} > 0, \quad x \in \mathbb{R}^d.$$

On the other hand, by the definition of $\{x^{(m)}\}$ we have

$$\frac{\partial v(0)}{\partial x_d} = \lim_{m\to\infty} \frac{\partial u(x^{(m)})}{\partial x_d} = 0,$$

a contradiction. This proves (4.1), and so σ satisfies (S1)-(S3) with $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon\}$. Theorem 1 now follows from Lemma 4.1 and the fact that the uniform convergence condition implies that the hyperplanes on which u is constant must be orthogonal to $e^{(d)}$. □

Remark 4.2. We can also prove directly by using suitable comparison functions that σ decays like $\exp(\mu_{\pm 1} x_d)$ near $x_d = \pm\infty$ for some $\mu_{\pm 1}$, and hence that h is bounded. Then a weaker version of the Liouville theorem, not using Proposition 2.7, also leads to Theorem 1.

Proof of Theorem 2. We define σ, ψ, h as in the proof of Theorem 1. In order to show that σ satisfies conditions (S1)-(S3) in Theorem 3, we let $\gamma(x'), x' \in \mathbb{R}^{d-1}$ be a level surface of $u(x', x_d)$, say $u(x', \gamma(x')) = 0, x' \in \mathbb{R}^{d-1}$. The function $\gamma(x')$ is well defined in $\mathbb{R}^{d-1}$ since $u(x)$ is strictly monotone in x_d, and γ is C^2 by the implicit function theorem. The cone condition implies that $|\nabla\gamma(x')| \le K_0$ for $x' \in \mathbb{R}^{d-1}$ for some $K_0 < \infty$. Since $F''(\pm 1) \ge \mu > 0$, we can choose $0 < \delta < 1$ and $\varepsilon_1 > 0$ such that $F''(u) > 2\varepsilon_1 > 0$ when $-1 < u < -1 + \delta$ or $1 - \delta < u < 1$. Let $\gamma_1(x'), \gamma_2(x'), x' \in \mathbb{R}^{d-1}$ be the level surfaces of $u(x)$ with $u(x', \gamma_1(x')) = 1 - \delta, u(x', \gamma_2(x')) = -1 + \delta$ respectively. We claim that there exists $\varepsilon_2 > 0$ such that

$$\sigma(x) > \varepsilon_2/2, \quad x \in \{x = (x', x_d) \in \mathbb{R}^d : \gamma_2(x') \le x_d \le \gamma_1(x'), x' \in \mathbb{R}^{d-1}\}. \tag{4.2}$$

We prove this claim by contradiction. If it is not true, then there exists a sequence $\{x^{(m)}\} = \{(x^{(m)'}, x_d^{(m)})\}$ such that $-1 + \delta \le u(x^{(m)}) \le 1 - \delta$ and $\lim_{m\to\infty} \sigma(x^{(m)}) = 0$. As in the proof of Theorem 1, we define a sequence of solutions to (0.3)

$$u^{(m)}(x) := u(x^{(m)} + x), \quad x \in \mathbb{R}^d,$$

and $\|u^{(m)}\|_{C^{3+\varepsilon}(\mathbb{R}^d)} \le K_3 < \infty$. Therefore, as before, there exists a subsequence of $u^{(m)}$, which we still denote by $u^{(m)}$, and a solution $v(x) \in C^{3+\varepsilon}(\mathbb{R}^d)$ of (0.3) such that $\|u^{(m)} - v\|_{C^{2+\varepsilon}(\Omega)} \to 0$ as $m \to \infty$ for any bounded set Ω.

Note that

$$-1 + \delta \le v(0) \le 1 - \delta \tag{4.3}$$

and

$$\frac{\partial v(0)}{\partial x_d} = \lim_{m\to\infty} \frac{\partial u(x^{(m)})}{\partial x_d} = 0.$$

Since $\varphi = \frac{\partial v(x)}{\partial x_d} \ge 0, x \in \mathbb{R}^d$ satisfies

$$-\Delta\varphi + F''(v(x))\varphi = 0, \quad x \in \mathbb{R}^d,$$

the strong maximum principle (see [GT]) yields $\frac{\partial v(x)}{\partial x_d} \equiv 0, x \in \mathbb{R}^d$.

Since $|v(0)| \le 1 - \delta$, v cannot be identically 1 or -1. It follows that $-1 < v(x) < 1$, $x \in \mathbb{R}^d$. The Lipschitzian condition on u leads to the same condition on v, i.e.

$$|\nabla_{x'} v(x)| \le L(v) \frac{\partial v(x)}{\partial x_d}, \quad x \in \mathbb{R}^d.$$

Therefore $\nabla v \equiv 0$, $x \in \mathbb{R}^d$. Hence $v(x)$ must be a constant, and so $v(x) \equiv u_0$, $x \in \mathbb{R}^d$, where u_0 is the unique critical point of F in $(-1, 1)$. We now show that this is impossible.

For any ball $B_R(0) \subset \mathbb{R}^d$, we know that the first eigenvalue $\lambda_1 > 0$ and eigenfunction $\varphi_1(x) > 0, x \in B_R(0)$ of $-\Delta$ in the Sobolev space $H_0^1(B_R(0))$ satisfy

$$\begin{cases} \Delta\varphi_1(x) + \lambda_1\varphi_1(x) = 0, \quad x \in B_R(0), \\ \varphi_1(x) = 0, x \in \partial B_R(0). \end{cases}$$

Since $F''(u_0) < 0$, we can choose R sufficiently large such that $\lambda_1 < -F''(u_0)/2$. On the other hand, when m is large enough we have $-F''(u^{(m)}(x)) \geq -F''(u_0)/2, x \in B_R(0)$. Since $\sigma^{(m)} := \frac{\partial u^{(m)}(x)}{\partial x_d} > 0$, $x \in B_R(0)$ satisfies

$$-\Delta\sigma^{(m)}(x) + F''(u^{(m)}(x))\sigma^{(m)}(x) = 0, \quad x \in B_R(0),$$

the quotient $\varphi_1(x)/\sigma^{(m)}(x) > 0$ satisfies

$$\Delta\varphi + 2\frac{\nabla\sigma^{(m)}}{\sigma^{(m)}} \cdot \nabla\varphi + V(x)\varphi = 0, \quad x \in B_R(0) \tag{4.4}$$

where $V(x) = \lambda_1 + F''(u^{(m)}(x)) \leq 0, x \in B_R(0)$. This contradicts the maximum principle for (4.4) since $\varphi_1(x)/\sigma^{(m)}(x)$ vanishes on $\partial B_R(0)$. Therefore we have proven (4.2).

Since u is bounded it follows immediately from (4.2) that there exists $K_1 < \infty$ such that

$$0 < \gamma_1(x') - \gamma(x') < K_1, \quad 0 < \gamma(x') - \gamma_2(x') < K_1, \quad x' \in \mathbb{R}^{d-1}, \tag{4.5}$$

so that

$$\sigma^{-1}\Delta\sigma = F''(u) \geq 2\varepsilon_2 \quad \text{on } I(-K_1, K_1)^c.$$

Let $K_2 = K_1 + 1$; it remains to show that for some $\varepsilon_0 > 0$

$$\sigma(x) > \varepsilon_0/2, \quad x \in I(-K_2, K_2). \tag{4.6}$$

We define $u^{(m)}$ and $v(x)$ as in the proof of (4.2). As before, we have $\frac{\partial v(0)}{\partial x_d} = 0$ and $\frac{\partial v(x)}{\partial x_d} \geq 0, x \in \mathbb{R}^d$. By the maximum principle (see [GT]) this implies that

$$\frac{\partial v(x)}{\partial x_d} \equiv 0, x \in \mathbb{R}^d. \tag{4.7}$$

But from (4.5) and the definition of γ_1, γ_2, we have

$$v(0, x_d) > 1 - \delta, \text{ if } x_d > 2K_1, \quad v(0, x_d) < -1 + \delta \text{ if } x_d < -2K_1.$$

This is a contradiction to (4.7), and therefore (4.6) holds.

So γ and σ satisfy the hypotheses of Theorem 3, and Theorem 2 now follows by Lemma 4.1. □

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