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Department of Mathematics
University of British Columbia
Vancouver, B.C.

Dear Martin:

Here is a proof of your conjecture. It requires some knowledge of reflection groups, which I'll be glad to explain in the fall.

Using Boerbaki's notation, let W be a finite reflection group generated by a set $\{s_1, \ldots, s_\ell\}$ of reflections in the walls of a fundamental chamber C and $\{\alpha_1, \ldots, \alpha_\ell\}$ the corresponding basis of a root system R. The reflections in W correspond to positive roots $\alpha \in R$.

Let p be a point and P the polytope with vertices $\{w(p) \mid w \in W\}$. Your hypothesis is that any two vertices of P can be reflected into each other by some reflection in W. In particular, R is irreducible and we can suppose that $p \in C$, i.e. that $(p, \alpha_i) \geq 0$ for $i = 1, \ldots, \ell$.

For $\alpha \in R$, let $p_\alpha = 2\frac{(p,\alpha)}{(\alpha,\alpha)}$, so that $s_\alpha(p) = p - p_\alpha \alpha$. In order for $s_\alpha(p)$ to be reflected to $s_\beta(p)$ by a reflection s_γ, $p_\alpha \alpha - p_\beta \beta$ must be a multiple of γ. However, this is not easy to achieve since the angles between roots are quite restricted. We assume that $(\alpha, \alpha) = 2$ for the 'long' roots in R and $(\alpha, \alpha) = 1$ for the 'short' ones, which exist only for some types of R.

Note that

(a) if α, β are positive roots of the same length and inclined at $\pi/3$ (or $\pi/5$) to each other, then $x\alpha - y\beta$ is a multiple of a root only if either $x = 0$, $y = 0$ or $x = y$;

(b) if α, β are positive roots in R inclined at $\pi/2$, then $x\alpha - y\beta$ is a multiple of a root only if either $x = 0$, $y = 0$, or $\pm x = y$, provided in the last case that α and β are of the same length, while $\alpha \pm \beta$ or $(\alpha \pm \beta)/2$ is also a root, which must be of different length.

It is convenient to use the "largest root" $\tilde{\alpha}$, which is a positive combination of $\alpha_1, \ldots, \alpha_\ell$, so that $P\tilde{\alpha} > 0$ if $p \neq 0$; furthermore, $\tilde{\alpha}$ is always 'long' and is orthogonal to all but 1 or 2 of $\alpha_1, \ldots, \alpha_\ell$.

The case $\ell = z$ yields *regular polygons*. If $\ell \geq 3$ and R is of type A, D, E (or H), then there are no short roots. Applying (b) to $\tilde{\alpha}$ and an α_i orthogonal to $\tilde{\alpha}$, we conclude that $P\alpha_i = 0$. The remaining α_i are inclined at $\pi/3$ (or $\pi/5$) to $\tilde{\alpha}$, so that either $P\alpha_i = 0$ or $P\tilde{\alpha} = P\alpha_i$ by (a). However, one sees from the table of the $\tilde{\alpha}$ that such α_i occur with a coefficient > 1 in $\tilde{\alpha}$ except for type A_l, when $\tilde{\alpha} = \alpha_1 + \ldots + \alpha_\ell$ (and $i = 1$ or ℓ). Therefore we cannot have $P\tilde{\alpha} = P\alpha_i$ since all $P\alpha_j$ are ≥ 0. For type A_ℓ, only one of $P\alpha_1$ or $P\alpha_\ell$ can be $\neq 0$, and p is therefore a multiple of the fundamental weight ω_1 or ω_ℓ. Thus p is fixed by all s_i except s_1 or s_l and P is a *regular simplex*.

There remain types B_ℓ or F_4, when short roots exist (we don't need to consider type C_ℓ).

For type B_ℓ, (b) shows that $P\alpha_i = 0$ for $i \geq 3$. (However, $\tilde{\alpha} - \alpha_1$ is equal to 2 times the 'short' root $\alpha_2 + \ldots + \alpha_{\ell-1} + \alpha_\ell$.) On the other hand, comparing $\tilde{\alpha}$ and α_2 by (a) shows that $p_{\alpha_2} = 0$. Thus only $P\alpha_1$ can be non zero and p is a multiple of the fundamental root ω_1, being fixed by all s_i for $i \neq 1$. This yields the *regular cross-polytope* (an octahedron for $\ell = 3$).

For type F_4, (b) shows that $p\alpha_3 = P\alpha_4 = 0$. However, $\tilde{\alpha} - \alpha_2$ is equal to 2 times the 'short' root $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$. The equation $p\tilde{\alpha} = P\alpha_2$ says that $2p_{\alpha_1} + 3p_{\alpha_2} = P\alpha_2$, so that $p\alpha_1 = P\alpha_2 = 0$. Thus only $p\alpha_i$ may be non zero. Comparing $\tilde{\alpha}$ with α_1 by (a) shows that in fact $P\alpha_1 = 0$. Q.E.D.

P.S. I expect that this doesn't make too much sense, but an attempt to explain the terms was rapidly starting to produce a book!

George Maxwell