Random walk on the incipient infinite cluster for oriented percolation in high dimensions

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Abstract

We consider simple random walk on the incipient infinite cluster for the spread-out model of oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$. In dimensions $d > 6$, we obtain bounds on exit times, transition probabilities, and the range of the random walk, which establish that the spectral dimension of the incipient infinite cluster is $\frac{4}{3}$, and thereby prove a version of the Alexander–Orbach conjecture in this setting. The proof divides into two parts. One part establishes general estimates for simple random walk on an arbitrary infinite random graph, given suitable bounds on volume and effective resistance for the random graph. A second part then provides these bounds on volume and effective resistance for the incipient infinite cluster in dimensions $d > 6$, by extending results about critical oriented percolation obtained previously via the lace expansion.

1 Introduction and main results

1.1 Introduction

The problem of random walk on a percolation cluster — the 'ant in the labyrinth' [17] — has received much attention both in the physics and the mathematics literature. Recently, several papers have considered random walk on a supercritical percolation cluster [5, 9, 32, 33]. Roughly speaking, supercritical percolation clusters on $\mathbb{Z}^d$ are d-dimensional, and these papers prove, in various ways, that a random walk on a supercritical percolation cluster behaves in a diffusive fashion similar to a random walk on the entire lattice $\mathbb{Z}^d$.

Although a mathematically rigorous understanding of *critical* percolation clusters is restricted to examples in dimensions $d = 2$ and $d > 6$, or $d > 4$ in the case of *oriented* percolation, it is

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generally believed that critical percolation clusters in dimension d have dimension less than d, and that random walk on a large critical cluster behaves subdiffusively. Critical percolation clusters are believed to be finite in all dimensions, and are known to be finite in the oriented setting [11]. To avoid finite-size issues associated with random walk on a finite cluster, it is convenient to consider random walk on the incipient infinite cluster (IIC), which can be understood as a critical percolation cluster conditioned to be infinite. The IIC has been constructed so far only when $d = 2$ [28], when $d > 6$ (in the spread-out case) [23], and when $d > 4$ for oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ (again in the spread-out case) [20]. See [34] for a summary of the high-dimensional results. Also, it is not difficult to construct the IIC on a tree [7, 29].

Random walk on the IIC has been proved to be subdiffusive on $\mathbb{Z}^2$ [29] and on a tree [7, 29]. See also [13, 14] for related results in the continuum limit. In this paper, we prove several estimates for random walk on the IIC for spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ in dimensions $d > 6$. These estimates, which show subdiffusive behaviour, establish that the spectral dimension of the IIC is $\frac{4}{3}$, thereby proving the Alexander–Orbach [3] conjecture in this setting. For random walk on ordinary (unoriented) percolation for $d < 6$ the Alexander–Orbach conjecture is generally believed to be false [26, Section 7.4].

The upper critical dimension for oriented percolation is 4. Because of this, we initially expected that the spectral dimension of the IIC would be equal to $\frac{4}{3}$ for oriented percolation in all dimensions $d > 4$, but not for $d < 4$. However, our methods require that we take $d > 6$. The random walk is allowed to travel backwards in 'time' (as measured by the oriented percolation process, see Figure 1), and this allows the walk to move between vertices that are not connected in the oriented sense. It may be that this effect raises the upper critical dimension for the random walk in the oriented setting to $d = 6$. Or it may be that our conclusions for the random walk remain true for all dimensions $d > 4$, despite the fact that our methods force us to assume $d > 6$. This leads to the open question: Do our results actually apply in all dimensions $d > 4$, or does different behaviour apply for $4 < d \leq 6$?

1.2 The IIC

In this section, we define the oriented percolation model and recall the construction of the IIC for spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ in dimensions $d > 4$ [20]. For simplicity, we will consider only the most basic example of a spread-out model. (In the physics literature, oriented percolation is usually called *directed* percolation; see [27].)

The spread-out oriented percolation model is defined as follows. Consider the graph with vertices $\mathbb{Z}^d \times \mathbb{Z}_+$ and directed bonds $((x, n), (y, n+1))$, for $n \geq 0$ and $x, y \in \mathbb{Z}^d$ with $0 \leq \|x - y\|_\infty \leq L$. Here L is a fixed positive integer and $\|x\|_\infty = \max_{i=1,\ldots,d} |x_i|$ for $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. Let $p \in [0, 1]$. We associate to each directed bond $((x, n), (y, n + 1))$ an independent random variable taking the value 1 with probability p and 0 with probability $1 - p$. We say a bond is *occupied* when the corresponding random variable is 1, and *vacant* when the random variable is 0. Given a configuration of occupied bonds, we say that (x, n) is *connected to* (y, m), and write $(x, n) \longrightarrow (y, m)$, if there is an oriented path from (x, n) to (y, m) consisting of occupied bonds, or if $(x, n) = (y, m)$. Let $C(x, n)$ denote the forward cluster of (x, n), i.e., $C(x, n) = \{(y, m) : (x, n) \longrightarrow (y, m)\}$, and let $|C(x, n)|$ denote its cardinality.

The joint probability distribution of the bond variables will be denoted $\mathbb{P}$, with corresponding expectation denoted $\mathbb{E}$; these depend on p and are defined on a probability space $(\Omega, \tilde{\mathcal{F}}, \mathbb{P})$. Let

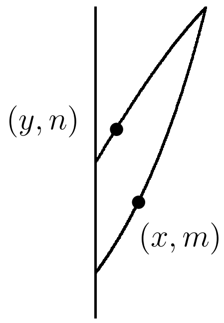

Figure 1: Although the vertex (x, m) is not connected to (y, n), or vice versa, in the sense of oriented percolation, it is nevertheless possible for a random walk to move from one of these vertices to the other.

$\theta(p) = \mathbb{P}(|C(0,0)| = \infty)$. For all dimensions $d \geq 1$ and for all $L \geq 1$, there is a critical value $p_c = p_c(d, L) \in (0, 1)$ such that $\theta(p) = 0$ for $p \leq p_c$ and $\theta(p) > 0$ for $p > p_c$. In particular, there is no infinite cluster when $p = p_c$ [11, 18]. For the remainder of this paper, we fix $p = p_c$, so that $\mathbb{P} = \mathbb{P}_{p_c}$.

To define the IIC, some terminology is required. A *cylinder event* is an event that is determined by the occupation status of a finite set of bonds. We denote the algebra of cylinder events by $\mathcal{F}_0$, and define $\mathcal{F}$ to be the σ-algebra generated by $\mathcal{F}_0$. The most natural definition of the IIC is as follows. Let $\{(x, m) \longrightarrow n\}$ denote the event that there exists (y, n) such that $(x, m) \longrightarrow (y, n)$. Let

$$\mathbb{Q}_n(E) = \mathbb{P}(E|(0,0) \longrightarrow n) \quad (E \in \mathcal{F}_0) \tag{1.1}$$

and define the IIC by

$$\mathbb{Q}_\infty(E) = \lim_{n\to\infty} \mathbb{Q}_n(E) \quad (E \in \mathcal{F}_0), \tag{1.2}$$

assuming the limit exists.

Let $d + 1 > 4 + 1$ and $p = p_c$. The combined results of [20, 21, 22] show that there is an $L_0 = L_0(d)$ such that for $L \geq L_0$ the limit (1.2) exists for every cylinder event $E \in \mathcal{F}_0$. Moreover, $\mathbb{Q}_\infty$ extends to a probability measure on the σ-algebra $\mathcal{F}$, and, writing $\mathcal{C} = C(0,0)$, $\mathcal{C}$ is $\mathbb{Q}_\infty$–a.s. an infinite cluster. We call $(\mathcal{C}, \mathbb{Q}_\infty)$ the IIC, and this provides the random environment for our random walk. We write $\mathbb{E}_\infty$ for expectation with respect to $\mathbb{Q}_\infty$. It will be convenient to remove a $\mathbb{Q}_\infty$–null set $\mathcal{N}$ from the configuration space Ω, so that for all $\omega \in \Omega_0 = \Omega - \mathcal{N}$ the cluster $\mathcal{C}(\omega)$ is infinite (and connected).

1.3 Random walk on graphs and the IIC

Let $\Gamma = (G, E)$ be an infinite graph, with vertex set G and edge set E. The edges $e \in E$ are *not* oriented. We assume that Γ is connected. We write $x \sim y$ if $\{x, y\} \in E$, and assume that (G, E) is locally finite, i.e., $\mu_y < \infty$ for each $y \in G$, where μ_y is the number of bonds that contain y. We extend μ to a measure on G.

Let $X = (X_n, n \in \mathbb{Z}_+, P^x, x \in G)$ be the discrete-time simple random walk on Γ. Then X has transition probabilities

$$P^x(X_1 = y) = \frac{1}{\mu_x}, \quad y \sim x. \tag{1.3}$$

We define the transition density (or discrete-time heat kernel) of X by

$$p_n(x,y) = P^x(X_n = y)\frac{1}{\mu_y}; \tag{1.4}$$

we have $p_n(x,y) = p_n(y,x)$.

The *spectral dimension of* G, denoted $d_s(G)$, is defined by

$$d_s(G) = -2 \lim_{n\to\infty} \frac{\log p_{2n}(x,x)}{\log n}, \tag{1.5}$$

if this limit exists. Here $x \in G$; it is easy to see that the limit is independent of the base point x. Note that $d_s(\mathbb{Z}^d) = d$.

To define the simple random walk on the IIC, for each $\omega \in \Omega$ we first define the random graph $(\mathcal{C}(\omega), E(\omega))$, where $\{(x,j),(y,k)\} \in E(\omega)$ if and only if $k = j+1$, $(x,j) \in \mathcal{C}$, and $(x,j) \longrightarrow (y, j+1)$. We use $\boldsymbol{x}$ to denote a space-time vertex $(x,j) \in \mathcal{C}$, and write $\mathbf{0} = (0,0)$. For $\boldsymbol{x} = (x,j)$ we write $|\boldsymbol{x}| = j$. For each $\omega \in \Omega_0$ we can then define the simple random walk $X = (X_n, n \in \mathbb{Z}_+, P^{\boldsymbol{x}}_\omega, \boldsymbol{x} \in \mathcal{C}(\omega))$. Let $p^\omega_n(\boldsymbol{x},\boldsymbol{y})$ denote the transition density of X, and

$$\tau_R = \min\{n \geq 0 : |X_n| \geq R\}. \tag{1.6}$$

Let

$$W_n = \{X_0, \dots, X_n\} \tag{1.7}$$

be the set of vertices hit by X up to time n, and let $|W_n|$ denote its cardinality. For 'regular' recurrent graphs, such as the IIC, one expects that $|W_n| \approx n^{d_s/2}$.

The original Alexander-Orbach conjecture [3] was that, in all dimensions,

$$|W_n| \approx n^{2/3}, \tag{1.8}$$

so that $d_s = \frac{4}{3}$ in all dimensions. As noted already above, the conjecture is now not believed to hold in low dimensions. Our main result is the following theorem, which shows that the Alexander–Orbach conjecture does hold for the IIC for spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ for $d > 6$.

Theorem 1.1. *For $d > 6$, there is an $L_1 = L_1(d) \geq L_0(d)$ such that for all $L \geq L_1$, there exists $\Omega_0 \subset \Omega$ with $\mathbb{Q}_\infty(\Omega_0) = 1$, and $\alpha_1, \alpha_2 \in (0,\infty)$ such that the following hold.*
(a) For each $\omega \in \Omega_0$ and $\boldsymbol{x} = (x,n) \in \mathcal{C}(\omega)$ there exists $N_{\boldsymbol{x}}(\omega) < \infty$ such that

$$(\log n)^{-\alpha_1} n^{-2/3} \leq p^\omega_{2n}(\boldsymbol{x},\boldsymbol{x}) \leq (\log n)^{\alpha_1} n^{-2/3}, \quad n \geq N_{\boldsymbol{x}}(\omega). \tag{1.9}$$

In particular, $d_s(\mathcal{C}) = \frac{4}{3}$, $\mathbb{Q}_\infty$–a.s., and the IIC *is recurrent.*
(b) For each $\omega \in \Omega_0$ and $\boldsymbol{x} = (x,n) \in \mathcal{C}(\omega)$ there exists $R_{\boldsymbol{x}}(\omega) < \infty$ such that

$$(\log R)^{-\alpha_2} R^3 \leq E^{\boldsymbol{x}}_\omega \tau_R \leq (\log R)^{\alpha_2} R^3, \quad R \geq R_{\boldsymbol{x}}(\omega). \tag{1.10}$$

In particular

$$\lim_{R\to\infty} \frac{\log E_\omega^{\boldsymbol{x}} \tau_R}{\log R} = 3.$$

(c) For each $\omega \in \Omega_0$ *and* $\boldsymbol{x} \in \mathcal{C}(\omega)$,

$$\lim_{n\to\infty} \frac{\log |W_n|}{\log n} = \frac{2}{3}, \quad P_\omega^{\boldsymbol{x}}\text{-}a.s.$$

Remark. The constants α_1, α_2 in (1.9)–(1.10) are independent of d and L.
Further results for the random walk on the IIC are given below, in a more general context, in Propositions 1.5–1.6, and Theorems 1.7–1.8.

One cannot expect (1.9) or (1.10) to hold with $\alpha_1 = 0$ or $\alpha_2 = 0$, since it is known that $\log\log$ fluctuations occur in the analogous limits for the IIC on regular trees [7].

The proof of Theorem 1.1 is performed in two principal steps. In the first step, general volume and resistance criteria are specified, which imply the conclusions of Theorem 1.1 for simple random walk in any random environment that satisfies the criteria. This is summarized in Section 1.4, and the details are carried out in Section 2. In the second step, these particular criteria are established for critical spread-out oriented percolation in dimensions $d > 6$, using an extension of results of [20, 21, 22, 25] that were obtained using the lace expansion. This step is carried out in Sections 4–5. Section 3 makes the necessary connections between the random walk and lace expansion results.

Throughout the paper, we use c, c' to denote strictly positive finite constants whose values are not significant and may change from line to line. We write c_i for positive constants whose values are fixed within theorems and lemmas.

1.4 Random walk in a random environment

In this section, we consider random walks on more general random graphs than the IIC. Let $\Gamma = (G, E)$ be a fixed infinite, connected graph, with unoriented edges. We define the vertex degrees μ_x as before, and extend μ to a measure on G. The natural metric on Γ, obtained by counting the number of steps in the shortest path between points, is written $d(x, y)$ for $x, y \in G$. We write

$$B(x, r) = \{y : d(x, y) < r\}, \qquad V(x, r) = \mu(B(x, r)), \quad r \in (0, \infty). \tag{1.11}$$

Following terminology used for manifolds, we call $V(x, r)$ the *volume* of the ball $B(x, r)$. We will assume G contains a marked vertex, which we denote 0, and we write

$$B(R) = B(0, R), \qquad V(R) = V(0, R). \tag{1.12}$$

The discrete-time simple random walk on Γ, $X = (X_n, n \in \mathbb{Z}_+, P^x, x \in G)$ is defined as before. For $A \subset G$, we write

$$T_A = \inf\{n \geq 0 : X_n \in A\}, \qquad \tau_A = T_{A^c}, \tag{1.13}$$

and let

$$\tau_R = \tau_{B(0,R)} = \min\{n \geq 0 : X_n \notin B(0, R)\}. \tag{1.14}$$

We define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{\substack{x,y \in G \\ x \sim y}} (f(x) - f(y))(g(x) - g(y)), \tag{1.15}$$

where $x \sim y$ means $\{x, y\} \in E$. If we regard Γ as an electrical network with a unit resistor on each edge in E, then $\mathcal{E}(f, f)$ is the energy dissipation when the vertices of G are at a potential f. Set $H^2 = \{f \in \mathbb{R}^G : \mathcal{E}(f, f) < \infty\}$. Let A, B be disjoint subsets of G. The effective resistance between A and B is defined by:

$$R_{\text{eff}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f \in H^2, f|_A = 1, f|_B = 0\}. \tag{1.16}$$

Let $R_{\text{eff}}(x, y) = R_{\text{eff}}(\{x\}, \{y\})$, and $R_{\text{eff}}(x, x) = 0$. For general facts on effective resistance and its connection with random walks see [2, 15, 31]. The following lemma gives some basic properties of R_{eff}.

Lemma 1.2. *Let $\Gamma = (G, E)$ be an infinite connected graph.*
(a) R_{eff} is a metric on G.
(b) If $A' \subset A$, $B' \subset B$ then $R_{\text{eff}}(A', B') \geq R_{\text{eff}}(A, B)$.
(c) $R_{\text{eff}}(x, y) \leq d(x, y)$.
(d) If $x, y \in G - A$ then $R_{\text{eff}}(x, A) \leq R_{\text{eff}}(x, y) + R_{\text{eff}}(y, A)$.
(e) If $\Gamma' = (G', E')$ is a subgraph of Γ, with effective resistance R'_{eff}, and if $A' = A \cap G'$ and $B' = B \cap G'$, then $R'_{\text{eff}}(A', B') \geq R_{\text{eff}}(A, B)$.

Proof. For (a) see [30, Section 2.3]. The monotonicity in (b) and (e) is immediate from the variational definition of R_{eff}. (c) is easy, and there is a proof in [6, Lemma 2.1]. (d) follows from (a) by considering the graph Γ' in which all vertices in A are connected by short circuits, which reduces A to a single vertex a. □

We now consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a family of random graphs $\Gamma(\omega) = (G(\omega), E(\omega), \omega \in \Omega)$. We assume that, for each $\omega \in \Omega$, the graph $\Gamma(\omega)$ is infinite, locally finite, and connected, and contains a marked vertex $0 \in G$. We denote balls in $\Gamma(\omega)$ by $B_\omega(x, r)$, their volume by $V_\omega(x, r)$, and write

$$B(R) = B_\omega(R) = B_\omega(0, R), \qquad V(R) = V_\omega(R) = V_\omega(0, R). \tag{1.17}$$

We write $X = (X_n, n \geq 0, P^x_\omega, x \in G(\omega))$ for the simple random walk on $\Gamma(\omega)$, and denote by $p^\omega_n(x, y)$ its transition density with respect to $\mu(\omega)$. To define X we introduce a second measure space $(\overline{\Omega}, \overline{\mathcal{F}})$, and define X on the product $\Omega \times \overline{\Omega}$. We write $\overline{\omega}$ to denote elements of $\overline{\Omega}$. Let $W_n = \{X_0, X_1, \dots, X_n\}$ and let

$$S_n = \mu(W_n) = \sum_{x \in W_n} \mu_x. \tag{1.18}$$

The key ingredients in our analysis of the simple random walk are volume and resistance bounds. The following defines a set $J(\lambda)$ of values of R for which we have 'good' volume and effective resistance estimates. The set $J(\lambda)$ depends on Γ, and thus is a random set under $\mathbb{P}$.

Definition 1.3. *Let $\Gamma = (G, E)$ be as above. For $\lambda > 1$, let $J(\lambda)$ be the set of those $R \in [1, \infty]$ such that the following all hold:*
(1) $V(R) \le \lambda R^2$,
(2) $V(R) \ge \lambda^{-1} R^2$,
(3) $R_{\text{eff}}(0, B(R)^c) \ge \lambda^{-1} R$.

Note that Lemma 1.2 gives that $R_{\text{eff}}(0, B(R)^c) \le R$, so there is no need for an upper bound complementary to Definition 1.3(3). We now make the following assumption concerning the graphs $(\Gamma(\omega))$. This involves upper and lower bounds on the volume, as well as an estimate which says that R is likely to be in $J(\lambda)$ for large enough λ.

Assumption 1.4. *(1) There exists $p(\lambda)$ which goes to 0 as $\lambda \to \infty$ such that for each $R \ge 1$,*

$$\mathbb{P}(R \in J(\lambda)) \ge 1 - p(\lambda). \tag{1.19}$$

(2) $\mathbb{E}[V(R)] \le c_1 R^2$.
(3) $\mathbb{E}[1/V(R)] \le c_2 R^{-2}$.
(4) There exist $q_0, c_3 > 0$ such that

$$p(\lambda) \le \frac{c_3}{\lambda^{q_0}}. \tag{1.20}$$

Remark. Assumption 1.4(2,3), together with Markov's inequality, provides upper bounds of the form $c\lambda^{-1}$ for the probability of the complements of the events in Definition 1.3(1,2). This creates an apparent redundancy in our formulation, but we use this form of the Assumption because some of our conclusions for the random walk rely only on Assumption 1.4(1,4) and do not require the stronger volume bounds given by Assumption 1.4(2,3).

Note that Assumption 1.4 only involves statements about the volume and resistance from one point 0 in the graph. In general, this kind of information would not be enough to give much control of the random walk. However, the graphs considered here have quite strong recurrence properties, and are therefore simpler to handle than general graphs. We use techniques developed in [6, 7, 35, 36, 37].

We will prove in Theorem 3.6 that Assumption 1.4 holds for the IIC. As the reader of Sections 4–5 will see, obtaining volume and (especially) resistance bounds on the IIC from one base point is already difficult; it is fortunate that we did not need more.

We have the following four consequences of Assumption 1.4 for random graphs. They give control, in different ways, of the quantities $R^{-3}E^0_\omega \tau_R$, $n^{2/3}p_{2n}(0,0)$, $n^{-1/3}d(0, X_n)$ and $|W_n|$, which measure the rate of dispersion of the random walk X from the base point 0. In fact, some of the results apply also to the random walk started from an arbitrary point $x \in G(\omega)$. Some statements in the first proposition involve the annealed law

$$P^* = \mathbb{P} \times P^0_\omega. \tag{1.21}$$

Proposition 1.5. *Suppose Assumption 1.4(1) holds. Let $n \ge 1$, $R \ge 1$. Then*

$$\mathbb{P}(\theta^{-1} \le R^{-3} E^0_\omega \tau_R \le \theta) \to 1 \quad \text{as } \theta \to \infty, \tag{1.22}$$

$$\mathbb{P}(\theta^{-1} \le n^{2/3} p^\omega_{2n}(0,0) \le \theta) \to 1 \quad \text{as } \theta \to \infty, \tag{1.23}$$

$$P^*(d(0, X_n) n^{-1/3} < \theta) \to 1 \quad \text{as } \theta \to \infty. \tag{1.24}$$

$$P^*(\theta^{-1} < (1 + d(0, X_n)) n^{-1/3}) \to 1 \quad \text{as } \theta \to \infty. \tag{1.25}$$

In each case the convergence is uniform.

Since $P_\omega^0(X_{2n}=0) \approx n^{-2/3}$, we cannot replace $1+d(0,X_n)$ by $d(0,X_n)$ in (1.25).

Proposition 1.6. *Suppose Assumption 1.4(1,2,3) hold. Then*

$$c_1 R^3 \le \mathbb{E}(E_\omega^0 \tau_R) \le c_2 R^3 \text{ for all } R \ge 1, \tag{1.26}$$

$$c_3 n^{-2/3} \le \mathbb{E}(p_{2n}^\omega(0,0)) \le c_4 n^{-2/3} \text{ for all } n \ge 1, \tag{1.27}$$

$$c_5 n^{1/3} \le \mathbb{E}(E_\omega^0 d(0,X_n)) \text{ for all } n \ge 1. \tag{1.28}$$

We do not have an upper bound in (1.28); this is discussed further in Example 2.6 below.

The additional assumption that $p(\lambda)$ decays polynomially enable us to obtain limit theorems. Both of the following theorems refer to the random walk started at an arbitrary point $x \in G(\omega)$.

Theorem 1.7. *Suppose Assumption 1.4(1) and (4) hold. Then there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 < \infty$, and a subset Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that the following statements hold.*
(a) For each $\omega \in \Omega_0$ and $x \in G(\omega)$ there exists $N_x(\omega) < \infty$ such that

$$(\log n)^{-\alpha_1} n^{-2/3} \le p_{2n}^\omega(x,x) \le (\log n)^{\alpha_1} n^{-2/3}, \quad n \ge N_x(\omega). \tag{1.29}$$

In particular, $d_s(G) = \frac{4}{3}$, $\mathbb{P}$–a.s., and the random walk is recurrent.
(b) For each $\omega \in \Omega_0$ and $x \in G(\omega)$ there exists $R_x(\omega) < \infty$ such that

$$(\log R)^{-\alpha_2} R^3 \le E_\omega^x \tau_R \le (\log R)^{\alpha_2} R^3, \quad R \ge R_x(\omega). \tag{1.30}$$

Hence

$$\lim_{R\to\infty} \frac{\log E_\omega^x \tau_R}{\log R} = 3.$$

(c) Let $Y_n = \max_{0\le k\le n} d(0,X_k)$. For each $\omega \in \Omega_0$ and $x \in G(\omega)$ there exist $N_x(\omega,\varpi), R_x(\omega,\varpi)$ such that $P_\omega^x(N_x < \infty) = P_\omega^x(R_x < \infty) = 1$, and such that

$$(\log n)^{-\alpha_3} n^{-1/3} \le Y_n(\omega,\varpi) \le (\log n)^{\alpha_3} n^{-1/3}, \quad n \ge N_x(\omega,\varpi), \tag{1.31}$$

$$(\log R)^{-\alpha_4} R^3 \le \tau_R(\omega,\varpi) \le (\log R)^{\alpha_4} R^3, \qquad R \ge R_x(\omega,\varpi). \tag{1.32}$$

Theorem 1.8. *Suppose Assumption 1.4(1) and (4) hold. Then there exists a subset Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that for each $\omega \in \Omega_0$ and $x \in G(\omega)$,*

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = \frac{2}{3}, \quad P_\omega^x\text{–a.s.}. \tag{1.33}$$

Remark. If all the vertices in G have degree bounded by a constant c_0, as they do for the IIC with $c_0 = c_0(d,L)$, then $|W_n| \le S_n \le c_0|W_n|$ and hence Theorem 1.8 implies also that

$$\lim_{n\to\infty} \frac{\log |W_n|}{\log n} = \frac{2}{3}, \quad P_\omega^x\text{–a.s.} \tag{1.34}$$

Example 1.9. (i) Assumption 1.4 holds for random walk on the IIC for the binomial tree; see [7, Corollary 2.12]. Therefore the conclusions of Propositions 1.5–1.6 and Theorems 1.7–1.8 hold for random walk on this IIC. The results of [7] go beyond Theorem 1.7(a) and (b) in this context, but Theorem 1.7(c) and Theorem 1.8 here are new.

(ii) It is shown in [4] that the invasion percolation cluster on a regular tree is stochastically dominated by the IIC for the binomial tree. Consequently, upper bounds on the volume and lower bounds on the effective resistance of the invasion percolation cluster follow from the corresponding bounds for the IIC (recall Lemma 1.2(e)). In addition, the lower bound on the volume in Assumption 1.4(3) is proved for the invasion percolation cluster in [4]. Assumption 1.4(1,2,4) for the invasion percolation cluster therefore follows from its counterpart for the IIC for the binomial tree. So Assumption 1.4 holds for the invasion percolation cluster on a regular tree, and hence simple random walk on the invasion percolation cluster also obeys the conclusions of Propositions 1.5–1.6 and Theorems 1.7–1.8. See [4] for further details about invasion percolation.

(iii) We prove in Sections 3–5 that when $L \geq L_1$ the IIC obeys Assumption 1.4(2,3) if $d > 4$, and obeys Assumption 1.4(1,4) if $d > 6$. The restriction to $d > 6$ is required only for our estimate of the effective resistance.

(iv) Consider the incipient infinite branching random walk (IIBRW), obtained as the limit as $n \to \infty$ of critical branching random walk (say with binomial offspring distribution) conditioned to survive to at least n generations [19, Section 2]. We interpret the IIBRW as a random infinite subgraph of $\mathbb{Z}^d \times \mathbb{Z}_+$. There is the option of considering either one edge per particle jump, leading to the occurrence of multiple edges between vertices, or identifying any such multiple edges as a single edge; we believe both options will behave similarly in dimensions $d > 4$. Consider simple random walk on the IIBRW. Our volume estimates for the IIC for oriented percolation for $d > 4$ will adapt to give similar estimates for the IIBRW for $d > 4$. The effective resistance $R_{\text{eff}}(0, B(R)^c)$ for the IIBRW is lower than it is for the IIC on a tree, due to cycles in the IIBRW. It is an interesting open problem to obtain a lower bound on $R_{\text{eff}}(0, B(R)^c)$ for the IIBRW, to establish Assumption 1.4 and hence its consequences Propositions 1.5–1.6 and Theorems 1.7–1.8 for random walk on the IIBRW. Our main interest is the question: Does random walk on the IIBRW have the same behaviour in all dimensions $d > 4$, or is there different behaviour for $4 < d \leq 6$ and $d > 6$? An answer would shed light on the question raised at the end of Section 1.1. It would also be of interest to consider this question in the continuum limit: Brownian motion on the canonical measure of super-Brownian motion conditioned to survive for all time (see [19]).

The proof of Propositions 1.5–1.6 and Theorems 1.7–1.8 is based on methods from [6] and [7], and is given in Section 2. To prove Theorem 1.1, it suffices to show that $\mathbb{Q}_\infty$ obeys Assumption 1.4 when $d > 6$ and L is sufficiently large. We do this in Sections 3–5.

2 Random walk in a random environment

In Section 2.1, we prove several preliminary results for random walk on a fixed but general graph. Then in Section 2.2 we apply these results to prove Propositions 1.5–1.6 and Theorems 1.7–1.8. We adopt the convention that if we cite elsewhere the constant c_1 in Lemma 2.2 (for example), we denote it as $c_{2.2.1}$.

2.1 Results for general graphs

In this section, we fix an infinite locally-finite connected graph $\Gamma = (G, E)$, and use bounds on the quantities $V(R)$ and $R_{\text{eff}}(0, B(R)^c)$ to control $E^0 \tau_R$, $p_n(0,0)$ and $E^0 d(0, X_n)$.

The results in [6] (see [6, Theorem 1.3, Lemma 2.2]) cover the case where, for all $x \in G$ and $R \geq 1$,

$$c_1 R^2 \leq V(x,R) \leq c_2 R^2, \quad c_3 R \leq R_{\text{eff}}(x, B(x,R)^c) \leq c_4 R. \tag{2.1}$$

Here, we treat the case where we only have information available on the volume and effective resistance from one fixed point 0 in the graph. Not surprisingly this means that we can only handle the quantities $\tau_{B(x,r)}$, $p_n(x,x)$ and $d(x, X_n)$ for x close to 0.

We extend some results from [6, 7]. Reference [7] considers the continuous-time random walk, rather than the discrete-time one. While some results (such as bounds on mean hitting times) are exactly the same for the two processes, those arguments which use a differential inequality are more complicated in the discrete-time setup. However, [6] does handle the discrete-time process. To deal with issues related to the possible bipartite structure of the graph it proves helpful to consider $p_n(x,y) + p_{n+1}(x,y)$.

Proposition 2.1. *Let $x_0 \in G$ and $f_n(y) = p_n(x_0, y) + p_{n+1}(x_0, y)$.*
(a) Let $r \geq 1$ and $n = 2r^3$. Then

$$f_n(x_0) \leq c_1 n^{-2/3} (1 \vee (r^2 / V(x_0, r))).$$

(b) We have

$$|f_n(y) - f_n(x_0)|^2 \leq \frac{c_2}{n} d(x_0, y) p_{2\lfloor n/2 \rfloor}(x_0, x_0).$$

Proof. (a) The third equation in [6, Proposition 3.3] gives, for $m \geq 1$ and $r > 0$,

$$f_{2m}(x_0)^2 \leq \frac{c}{V(x_0, r)^2} + \frac{c r f_{2m}(x_0)}{m}.$$

Using $a + b \leq 2(a \vee b)$, we see that $f_{2m}(x_0) \leq (c'/V(x_0, r)) \vee (c' r/m)$, and the result follows by setting $m = r^3$.
(b) Using [6, Lemma 1.1] and Lemma 1.2,

$$|f_n(y) - f_n(x)|^2 \leq R_{\text{eff}}(x,y) \mathcal{E}(f_n, f_n) \leq d(x,y) \mathcal{E}(f_n, f_n).$$

We then use [6, Lemma 3.10] to bound $\mathcal{E}(f_n, f_n)$. □

Proposition 2.2. *Let $R \geq 1$, $m \geq 1$, $\varepsilon \leq 1/(2m)$. Write $B = B(0, R)$, $B' = B(0, \frac{1}{2}\varepsilon R)$, $V = V(0, R)$, $V' = V(0, \frac{1}{2}\varepsilon R)$.*
(a) For $x \in B$,

$$E^x \tau_R \leq 2RV. \tag{2.2}$$

(b) Suppose that

$$R_{\text{eff}}(x, B^c) \geq R/m \quad \text{for } x \in B(0, \varepsilon R). \tag{2.3}$$

Then for $x \in B'$,

$$E^x \tau_R \geq \frac{RV'}{4m}, \tag{2.4}$$

$$P^x(\tau_R > n) \geq \frac{V'}{8mV} - \frac{n}{2RV} \quad \textit{for } n \geq 0, \tag{2.5}$$

$$p_{2n}(x,x) \geq \frac{c_1 (V')^2}{m^2 V^3} \qquad \textit{for } n \leq \frac{RV'}{8m}. \tag{2.6}$$

Proof. This follows from [7, Proposition 4.4]. □

Recall the set $J(\lambda)$ defined in Definition 1.3. We will need to know that bounds in the following are polynomial in λ. To indicate this, we write $c_i(\lambda)$ to denote positive constants of the form $c_i(\lambda) = c_i \lambda^{\pm q_i}$. The sign of q_i is such that statements become weaker as λ increases. The following proposition controls the mean escape times and transition probabilities.

Proposition 2.3. *Let* $\lambda > 1$.
(1) Suppose that $R \in J(\lambda)$. *Then there exist* $c_1(\lambda), c_2(\lambda)$ *such that*

$$E^z \tau_R \leq 2\lambda R^3 \qquad \textit{for } z \in B(R), \tag{2.7}$$

$$p_n(0,0) + p_{n+1}(0,0) \leq c_1(\lambda) n^{-2/3} \quad \textit{if } n = 2R^3, \tag{2.8}$$

$$p_n(0,y) + p_{n+1}(0,y) \leq c_2(\lambda) n^{-2/3} \quad \textit{for } y \in B(R) \textit{ if } n = 2R^3. \tag{2.9}$$

(2) Suppose that $R \in J(\lambda)$ *and* $R/(4\lambda) \in J(\lambda)$. *Then there exist* $c_3(\lambda), \cdots, c_7(\lambda)$ *such that*

$$c_3(\lambda) R^3 \leq E^x \tau_R \quad \textit{for } x \in B(R/4\lambda), \tag{2.10}$$

$$P^0(\tau_R > c_4(\lambda) R^3) \geq c_5(\lambda), \tag{2.11}$$

$$p_{2n}(0,0) \geq c_6(\lambda) n^{-2/3} \quad \textit{for } \tfrac{1}{2} c_7(\lambda) R^3 \leq n \leq c_7(\lambda) R^3. \tag{2.12}$$

Proof. (1) (2.7) is immediate by Proposition 2.2(a).
As $R \in J(\lambda)$, $R^2/V(R) \leq \lambda$, so (2.8) follows from Proposition 2.1(a).
Using Proposition 2.1(b), and writing $f_n(y) = p_n(0,y) + p_{n+1}(0,y)$, $n' = 2\lfloor n/2 \rfloor$,

$$f_n(y) \leq f_n(0) + |f_n(y) - f_n(0)| \leq f_n(0) + (c d(0,y) n^{-1} p_{n'}(0,0))^{1/2}. \tag{2.13}$$

So, by (2.8), if $d(0,y) \leq n^{1/3}$ then we have (2.9), namely

$$f_n(y) \leq c'(\lambda) n^{-2/3}. \tag{2.14}$$

(2) Set $m = 2\lambda$, $\varepsilon = 1/m = 1/(2\lambda)$. Since $R \in J(\lambda)$, it follows from Lemma 1.2(c,d) that for $x \in B = B(0,R)$,

$$R/\lambda \leq R_{\text{eff}}(0, B^c) \leq R_{\text{eff}}(0,x) + R_{\text{eff}}(x, B^c) \leq d(0,x) + R_{\text{eff}}(x, B^c).$$

Hence $R_{\text{eff}}(x, B^c) \geq R/m$ if $d(0,x) \leq \varepsilon R$, and so the assumption of Proposition 2.2(b) holds. Since $R \in J(\lambda)$, $V(R) \leq \lambda R^2$. Also $\frac{1}{2}\varepsilon R = R/(4\lambda) \in J(\lambda)$, so $V' \geq \lambda^{-1}(R/4\lambda)^2 = R^2/(16\lambda^3)$; the bounds now follow from Proposition 2.2(b). For later use, we note that it suffices to take $c_4(\lambda) = (256\lambda^4)^{-1}$ and $c_5(\lambda) = (512\lambda^5)^{-1}$. (The constant $c_7(\lambda)$ is of the form $c_7 \lambda^{-q_7}$.) □

Next we apply similar arguments to control $d(0, X_n)$, beginning with a preliminary lemma. Recall that T_A was defined in (1.13) to be the hitting time of $A \subset G$.

Lemma 2.4. *Let $\lambda \geq 1$ and $0 < \varepsilon \leq 1/2\lambda$. If $R \in J(\lambda)$, and $y \in B(\varepsilon R)$ then*

$$P^y(\tau_R < T_0) \leq 2\varepsilon\lambda, \tag{2.15}$$
$$P^0(\tau_R < T_y) \leq \varepsilon\lambda. \tag{2.16}$$

Proof. If A and B are disjoint subsets of G and $x \notin A \cup B$, then (see [10, (4)])

$$P^x(T_A < T_B) \leq \frac{R_{\text{eff}}(x, B)}{R_{\text{eff}}(x, A)}.$$

Let $d(0, y) \leq \varepsilon R$. Then $R_{\text{eff}}(y, 0) \leq d(y, 0) \leq \varepsilon R$, while

$$R_{\text{eff}}(y, B(R)^c) \geq R_{\text{eff}}(0, B(R)^c) - R_{\text{eff}}(0, y) \geq R/\lambda - \varepsilon R \geq R/2\lambda.$$

So,

$$P^y(\tau_R < T_0) \leq \frac{R_{\text{eff}}(y, 0)}{R_{\text{eff}}(y, B(R)^c)} \leq 2\varepsilon\lambda.$$

Similarly,

$$P^0(\tau_R < T_y) \leq \frac{R_{\text{eff}}(0, y)}{R_{\text{eff}}(0, B(R)^c)} \leq \varepsilon\lambda.$$

□

Proposition 2.5. *For each $\lambda > 1$, there exist $c_1(\lambda), \cdots, c_7(\lambda)$ such that the following hold.*
(a) Let $\lambda \geq 1$, $\varepsilon < 1/(4\lambda)$ and $R, \varepsilon R, \varepsilon R/(4\lambda) \in J(\lambda)$. Then

$$P^y\Big(\tau_R \leq c_1(\lambda)(\varepsilon R)^3\Big) \leq c_2(\lambda)\varepsilon, \qquad \text{for } y \in B(\varepsilon R). \tag{2.17}$$

(b) Let $n \geq 1$, $M \geq 1$, and set $R = M n^{1/3}$. If $R, c_3(\lambda)R/M, c_3(\lambda)R/(4\lambda M) \in J(\lambda)$, then

$$P^0\Big(n^{-1/3} d(0, X_n) > M\Big) \leq \frac{c_4(\lambda)}{M}. \tag{2.18}$$

(c) Let $R = (n/2)^{1/3}$ and $\theta \in (0, 1]$. If $R \in J(\lambda)$ then

$$P^0\Big(X_n \in B(\theta R)\Big) \leq c_5(\lambda) R^{-2} V(\theta R). \tag{2.19}$$

(d) Let $R = (n/2)^{1/3}$. If $R, c_6(\lambda)R \in J(\lambda)$ then

$$P^0(\tau_{c_6(\lambda)R} \leq n) \geq P^0\Big(X_n \notin B(0, c_6(\lambda)R)\Big) \geq \tfrac{1}{2}. \tag{2.20}$$

Hence

$$E^0 d(0, X_n) \geq c_7(\lambda) n^{1/3}. \tag{2.21}$$

Proof. (a) Let $c_1(\lambda) = c_{2.3.4}(\lambda) < 1$. Then the desired inequality is trivial when $\varepsilon R \leq 1$, so assume that $\varepsilon R > 1$. Let $q(y) = P^y(\tau_R < T_0)$, so that, by substituting 2ε into ε in Lemma 2.4, if

$d(0,y) \le 2\varepsilon R$ then $q(y) \le 4\varepsilon\lambda$. Write $t_0 = c_1(\lambda)(\varepsilon R)^3$ and $a = P^0(\tau_R \le t_0)$. Now if $y \in B(2\varepsilon R)$ then

$$\begin{aligned} P^y(\tau_R \le t_0) &= P^y(\tau_R \le t_0, \tau_R < T_0) + P^y(\tau_R \le t_0, \tau_R > T_0) \\ &\le P^y(\tau_R \le T_0) + P^y(T_0 < \tau_R, \tau_R - T_0 \le t_0) \\ &\le q(y) + (1-q(y))a \le 4\varepsilon\lambda + a, \end{aligned} \tag{2.22}$$

using the strong Markov property for the second inequality. So, by a second application of the strong Markov property, and (2.11),

$$\begin{aligned} a = P^0(\tau_R \le t_0) &\le E^0[1_{\{\tau_{\varepsilon R} \le t_0\}} P^{X_{\tau_{\varepsilon R}}}(\tau_R \le t_0)] \\ &\le (1 - c_{2.3.5}(\lambda))(4\varepsilon\lambda + a), \end{aligned} \tag{2.23}$$

where we used the fact that $X_{\tau_{\varepsilon R}} \in B(\varepsilon R + 1) \subset B(2\varepsilon R)$ in the last inequality. Rewriting this gives $a \le 4\varepsilon\lambda(1 - c_{2.3.5}(\lambda))/c_{2.3.5}(\lambda)$. Substituting in (2.22) gives (2.17) with $c_2(\lambda) = 4\lambda/c_{2.3.5}(\lambda)$.
(b) Let $c_3(\lambda) = c_{2.3.4}(\lambda)^{-1/3}$, $c_4(\lambda) = c_2(\lambda)c_3(\lambda) = 4c_{2.3.4}(\lambda)^{-1/3}\lambda/c_{2.3.5}(\lambda)$, $M' = M/c_3(\lambda)$, and $\varepsilon = (M')^{-1}$. The desired inequality is trivial when $c_4(\lambda)/M \ge 1$, so assume that $c_4(\lambda)/M < 1$. Then,

$$\varepsilon = \frac{1}{M'} = \frac{c_3(\lambda)}{M} < \frac{c_3(\lambda)}{c_4(\lambda)} = \frac{1}{c_2(\lambda)} \le \frac{1}{4\lambda},$$

so the assumption in (a) is satisfied. Since

$$\begin{aligned} P^0(d(0,X_n)n^{-1/3} > M) &= P^0(d(0,X_n) > R) \\ &\le P^0(\tau_R \le n) = P^0(\tau_R \le c_{2.3.4}(\lambda)(\varepsilon R)^3), \end{aligned} \tag{2.24}$$

using (a) gives the desired estimate. Note that, inserting the values of $c_{2.3.4}(\lambda)$ and $c_{2.3.5}(\lambda)$, we have $c_4(\lambda) \le c\lambda^{22/3}$.
(c) By (2.9), writing $B' = B(0, \theta R) \subset B(0,R)$ and $f_n(0,y) = p_n(0,y) + p_{n+1}(0,y)$,

$$P^0(X_n \in B') = \sum_{y \in B'} p_n(0,y)\mu_y \le \sum_{y \in B'} f_n(0,y)\mu_y \le V(\theta R)c_5(\lambda)R^{-2}. \tag{2.25}$$

(d) Let $\theta = c_6(\lambda) \in (0,1]$ satisfy $\lambda c_5(\lambda)\theta^2 = \frac{1}{2}$. Then using (c) and $\theta R \in J(\lambda)$,

$$P^0(X_n \in B(\theta R)) \le V(\theta R)c_5(\lambda)R^{-2} \le \lambda(\theta R)^2 c_5(\lambda)R^{-2} = \tfrac{1}{2}. \tag{2.26}$$

This proves the first assertion. Also,

$$E^0 d(0,X_n) \ge \theta R P^0(X_n \notin B') \ge \tfrac{1}{2}\theta R \ge c_7(\lambda)n^{1/3}. \tag{2.27}$$

□

We do not have an upper bound on $E^0 d(0,X_n)$ to complement the lower bound of Proposition 2.5(d), assuming only volume and resistance bounds from a single base point, i.e., bounds on $V(0,R)$ and $R_{\text{eff}}(0, B(R)^c)$. Suppose that $J(\lambda) = [1,\infty)$ for some $\lambda \ge 1$, and let $Z_n = n^{-1/3}d(0,X_n)$. Then we are able to bound $E^0 Z_n^p$ for $p < 1$, since Proposition 2.5(b) gives

$$\begin{aligned} E^0[Z_n^p] &\le \sum_{m=1}^{\infty} (2^{m+1})^p P^0\Big(2^m \le n^{-1/3}d(0,X_n) < 2^{m+1}\Big) \\ &\le \sum_{m=1}^{\infty} (2^{m+1})^p P^0\Big(n^{-1/3}d(0,X_n) \ge 2^m\Big) \le c_1 \sum_{m=1}^{\infty} 2^{m(p-1)} = c_2 < \infty. \end{aligned}$$

On the other hand the following example indicates that, under our hypotheses, we cannot expect to have a uniform bound on $E^0(Z_n^p)$ when $p > 1$.

Example 2.6. We just sketch this argument.

Let Γ be the subgraph of $\mathbb{Z}^2$ with vertex set $G = G_0 \cup G_1$, where $G_0 = \{(n, 0), n \in \mathbb{Z}\}$, and $G_1 = \{(n, m) : 0 \le m \le n\}$. Let the edges be $\{(n, 0), (n+1, 0)\}$, for $n \in \mathbb{Z}$, and $\{(n, m), (n, m+1)\}$ if $n \ge 1$ and $0 \le m \le n-1$. Thus Γ consists of $\mathbb{Z}_-$ and a comb-type graph of vertical branches with base $\mathbb{Z}_+$. Write 0 for $(0, 0)$. It is easily checked that $V(0, R) \asymp R^2$, and $R_{\text{eff}}(0, B(0, R)^c) \ge R/4$. Thus there exists $\lambda_0 < \infty$ such that $J(\lambda_0) = [1, \infty)$. Let

$$H(a, b) = \{(n, m) \in G : a \le n \le b\}.$$

Let X_n be the simple random walk on Γ. If we time-change out the excursions of X away from $\mathbb{Z}$ then we obtain a simple random walk Y_n on $\mathbb{Z}$. Now let $R \ge 1$, and $r = R^{2/3} \in \mathbb{Z}$. Let $A = H(-r, r)$. By Proposition 2.3 we have $E^0\tau_A \approx r^3 \approx R^2$. Since X only moves horizontally when it is on the x-axis, $P^0(X_{\tau_A} = (-r, 0)) = 1/2$. If $X_{\tau_A} = (-r, 0)$ then the probability that X reaches $H(-\infty, -R)$ before returning to 0 is $r/R \approx R^{-1/3}$; also, if X does this then the time taken to do so will be of order R^2.

These arguments lead us to expect that if $n = R^2$ then

$$P^0\Big(X_n \in H(-\infty, -R/2)\Big) \ge cR^{-1/3}. \tag{2.28}$$

Given (2.28), it follows from Markov's inequality that

$$E^0 Z_n^p \ge n^{-p/3}(R/2)^p P^0\Big(X_n \in H(-\infty, -R/2)\Big) \ge cn^{(p-1)/6},$$

and the lower bound diverges if $p > 1$. This concludes Example 2.6.

2.2 Results for random graphs

We now consider a family of random graphs, as described in Section 1.4, and prove Propositions 1.5–1.6 and Theorems 1.7–1.8.

We begin by obtaining tightness of the quantities $R^{-3}E^0\tau_R$, $n^{2/3}p_{2n}(0, 0)$, and $n^{-1/3}d(0, X_n)$.

Proof of Proposition 1.5. We begin with (1.22). Let $\varepsilon > 0$. Choose $\lambda \ge 1$ such that $2p(\lambda) < \varepsilon$ – here $p(\lambda)$ is the function given by Assumption 1.4. Let $R \ge 1$ and set $F_1 = \{R, R/(4\lambda) \in J(\lambda)\}$.

Suppose first that $R/4\lambda \ge 1$. Then, by Assumption 1.4(1), $\mathbb{P}(F_1) \ge 1 - 2p(\lambda)$. For $\omega \in F_1$, by Proposition 2.3, there exists $c_1 < \infty$, $q_1 \ge 0$ such that

$$(c_1\lambda^{q_1})^{-1} \le R^{-3}E^x_\omega\tau_R \le c_1\lambda^{q_1} \text{ for } x \in B(R/4\lambda). \tag{2.29}$$

So, if $\theta_0 = c_1\lambda^{q_1}$ then for $\theta \ge \theta_0$,

$$\mathbb{P}\Big(\theta^{-1} \le R^{-3}E^0_\omega\tau_R \le \theta\Big) \ge \mathbb{P}(F_1) \ge 1 - 2p(\lambda) \ge 1 - \varepsilon. \tag{2.30}$$

Now consider the case when $R \le 4\lambda$. For each graph $\Gamma(\omega)$ let

$$Y(\omega) = \sup_{1 \le r \le 4\lambda} r^{-3}E^0_\omega\tau_r.$$

Then $Y(\omega) < \infty$ for each ω, so there exists θ_1 such that

$$\mathbb{P}(R^{-3}E^0_\omega \tau_R > \theta_1) \le \mathbb{P}(Y > \theta_1) \le \varepsilon.$$

If we take $\theta_1 > (4\lambda)^3$ then since $E^0_\omega \tau_R \ge 1$, we have $R^{-3}E^0_\omega \tau_R \ge \theta_1^{-1}$. So, for $\theta \ge \theta_1$, we also have $\mathbb{P}\Big(\theta^{-1} \le R^{-3}E^0_\omega \tau_R \le \theta\Big) \ge 1 - \varepsilon$, which completes the proof of (1.22).

We now turn to (1.23). Let $n \ge 1$, $\lambda \ge 1$, and let R_0, R_1 be defined by $n = c_{2.3.7}(\lambda) R_1^3 = 2R_0^3$. Let $F_2 = \{R_0, R_1, R_1/(4\lambda) \in J(\lambda)\}$. Suppose first that R_0 and $R_1/4\lambda$ are both greater than 1; then $\mathbb{P}(F_2) \ge 1 - 3p(\lambda)$. If $\omega \in F_2$ then by Proposition 2.3

$$(c_2 \lambda^{q_2})^{-1} \le n^{2/3} p^\omega_{2n}(0,0) \le c_2 \lambda^{q_2}.$$

So,

$$\mathbb{P}\Big((c_2 \lambda^{q_2})^{-1} \le n^{2/3} p^\omega_{2n}(0,0) \le c_2 \lambda^{q_2}\Big) \ge \mathbb{P}(F_2) \ge 1 - 3p(\lambda). \qquad (2.31)$$

The case when n is small is dealt with in the same way as in the proof of (1.22).

We now turn to (1.24). Let $n \ge 1$ and $\lambda \ge 1$. Let $M = \lambda^8$ and set

$$R_0 = Mn^{1/3}, \quad R_1 = c_{2.5.3}(\lambda) n^{1/3}, \quad R_2 = c_{2.5.3}(\lambda) n^{1/3}/(4\lambda), \qquad (2.32)$$

$F_3 = \{R_0, R_1 R_2 \in J(\lambda)\}$. Suppose first that n is large enough so that $R_i \ge 1$ for $0 \le i \le 2$. By Proposition 2.5(b), if $\omega \in F_3$ then

$$P^0_\omega\Big(n^{-1/3} d(0, X_n) > \lambda^8\Big) \le \frac{c_{2.5.4(\lambda)}}{\lambda^8} \le \frac{c\lambda^{22/3}}{\lambda^8} = \frac{c}{\lambda^{2/3}}.$$

Taking $\theta = \lambda^8$, we have

$$P^*\Big(n^{-1/3} d(0, X_n) > \theta\Big) \le \mathbb{P}(F_3^c) + \mathbb{E}\Big(P^0_\omega(n^{-1/3} d(0, X_n) > \lambda^8); F_3\Big) \le 3p(\theta^{1/8}) + c_3 \theta^{-1/12}. \quad (2.33)$$

Now let $\varepsilon > 0$. Choose θ_0 so that the right side of (2.33) is less than ε. Let $\lambda^8 = \theta_0$. Then there exists $n_1 = n_1(\varepsilon)$ such that if $n \ge n_1$ then R_0, R_1, R_2 (given by (2.32)) are all greater than 1. If $n \ge n_1$ then (2.33) implies that $P^*(n^{-1/3} d(0, X_n) > \theta_0) < \varepsilon$.

To handle the case when $n \le n_1$, for each ω let

$$Z_\theta(\omega) = \max_{1 \le n \le n_1} P^0_\omega(n^{-1/3} d(0, X_n) > \theta).$$

Then Z_θ is non-increasing in θ, and $\lim_{\theta \to \infty} Z_\theta(\omega) = 0$ for each ω. So, by monotone convergence

$$\lim_{\theta \to \infty} \mathbb{E} Z_\theta(\omega) = 0.$$

Thus there exists θ_1 such that

$$P^*(n^{-1/3} d(0, X_n) > \theta_1) \le \mathbb{E} Z_{\theta_1} < \varepsilon \quad \text{for all } n \le n_1.$$

Taking $\theta = \theta_0 \vee \theta_1$, we obtain (1.24).

Finally, we prove (1.25). Let $\varepsilon > 0$. Let $c_{2.5.5}(\lambda) = c_4 \lambda^{q_3 - 1}$. Choose λ so that $2p(\lambda) + c_4 \lambda^{-q_3} < \varepsilon$, and let $\theta_0 = \lambda^{q_3}$, $\delta = 1/\theta_0$. Choose R so that $2R^3 = n$, and $n_0 = n_0(\varepsilon)$ such that $n \ge n_0$ implies

$\delta R \geq 1$. Set $\theta_1 = 1 + n_0^{1/3}$, and $\theta = \theta_0 \vee \theta_1$. Suppose first that $n \geq n_0$, and set $F_4 = \{R, \delta R \in J(\lambda)\}$. If $\omega \in F_4$ then by Proposition 2.5(c) (using c to account for factors $2^{1/3}$) we have

$$P_\omega^0\Big(n^{-1/3} d(0, X_n) \leq \delta\Big) \leq c_{2.5.5}(\lambda)(cR)^{-2} V(c\delta R) \leq c_4 \lambda^{q_3} \delta^2 = c_4 \lambda^{-q_3}.$$

So,

$$\begin{aligned} P^*\Big(n^{-1/3}(1 + d(0, X_n)) < \theta^{-1}\Big) &\leq P^*\Big(n^{-1/3} d(0, X_n) < \theta_0^{-1}\Big) \\ &\leq \mathbb{P}(F_4^c) + \mathbb{E}\Big(P_\omega^0(n^{-1/3} d(0, X_n) < \theta_0^{-1}); F_4\Big) \\ &\leq 2p(\lambda) + c_4 \lambda^{-q_3} \leq \varepsilon. \end{aligned} \tag{2.34}$$

If $n \leq n_0$ then $n^{-1/3}(1 + d(0, X_n)) \geq n^{-1/3} \geq \theta_1^{-1}$, and so we deduce that, for all n,

$$P^*(n^{-1/3}(1 + d(0, X_n)) < \theta^{-1}) \leq \varepsilon,$$

which proves (1.25). □

Proof of Proposition 1.6. We begin with the upper bounds in (1.26)–(1.27). By Proposition 2.2(a) and Assumption 1.4(2),

$$\mathbb{E}(E_\omega^0 \tau_R) \leq \mathbb{E}(2RV(R)) \leq cR^3,$$

while by Proposition 2.1(a), if $R = (n/2)^{1/3}$ then using Assumption 1.4(3)

$$p_{2n}^\omega(0, 0) \leq cn^{-2/3} \mathbb{E}(1 + R^2/V(R)) \leq c' n^{-2/3}.$$

For each of the lower bounds, it is sufficient to find a set $F \subset \Omega$ of 'good' graphs with $\mathbb{P}(F) \geq c > 0$ such that, for all $\omega \in F$ we have suitable lower bounds on $E_\omega^0 \tau_R$, $p_{2n}^\omega(0, 0)$ or $E_\omega^0 d(0, X_n)$. For the lower bounds, we assume that $R \geq 1$ is large enough so that $R/4\lambda_0 \geq 1$, where λ_0 is chosen large enough that $p(\lambda_0) < 1/8$. We can then obtain the results for all n (chosen below to depend on R) and R by adjusting the constants c_1, c_3, c_5 in (1.26)–(1.28).

Let $F = \{R, R/(4\lambda_0) \in J(\lambda_0)\}$. Then $\mathbb{P}(F) \geq \frac{3}{4}$, and for $\omega \in F$, by (2.10), $E_\omega^0 \tau_R \geq c_1(\lambda_0) R^3$. So,

$$\mathbb{E}(E_\omega^0 \tau_R) \geq \mathbb{E}(E_\omega^0 \tau_R; F) \geq c_1(\lambda_0) R^3 \mathbb{P}(F) \geq c_2(\lambda_0) R^3.$$

Also, by (2.12), if $n \in [\frac{1}{2} c_{2.3.7}(\lambda_0) R^3, c_{2.3.7}(\lambda_0) R^3]$ then

$$p_{2n}^\omega(0, 0) \geq c_3(\lambda_0) n^{-2/3}.$$

Given $n \in \mathbb{N}$, choose R so that $n = c_{2.3.7}(\lambda_0) R^3$ and let F be as above. Then

$$\mathbb{E} p_{2n}^\omega(0, 0) \geq \mathbb{P}(F) c_3(\lambda_0) n^{-2/3} \geq c_4(\lambda_0) n^{-2/3},$$

giving the lower bound in (1.27).

A similar argument uses (2.21) to conclude (1.28). □

Proof of Theorem 1.7. We will take $\Omega_0 = \Omega_a \cap \Omega_b \cap \Omega_c$ where the sets Ω_* are defined in the proofs of (a), (b) and (c). By Assumption 1.4(4), $p(\lambda) = \mathbb{P}(R \notin J(\lambda)) \leq c_0 \lambda^{-q_0}$.

(a) We begin with the case $x = 0$, and write $w(n) = p_{2n}^\omega(0, 0)$. By (2.31) we have

$$\mathbb{P}((c_1 \lambda^{-q_1})^{-1} < n^{2/3} w_n \leq c_1 \lambda^{-q_1}) \geq 1 - 3p(\lambda).$$

Let $n_k = \lfloor e^k \rfloor$ and $\lambda_k = k^{2/q_0}$. Then, since $\sum p(\lambda_k) < \infty$, by Borel–Cantelli there exists $K_0(\omega)$ with $\mathbb{P}(K_0 < \infty) = 1$ such that $c_1^{-1} k^{-2q_1/q_0} \le n_k^{2/3} w(n_k) \le c_1 k^{2q_1/q_0}$ for all $k \ge K_0(\omega)$. Let $\Omega_a = \{K_0 < \infty\}$. For $k \ge K_0$ we therefore have

$$c_2^{-1}(\log n_k)^{-2q_1/q_0} n_k^{-2/3} \le w(n_k) \le c_2 (\log n_k)^{2q_1/q_0} n_k^{-2/3},$$

so that (1.29) holds for the subsequence n_k. The spectral decomposition (see for example [2, Chapter 3 (32)]) gives that $p_{2n}^\omega(0,0)$ is monotone decreasing in n. So, if $n > N_0 = e^{K_0} + 1$, let $k \ge K_0$ be such that $n_k \le n < n_{k+1}$. Then

$$w(n) \le w(n_k) \le c_2 (\log n_k)^{2q_1/q_0} n_k^{-2/3} \le 2e^{2/3} c_2 (\log n)^{2q_1/q_0} n^{-2/3}.$$

Similarly $w(n) \ge w(n_{k+1}) \ge c_3 n^{-2/3} (\log n)^{-2q_1/q_0}$. Taking $q_2 > 2q_1/q_0$, so that the constants c_2, c_3 can be absorbed into the $\log n$ term, we obtain

$$(\log n)^{-q_2} n^{-2/3} \le p_{2n}^\omega(0,0) \le (\log n)^{q_2} n^{-2/3} \quad \text{for all } n \ge N_0(\omega). \tag{2.35}$$

That $\lim_n \log p_{2n}^\omega(0,0)/\log n = -2/3$, $\mathbb{P}$–a.s. is then immediate. Since $\sum_n p_{2n}^\omega(0,0) = \infty$, X is recurrent.

If $x, y \in \mathcal{C}(\omega)$ and $k = d_\omega(x,y)$, then using the Chapman–Kolmogorov equations

$$p_{2n}^\omega(x,x)(p_k^\omega(x,y)\mu_x(\omega))^2 \le p_{2n+2k}^\omega(y,y).$$

Let $\omega \in \Omega_a$, $x \in \mathcal{C}(\omega)$, write $k = d_\omega(0,x)$, $h^\omega(0,x) = (p_k^\omega(x,0)\mu_x(\omega))^{-2}$, and let $n \ge N_0(\omega) + 2k$. Then

$$\begin{aligned} p_{2n}^\omega(x,x) &\le h^\omega(0,x) p_{2n+2k}^\omega(0,0) \\ &\le h^\omega(0,x)(\log(n+k))^{q_2}(n+k)^{-2/3} \\ &\le h^\omega(0,x)(\log(2n))^{q_2} n^{-2/3} \le (\log n)^{1+q_2} n^{-2/3} \end{aligned}$$

provided $\log n \ge 2^{q_2} h^\omega(0,x)$. Taking $N_x(\omega) = \exp(2^{q_2} h^\omega(0,x)) + 2d_\omega(0,x) + N_0(\omega)$, and $\alpha_1 = 1 + q_2$, this gives the upper bound in (1.29). The lower bound is obtained in the same way.

(b) Let $R_n = e^n$ and $\lambda_n = n^{2/q_0}$. Let $F_n = \{R_n, R_n/4\lambda_n \in J(\lambda_n)\}$. Then (provided $R_n/4\lambda_n \ge 1$) we have $\mathbb{P}(F_n^c) \le 2p(\lambda_n) \le 2n^{-2}$. So, by Borel–Cantelli, if $\Omega_b = \liminf F_n$, then $\mathbb{P}(\Omega_b) = 1$. Hence there exists M_0 with $M_0(\omega) < \infty$ on Ω_b, and such that $\omega \in F_n$ for all $n \ge M_0(\omega)$.

Now fix $\omega \in \Omega_b$, and let $x \in \mathcal{C}(\omega)$. Write $F(R) = E_\omega^x \tau_R$. By (2.29) there exist constants c_4, q_4 such that

$$(c_4 \lambda_n^{q_4})^{-1} \le R_n^{-3} F(R_n) \le c_4 \lambda_n^{q_4}. \tag{2.36}$$

provided $n \ge M_0(\omega)$ and n is also large enough so that $x \in B(R_n/4\lambda_n)$. Writing $M_x(\omega)$ for the smallest such n,

$$c_4^{-1}(\log R_n)^{-2q_4/q_0} R_n^3 \le F(R_n) \le c_4 (\log R_n)^{2q_4/q_0} R_n^3, \quad \text{for all } n \ge M_x(\omega).$$

As $F(R)$ is monotonic, the same argument as in (a) enables us to replace $F(R_n)$ by $F(R)$, for all $R \ge R_x = 1 + e^{M_x}$. Taking $\alpha_2 > 2q_4/q_0$ we obtain (1.30).

(c) Recall that $Y_n = \max_{0\le k\le n} d(0, X_k)$. We begin by noting that

$$\{Y_n \ge R\} = \{\tau_R \le n\}. \tag{2.37}$$

Using this, (1.31) follows easily from (1.32).

It remains to prove (1.32). Since τ_R is monotone in R, as in (b) it is enough to prove the result for the subsequence $R_n = e^n$.

The estimates in (b) give the upper bound. In fact, if $\omega \in \Omega_b$, and $n \ge M_x(\omega)$, then by (2.36)

$$P^x_\omega(\tau_{R_n} \ge n^2 c_4 \lambda_n^{q_4} R_n^3) \le \frac{F(R_n)}{n^2 c_4 \lambda_n^{q_4} R_n^3} \le n^{-2}.$$

So, by Borel–Cantelli (with respect to the law P^x_ω), there exists $N'_x(\omega, \overline{\omega})$ with

$$P^x_\omega(N'_x < \infty) = P^x_\omega(\{\overline{\omega} : N'_x(\omega, \overline{\omega}) < \infty\}) = 1$$

such that

$$\tau_{R_n} \le c_5 (\log R_n)^{q_5} R_n^3, \qquad \text{for all } n \ge N'_x.$$

For the lower bound, write $c_{2.5.1}(\lambda) = c_6 \lambda^{-q_6}$, $c_{2.5.2}(\lambda) = c_7 \lambda^{q_7}$. Let $\lambda_n = n^{2/q_0}$, and $\varepsilon_n = n^{-2}\lambda_n^{-q_6-q_7}$. Set $G_n = \{R_n, \varepsilon_n R_n, \varepsilon_n R_n/(4\lambda_n) \in J(\lambda_n)\}$. Then, for n sufficiently large so that $\varepsilon_n R_n/(4\lambda_n) \ge 1$, we have $\mathbb{P}(G_n^c) \le 3p(\lambda_n) \le 3c_0 n^{-2}$. Let $\Omega_c = \Omega_b \cap (\liminf G_n)$; then by Borel–Cantelli $\mathbb{P}(\Omega_c) = 1$ and there exists M_1 with $M_1(\omega) < \infty$ for $\omega \in \Omega_c$ such that $\omega \in G_n$ whenever $n \ge M_1(\omega)$. By Proposition 2.5(a), if $n \ge M_1$ and $x \in B(\varepsilon_n R_n)$ then

$$P^x_\omega(\tau_{R_n} \le c_6 \lambda_n^{-q_6} \varepsilon_n^3 R_n^3) \le c_7 \lambda_n^{q_7} \varepsilon_n \le c_7 n^{-2}. \tag{2.38}$$

So, using Borel–Cantelli, we deduce that (for some q_8)

$$\tau_{R_n} \ge c_6 \lambda^{-q_6} \varepsilon_n^3 R_n^3 \ge n^{-q_8} R_n^3 = (\log R_n)^{-q_8} R_n^3,$$

for all $n \ge N''_x(\omega, \overline{\omega})$. This completes the proof of (1.32). □

Proof of Theorem 1.8. We first consider the case $x = 0$. Let $c_1 \in (0,1)$, $c_2 \ge 2$, $q_1 \ge 1$, $q_2 \ge 1$ be chosen so that

$$c_{2.5.1}(\lambda) \ge c_1 \lambda^{-q_1}, \quad c_{2.5.2}(\lambda) \le c_2 \lambda^{q_2}.$$

Let $R_k = e^k$, and $\lambda_k = k^{q_3}$ where $q_3 \ge 2$ is chosen large enough so that $\sum p(\lambda_k) < \infty$. Let $\varepsilon_k = c_2^{-1} \lambda_k^{-q_2} k^{-q_3}$. Set

$$F_k = \{R_k, \varepsilon_k R_k, \varepsilon R_k / 4\lambda_k \in J(\lambda_k)\}.$$

For $\omega \in F_k$ we have by Proposition 2.5(a)

$$P^0_\omega(\tau_{R_k} \le c_1 \lambda_k^{-q_1} (\varepsilon_k R_k)^3) \le c_2 \lambda_k^{q_2} \varepsilon_k = k^{-q_3}.$$

Set $n(k) = c_1 \lambda_k^{-q_1} (\varepsilon_k R_k)^3$. Then

$$P^*(\{\tau_{R_k} \le n(k)\} \cup F_k^c) \le \mathbb{P}(F_k^c) + k^{-q_3} \le 3p(\lambda_k) + k^{-q_3}. \tag{2.39}$$

Therefore by Borel–Cantelli, we deduce that, P^*–a.s., for all sufficiently large k, $\tau_{R_k} > n(k)$ and F_k holds. So, for large k,

$$S_{n(k)} \le S_{\tau_{R_k}} \le V(R_k) \le \lambda_k R_k^2.$$

If n is sufficiently large, then choosing k so that $n(k-1) < n \le n(k)$,

$$\frac{\log S_n}{\log n} \le \frac{\log S_{n(k)}}{\log n(k-1)} \le \frac{2k + \log \lambda_k}{3(k-1) + \log(c_1 \varepsilon_{k-1}^3 \lambda_{k-1}^{-q_1})} \le \tfrac{2}{3} + \frac{c \log k}{k},$$

and this gives the upper bound in (1.33) for the case $x = 0$.

For the lower bound, let $\xi(x, R) = 1_{\{T_x > \tau_R\}}$. If $R \in J(\lambda)$ and $\varepsilon < 1/2\lambda$ then by Lemma 2.4,

$$P_\omega^0(\xi(x, R) = 1) \le \varepsilon\lambda, \qquad \text{for } x \in B(\varepsilon R).$$

Set

$$Y_k = V(\varepsilon_k R_k)^{-1} \sum_{x \in B(\varepsilon_k R_k)} \xi(x, R).$$

Then if $\omega \in F_k$,

$$P_\omega^0(Y_k \ge \tfrac{1}{2}) \le 2E_\omega^0 Y_k \le 2\varepsilon_k \lambda_k \le k^{-q_3}.$$

Let $m(k) = k^{q_3} \lambda_k R_k^3$. Then if $\omega \in F_k$, by (2.7),

$$P_\omega^0(\tau_{R_k} \ge m(k)) \le 2\lambda_k R_k^3 m(k)^{-1} = 2k^{-q_3}.$$

Thus

$$P^*(F_k^c \cup \{Y_k \ge \tfrac{1}{2}\} \cup \{\tau_{R_k} \ge m(k)\}) \le 3p(\lambda_k) + 3k^{-q_3},$$

so by Borel–Cantelli, P^*–a.s. there exists a $k_0(\omega) < \infty$ such that, for all $k \ge k_0$, F_k holds, $\tau_{R_k} \le m(k)$, and $Y_k \le 1/2$. So, for $k \ge k_0$,

$$S_{m(k)} \ge S_{\tau_{R_k}} \ge \sum_{x \in B(\varepsilon_k R_k)} (1 - \xi(x, R)) = V(\varepsilon_k R_k)(1 - Y_k) \ge \tfrac{1}{2} \lambda_k^{-1} (\varepsilon_k R_k)^2.$$

Let n be large enough so that $m(k) \le n < m(k+1)$ for some $k \ge k_0$. Then

$$\frac{\log S_n}{\log n} \ge \frac{\log S_{m(k)}}{\log m(k+1)} \ge \frac{2k - c \log k}{3(k+1) + c' \log(k+1)},$$

and the lower bound in (1.33) follows. This proves (1.33) when $x = 0$.

Now let

$$\Omega_0 = \{\omega : G(\omega) \text{ is recurrent and } P_\omega^0(\lim_n (\log S_n / \log n) = \tfrac{2}{3}) = 1\}.$$

We have $\mathbb{P}(\Omega_0) = 1$. If $\omega \in \Omega_0$, and $x \in G(\omega)$ then X hits 0 with P_ω^x–probability 1. Since the limit does not depend on the initial segment $X_0, \dots, X_{T_0}$, we obtain (1.33). □

Remark. Note that the constants c_i in Proposition 1.6 and α_i in Theorem 1.7 depend only on the constants c_1, c_2, c_3, q_0 in Assumption 1.4.

3 Verification of Assumption 1.4 for the IIC

In Section 3.1, we state three propositions which give estimates for the volume and effective resistance for the IIC. Propositions 3.1–3.2, which pertain to the volume growth of $\mathcal{C}$, are proved in Section 4. Proposition 3.3, which will be used to estimate the effective resistance, is proved in Section 5. In Section 3.2, we use the three propositions to verify Assumption 1.4 for the IIC, and complete the proof of our main result Theorem 1.1.

3.1 Three propositions

We will use the following notation for the IIC. Let $U(R) = \{(x,n) : n \geq R\}$, $B(R) = \{(x,n) \in \mathcal{C} : 0 \leq n < R\}$, and $\partial B(R) = \{(x,R) : (x,R) \in \mathcal{C}\}$. We note that, using the graph distance d on $\mathcal{C}$, $B(R)$ is just the ball $B(\mathbf{0}, R)$, and $\partial B(R)$ is its exterior boundary. Let

$$Z_R = (2\tau_1 A^2 V^* R^2)^{-1} V(R), \tag{3.1}$$

where A, V^* and τ_1 are constants (all near 1 for large L) defined in Section 4.1.1.

Proposition 3.1. *Let $d > 4$ and $L \geq L_0$. Under the* IIC *measure, the random variables Z_R converge in distribution to a strictly positive limit Z, whose distribution is independent of d and L. Moreover, all moments converge, i.e., $\mathbb{E}_\infty Z_R^l \to \mathbb{E} Z^l$ for each $l \in \mathbb{N}$. In particular,*

$$c_1(d) R^2 \leq \mathbb{E}_\infty V(R) \leq c_2(d) R^2.$$

Moreover, c_1 and c_2 do not depend on d, if we further require that $L \geq L_1$, for some $L_1 = L_1(d)$.

Proposition 3.2. *Let $d > 4$ and $L \geq L_0$.*

$$\mathbb{Q}_\infty(V(R) R^{-2} < \lambda) \leq c_1(d) \exp\{-c_2(d) \lambda^{-1/2}\}, \qquad R \geq 1. \tag{3.2}$$

Moreover, c_1 and c_2 do not depend on d, if we further require that $L \geq L_1$, for some $L_1 = L_1(d)$.

The third proposition gives an estimate on the expected number of edges at level $n-1$ that need to be cut in order to disconnect 0 from level R. We say that $(x,n), (x',n') \in \mathcal{C}$ are *RW-connected*, if there is a path, not necessarily oriented, in $\mathcal{C}$ from (x,n) to (x',n'). We reserve the term *connected* to mean oriented connection, that is $(x,n) \longrightarrow (x',n')$. Let

$$D(n) = \left\{ e = ((w, n-1), (x,n)) \subset \mathcal{C} : \begin{array}{l} (x,n) \text{ is RW-connected to} \\ \text{level } R \text{ by a path in } \mathcal{C} \cap U(n) \end{array} \right\}, \quad 0 < n \leq R. \tag{3.3}$$

It follows from the definition that all edges in $D(n)$ need to be cut in order to RW-disconnect 0 from level R. Also, cutting all the edges in $D(n)$ RW-disconnects 0 from $B(R)^c$, since for any RW-path from 0 to $B(R)^c$ the last crossing of level n occurs at an edge in $D(n)$.

Proposition 3.3. *Assume $d > 6$. There exists $L_1 = L_1(d) \geq L_0(d)$ such that for $L \geq L_1$,*

$$\mathbb{E}_\infty(|D(n)|) \leq c_1(a), \qquad 0 < n \leq aR, \quad 0 < a < 1. \tag{3.4}$$

The constant $c_1(a)$ is independent of the dimension d.

Remark. Proposition 3.3 is the only place where we need $d > 6$ rather than $d > 4$.

3.2 Verification of Assumption 1.4 for the IIC

We begin with a lemma that relates $|D(n)|$ and the effective resistance.

Lemma 3.4. *For oriented percolation in any dimension $d \geq 1$,*

$$R_{\text{eff}}(0, \partial B(R)) \geq \sum_{n=1}^{3R/4} \frac{1}{|D(n)|}. \tag{3.5}$$

Proof. Let $\xi_n = |D(n)|$. Note that $\xi_n \geq 1$ since $(0,0) \longrightarrow R$.

For $(x,n) \in \mathcal{C}$ let $b(x,n)$ be the largest k such that (x,n) is RW-connected to level R by a path in $\mathcal{C} \cap U(k)$. Let $e = ((w,n-1),(x,n)) \subset \mathcal{C}$. If $e \in D(n)$ then $b(x,n) = n$ and $b(w,n-1) = n-1$. Conversely, if $b(x,n) = n$ then $b(w,n-1) = n-1$ and $e \in D(n)$.

Now let $h : [0,R] \to [0,1]$ be non-decreasing with $h(0) = 0$, $h(R-1) = 1$. Define for $(x,n) \in B(R)$

$$f(x,n) = h(b(x,n)). \tag{3.6}$$

By (1.16), we have

$$R_{\text{eff}}(0,\partial B(R))^{-1} \leq \mathcal{E}(f,f) = \sum_{n=1}^{R} \sum_{((w,n-1),(x,n))\in\mathcal{C}} (f(w,n-1) - f(x,n))^2. \tag{3.7}$$

Let $e = ((w,n-1),(x,n))$ be an edge in $\mathcal{C}$. Then $b(w,n-1) = \min\{n-1, b(x,n)\}$. If $b(w,n-1) < n-1$ then $b(x,n) = b(w,n-1)$, and so $f(w,n-1) = f(x,n)$. If $b(w,n-1) = n-1$ then $b(x,n)$ is either $n-1$ or n. If $b(w,n-1) = n-1$ and $b(x,n) = n-1$ then $f(w,n-1) = f(x,n)$. Thus the only contributions to the sum (3.7) are from bonds in $\cup_{n=1}^{R} D(n)$. Therefore

$$\mathcal{E}(f,f) = \sum_{n=1}^{R} \xi_n (h(n) - h(n-1))^2. \tag{3.8}$$

Now let $K = \sum_{i=1}^{3R/4} \xi_i^{-1}$ and

$$h(n) = K^{-1} \sum_{i=1}^{n} \xi_i^{-1}, \quad 0 \leq n \leq 3R/4, \tag{3.9}$$

so that $h(3R/4) = 1$. Let $h(n) = 1$ for $n \geq 3R/4$. Then $\mathcal{E}(f,f) = K^{-1}$, so $R_{\text{eff}}(0,\partial B(R)) \geq K$. □

Now we combine Proposition 3.3 and Lemma 3.4 to show that it is unlikely that the effective resistance $R_{\text{eff}}(0,\partial B(R))$ is less than a small multiple of R.

Proposition 3.5. *For $d > 6$ and $L \geq L_1$,*

$$\mathbb{Q}_\infty(R_{\text{eff}}(0,\partial B(R)) \leq \varepsilon R) \leq c\varepsilon. \tag{3.10}$$

Proof. Let $r = 3R/4$. By Lemma 3.4 and the Cauchy–Schwarz inequality,

$$R_{\text{eff}}(0,\partial B(R))^{-1} \leq \Big(\sum_{n=1}^{r} |D(n)|^{-1} \Big)^{-1} \leq r^{-2} \sum_{n=1}^{r} |D(n)|. \tag{3.11}$$

Therefore, by Proposition 3.3,

$$\begin{aligned}
\mathbb{Q}_\infty(R_{\text{eff}}(0,\partial B(R)) \leq \varepsilon R) &= \mathbb{Q}_\infty(R_{\text{eff}}(0,\partial B(R))^{-1} \geq \varepsilon^{-1}R^{-1}) \\
&\leq \mathbb{Q}_\infty(\sum_{n=1}^{r} |D(n)| \geq c_1 r/\varepsilon) \\
&\leq (\varepsilon/c_1 r)\mathbb{E}_\infty(\sum_{n=1}^{r} |D(n)|) \leq c_2\varepsilon.
\end{aligned}$$

□

Theorem 3.6. *For $d > 6$, there is an $L_1 = L_1(d) \geq L_0(d)$ such that for all $L \geq L_1$, Assumption 1.4(1)–(4) hold with $q_0 = 1$ and constants c_1, c_2, c_3 independent of d and L.*

Proof. Let $W_R = V(R)/R^2$. By Proposition 3.1 we have (2) and

$$\mathbb{Q}_\infty(W_R \geq \lambda) \leq \lambda^{-1}\mathbb{E}_\infty W_R \leq c\lambda^{-1}. \tag{3.12}$$

Also, Proposition 3.2 gives

$$\mathbb{Q}_\infty(W_R < \lambda^{-1}) \leq c\exp(-c'\lambda^{1/2}), \tag{3.13}$$

and (3) is then immediate after integration. Combining (3.12)–(3.13) and (3.10) gives (1), and also, since each of the bounds is less than $c\lambda^{-1}$ for large λ, (4), with $q_0 = 1$. □

Proof of Theorem 1.1. This is immediate from Theorems 3.6 and 1.7–1.8, on noting that if $X_0 = \mathbf{0}$ then τ_R is just the exit time from the ball $B(R)$. Since the constants in Assumption 1.4 are independent of d, L (provided $d > 6$ and $L \geq L_1(d)$), the constants α_1, α_2 are also independent of d and L. □

4 IIC volume estimates: Proof of Propositions 3.1–3.2

In Section 4.1 we prove Proposition 3.1, and in Section 4.2 we prove Proposition 3.2.

4.1 Volume convergence: Proof of Proposition 3.1

4.1.1 The IIC r-point functions

We assume throughout that $d > 4$ and that L is large; these assumptions will often not be mentioned explicitly in the following. Throughout, $\beta = L^{-d}$, K denotes a constant that only depends on d, and $\bar{K}$ denotes an absolute constant.

The critical oriented percolation two-point function $\tau_n(x)$ is defined by

$$\tau_n(x) = \mathbb{P}_{p_c}((0,0) \longrightarrow (x,n)). \tag{4.1}$$

Let $\tau_n = \sum_{x\in\mathbb{Z}^d} \tau_n(x)$. By [25, Theorem 1.1],

$$\sup_{x\in\mathbb{Z}^d} \tau_n(x) \leq K\beta(n+1)^{-d/2}, \quad n \geq 1, \tag{4.2}$$

$$\tau_n = A(1 + \mathcal{O}(n^{(4-d)/2})), \quad \text{as } n \to \infty, \tag{4.3}$$

where $|A - 1| \leq K\beta$. Although not explicitly stated in [25], it follows from [24, Eqn. (4.2)], that the error term in (4.3) is bounded by $K\beta n^{(4-d)/2}$. (Note that $f_n(0, z_c)$ of [24] corresponds to our τ_n.) Hence for $L \geq L_1 = L_1(d)$, we have

$$\bar{K}^{-1} \leq A \leq \bar{K}, \qquad |\tau_n - A| \leq \bar{K}n^{(4-d)/2}, \quad n \geq 1, \qquad \bar{K}^{-1} \leq \tau_n \leq \bar{K}, \quad n \geq 0. \tag{4.4}$$

Also, noting that τ_1 is called p_c in [25], we see from [25, Eqn. (1.12)] that $|\tau_1 - 1| \leq K\beta \leq \bar{K}$ for $L \geq L_1(d)$ sufficiently large.

Let $\vec{y} = (y_1, \ldots, y_{r-1})$ and $\vec{m} = (m_1, \ldots, m_{r-1})$ with $y_i \in \mathbb{Z}^d$, $m_i \in \mathbb{Z}_+$. For $r \geq 2$, the IIC r-point function is defined by

$$\rho^{(r)}_{\vec{m}}(\vec{y}) = \mathbb{Q}_\infty((0,0) \longrightarrow (y_i, m_i) \text{ for all } i = 1, \ldots, r-1). \tag{4.5}$$

Let

$$\hat{\rho}^{(r)}_{\vec{m}} = \sum_{y_1,\ldots,y_{r-1} \in \mathbb{Z}^d} \rho^{(r)}_{\vec{m}}(\vec{y}). \tag{4.6}$$

Let V^* denote the *vertex factor* of [25, Theorem 1.2] (written V in [25] but written V^* here to avoid confusion with the volume). The vertex factor is a constant with $|V^* - 1| \leq K\beta$, and we assume that L_1 has been chosen so that $\bar{K}^{-1} \leq V^* \leq \bar{K}$. Let A be the constant of (4.3). Let $r \geq 2$, $\vec{t} = (t_1, \ldots, t_{r-1}) \in (0,1]^{r-1}$, and for a positive integer m, let $m\vec{t}$ be the vector with components $\lfloor mt_i \rfloor$. It follows from [20, (5.15)] that

$$\lim_{m \to \infty} \frac{1}{(mA^2V^*)^{r-1}} \hat{\rho}^{(r)}_{m\vec{t}} = \hat{M}^{(r)}_{1,\vec{t}}, \tag{4.7}$$

where the limit $\hat{M}^{(r)}_{1,\vec{t}}$ is the r^{th} moment of the canonical measure $\mathbb{N}$ of super-Brownian motion X_t, namely

$$\hat{M}^{(r)}_{s_1,\ldots,s_r} = \mathbb{N}\Big(X_{s_1}(\mathbb{R}^d) \cdots X_{s_r}(\mathbb{R}^d)\Big). \tag{4.8}$$

For $r = 1$, we have simply

$$\hat{M}^{(1)}_s = \mathbb{N}\Big(X_s(\mathbb{R}^d)\Big) = 1. \tag{4.9}$$

For $r > 2$ and $\bar{s} = (s_1, \ldots, s_r)$ with each $s_i > 0$, the $\hat{M}^{(l)}_{\bar{s}}$ are given recursively by

$$\hat{M}^{(r)}_{\bar{s}} = \int_0^{\underline{s}} ds\, \hat{M}^{(1)}_s \sum_{I \subset J_1 : |I| \geq 1} \hat{M}^{(i)}_{\bar{s}_I - s} \hat{M}^{(r-i)}_{\bar{s}_{J \setminus I} - s}, \tag{4.10}$$

where $i = |I|$, $J = \{1, \ldots, l\}, J_1 = J \backslash \{1\}$, $\underline{s} = \min_i s_i$, $\vec{s}_I$ denotes the vector consisting of the components s_i of $\vec{s}$ with $i \in I$, and $\vec{s}_I - s$ denotes subtraction of s from each component of $\vec{s}_I$. The explicit solution to the recursive formula (4.10) can be found, e.g., in [25, (1.25)]. In particular, $\hat{M}^{(2)}_{s_1,s_2} = s_1 \wedge s_2$. It is shown in [20, Lemma 4.2] that for $r \geq 1$ and $t > 0$,

$$\hat{M}^{(r)}_{t,\ldots,t} = t^{r-1} 2^{-(r-1)} r!. \tag{4.11}$$

To this we add the following elementary fact.

Lemma 4.1. *For $r \geq 1$, $\hat{M}^{(r)}_{s_1,\ldots,s_r}$ is nondecreasing in each s_i.*

Proof. The proof is by induction on r. For $r = 1$, $\hat{M}^{(1)}_{s_1} = 1$ by (4.9), which is nondecreasing. Assume the result holds for all $j \leq r$. Then it holds also for $r + 1$ by (4.10), since increasing an s_i can only increase the integrand (by the induction hypothesis) or the domain of integration in (4.10). □

4.1.2 Proof of Proposition 3.1

Recall from (3.1) that Z_R is defined by

$$Z_R = (2\tau_1 A^2 V^* R^2)^{-1} V(R). \tag{4.12}$$

Let Y_t denote the canonical measure of super-Brownian motion conditioned to survive for all time (see [19]). Let

$$Z = \int_0^1 dt\; Y_t(\mathbb{R}^d), \tag{4.13}$$

so that Z is a positive random variable. It is clear that the distribution of Z does not depend on L. It also does not depend on d, since it is equal to the mass up to time 1 of the continuum random tree conditioned to survive forever. To prove Proposition 3.1, it suffices to prove that all moments of Z_R converge to those of Z and that Z has a moment generating function with nonzero radius of convergence (see [12, Theorem 30.2]).

Let

$$\tilde{Z}_R = (A^2 V^* R^2)^{-1} |B(R)|. \tag{4.14}$$

Thus $\tilde{Z}_R$ is defined in terms of the vertices in $B(R)$, whereas Z_R is defined in terms of the edges. We use (4.7) to prove that $\lim_{R\to\infty} \mathbb{E}\tilde{Z}_R^l = \mathbb{E}Z^l$ for all $l \geq 1$, and then adapt this to Z_R.

Let $l \geq 1$. By definition,

$$\begin{aligned} \mathbb{E}\tilde{Z}_R^l &= \frac{1}{(A^2V^*R^2)^l} \sum_{n_1=0}^{R-1} \cdots \sum_{n_l=0}^{R-1} \sum_{x_1\in\mathbb{Z}^d} \cdots \sum_{x_l\in\mathbb{Z}^d} \rho^{(l+1)}_{n_1,\ldots,n_l}(x_1,\ldots,x_l) \\ &= \frac{1}{R}\sum_{n_1=0}^{R-1} \cdots \frac{1}{R}\sum_{n_l=0}^{R-1} \frac{1}{(A^2V^*R)^l} \hat{\rho}^{(l+1)}_{\vec{t}R}, \end{aligned} \tag{4.15}$$

where $\vec{t} = (n_1 R^{-1}, \ldots, n_l R^{-1})$. The summand on the right hand side is bounded by a constant, by standard tree-graph inequalities [1] (see [20, Section 5.1] for the details when $l = 1$). Therefore, by (4.7) and the dominated convergence theorem,

$$\lim_{R\to\infty} \mathbb{E}\tilde{Z}_R^l = \int_0^1 dt_1 \cdots \int_0^1 dt_l \hat{M}^{(l+1)}_{1,\vec{t}}. \tag{4.16}$$

But the right-hand side is $\mathbb{E}Z^l$ (see [19, Section 3.4]).

The next lemma implies that it is also the case that $\lim_{R\to\infty} \mathbb{E}Z_R^l = \mathbb{E}Z^l$ for all $l \geq 1$.

Lemma 4.2. *For all $l \geq 1$,*

$$(1 - 2/R)^{2l} \mathbb{E}\tilde{Z}_{R-2}^l \leq \mathbb{E}Z_R^l \leq \mathbb{E}\tilde{Z}_{R-1}^l + c(d, L, l)R^{-1}. \tag{4.17}$$

Given Lemma 4.2, it remains to show that Z has an exponential moment. But by definition, and by Lemma 4.1 and (4.11), the moments of Z obey

$$\mathbb{E}Z^l = \int_0^1 dt_1 \cdots \int_0^1 dt_l \hat{M}^{(l+1)}_{1,\vec{t}} \leq \int_0^1 dt_1 \cdots \int_0^1 dt_l \hat{M}^{(l+1)}_{1,1,\ldots,1} = 2^{-l}(l+1)!. \tag{4.18}$$

Therefore, the moment generating function has at least radius of convergence 2. This shows that Z_R converges weakly to Z. Note that for $L \geq L_1$, the constants A, V^* and τ_1 satisfy bounds independent of d, hence c_1 and c_2 in Proposition 3.1 do not depend on d. This completes the proof of Proposition 3.1.

Proof of Lemma 4.2. For $l \geq 1$, we define

$$\sigma_{\vec{m}}^{(l+1)}(\vec{x}, \vec{y}) = \mathbb{Q}_\infty((0,0) \longrightarrow (x_i, m_i) \longrightarrow (y_i, m_i + 1) \text{ for all } i = 1, \dots, l).$$

Note that

$$2|\text{edges in } B(R-1)| \leq \sum_{(x,m)\in B(R)} \mu_{(x,m)} = V(R) \leq 2|\text{edges in } B(R)|, \tag{4.19}$$

since edges on the boundary of $B(R)$ are counted once in $V(R)$, while other edges are counted twice. Therefore

$$\mathbb{E}Z_R^l \geq \frac{1}{(\tau_1 A^2 V^* R^2)^l} \sum_{n_1=0}^{R-2} \cdots \sum_{n_l=0}^{R-2} \sum_{x_1,y_1\in\mathbb{Z}^d} \cdots \sum_{x_l,y_l\in\mathbb{Z}^d} \sigma_{n_1,\dots,n_l}^{(l+1)}(x_1, \dots, x_l, y_1, \dots, y_l), \tag{4.20}$$

with a corresponding upper bound if the summations over the n_i's extend to $R-1$.

Lower bound. The Harris–FKG inequality [16] implies that for increasing events A and B we have $\mathbb{Q}_n(A \cap B) \geq \mathbb{Q}_n(A)\mathbb{P}(B)$. If A and B are cylinder events, then by passing to the limit, we have $\mathbb{Q}_\infty(A \cap B) \geq \mathbb{Q}_\infty(A)\mathbb{P}(B)$. Hence

$$\sigma_{\vec{n}}^{(l+1)}(\vec{x}, \vec{y}) \geq \rho_{\vec{n}}^{(l+1)}(\vec{x}) \prod_{i=1}^{l} \tau_1(y_i - x_i). \tag{4.21}$$

With (4.15), this gives $\mathbb{E}Z_R^l \geq [(R-2)/R]^{2l}\mathbb{E}\tilde{Z}_{R-2}^l$.

Upper bound. Let

$$A_{\vec{m}}(\vec{x}) = \{(0,0) \longrightarrow \infty,\ (0,0) \longrightarrow (x_i, m_i),\ i = 1, \dots, l\}.$$

Let $F_{\vec{m}}(\vec{x}, \vec{y})$ denote the event that the following $l+1$ events occur on disjoint sets of edges:

$$A_{\vec{m}}(\vec{x}),\ \{(x_1, m_1) \longrightarrow (y_1, m_1 + 1)\}, \dots, \{(x_l, m_l) \longrightarrow (y_l, m_l + 1)\}. \tag{4.22}$$

Then

$$\sigma_{\vec{m}}^{(l+1)}(\vec{x}, \vec{y}) \leq \mathbb{Q}_\infty(F_{\vec{m}}(\vec{x}, \vec{y})) + \mathbb{Q}_\infty(A_{\vec{m}}(\vec{x}) \setminus F_{\vec{m}}(\vec{x}, \vec{y})). \tag{4.23}$$

By the BK inequality [8],

$$\mathbb{Q}_\infty(F_{\vec{m}}(\vec{x}, \vec{y})) \leq \mathbb{Q}_\infty(A_{\vec{m}}(\vec{x})) \prod_{i=1}^{l} \tau_1(y_i - x_i) = \rho_{\vec{m}}^{(l+1)}(\vec{x}) \prod_{i=1}^{l} \tau_1(y_i - x_i). \tag{4.24}$$

The sum of this bound over $\vec{x}$ and $\vec{y}$ is $\hat{\rho}_{\vec{m}}^{(l+1)}\tau_1^l$. With (4.15), this gives a contribution $\mathbb{E}\tilde{Z}_{R-1}^l$ to the upper bound version of (4.20).

We claim that on the event $A_{\vec{m}}(\vec{x}) \setminus F_{\vec{m}}(\vec{x}, \vec{y})$, there exists $1 \leq i \leq l$ such that either $(x_i, m_i) \longrightarrow (x_j, m_j)$ for some $j \neq i$, or $(x_i, m_i) \longrightarrow \infty$. To see this, we may assume that all the (x_i, m_i)'s are different, otherwise there is nothing to prove. Under this assumption, the last l events in (4.22) occur disjointly. As in a tree-graph bound [1], choose a set of disjoint paths showing that $A_{\vec{m}}(\vec{x})$

occurs. Then at least one of the paths uses an edge $((x_i,m_i),(y_i,m_i+1)$, otherwise $F_{\vec{m}}(\vec{x},\vec{y})$ would occur. This path includes a connection $(x_i,m_i) \longrightarrow (x_j,m_j)$ or $(x_i,m_i) \longrightarrow \infty$, proving the claim.

By the claim, the second term on the right hand side of (4.23) is at most

$$\sum_{1\le i\le l}\left[\sum_{j\neq i}\mathbb{Q}_\infty(A_{\vec{m}}(\vec{x}),\,(x_i,m_i)\longrightarrow(x_j,m_j))+\mathbb{Q}_\infty(A_{\vec{m}}(\vec{x}),\,(x_i,m_i)\longrightarrow\infty)\right]. \tag{4.25}$$

Each term in (4.25) can be bounded using a tree-graph inequality where the number of internal vertices in the tree-graph bound is $l-1$, one less than it would be for $\rho^{(l+1)}$. This implies that the sum of (4.25) over $\vec{x}$ and $\vec{y}$ is bounded by $c(d,L,l)R^{l-1}$. It follows that

$$\mathbb{E}Z_R^l \le \mathbb{E}\tilde{Z}_{R-1}^l + c(d,L,l)R^{-1},$$

which gives the desired upper bound and completes the proof of (4.17). □

4.2 Volume estimate: Proof of Proposition 3.2

4.2.1 Rate of convergence to the IIC

We will use the following alternate definition of the IIC. Recall that $\mathcal{F}_0$ denotes the cylinder events, and let

$$\mathbb{P}_n(E) = \frac{1}{\tau_n}\sum_{x\in\mathbb{Z}^d}\mathbb{P}(E\cap\{(0,0)\longrightarrow(x,n)\}) \quad (E\in\mathcal{F}_0). \tag{4.26}$$

Let

$$\mathbb{P}_\infty(E) = \lim_{n\to\infty}\mathbb{P}_n(E) \quad (E\in\mathcal{F}_0), \tag{4.27}$$

assuming the limit exists. The combined results of [20, 21, 22] show that for $d>4$ and $L\ge L_0(d)$ the limit (4.27) exists and extends to a measure on the full σ-algebra $\mathcal{F}$, and that $\mathbb{P}_\infty = \mathbb{Q}_\infty$. Thus (4.27) provides an alternate definition of the IIC.

Let $\mathcal{E}_m$ denote the set of cylinder events measurable with respect to the set of edges up to level $m-1$. In [20, Eqn. (2.19)], the following representation was obtained for $\mathbb{P}_n(E)$, $E\in\mathcal{E}_m$:

$$\mathbb{P}_n(E) = \frac{1}{\tau_n}\left[\sum_{l=m}^{n-1}\varphi_l(E)\tau_1\tau_{n-l-1}+\varphi_n(E)\right], \tag{4.28}$$

where $\varphi_l(E)$ is a function arising in the lace expansion. The factor τ_1 was called p_c in [20]. By [20, Lemma 2.2], φ_l satisfies

$$|\varphi_l(E)| \le K\beta m(l-m+1)^{-d/2}, \qquad l\ge m+1. \tag{4.29}$$

However, a very slight modification of the proof of [20, Lemma 2.2] actually shows that

$$|\varphi_l(E)| \le K\beta(l-m+1)^{(2-d)/2}, \quad l\ge m\ge 1, \tag{4.30}$$

and we will use this variant. The IIC measure is given in [20, Eqn. (2.29)] as

$$\mathbb{P}(E) = \sum_{l=m}^{\infty}\tau_1\varphi_l(E), \qquad E\in\mathcal{E}_m. \tag{4.31}$$

The proof of Proposition 3.2 relies on the following lemma, which bounds the rate at which the measure $\mathbb{P}_{2m}$ converges to $\mathbb{P}_\infty$.

Lemma 4.3. *Let $d > 4$. For $E \in \mathcal{E}_m$,*

$$|\mathbb{P}_{2m}(E) - \mathbb{P}_\infty(E)| = \mathcal{O}((m+1)^{(4-d)/2}) \tag{4.32}$$

where the constant in the error term is uniform in E and $L \geq L_0$. The error term can be guaranteed to be uniform in d as well, by further requiring that $L \geq L_1$ for some $L_1 = L_1(d)$.

Proof. By the triangle inequality,

$$|\mathbb{P}_{2m}(E) - \mathbb{P}_\infty(E)| \leq \left|\mathbb{P}_{2m}(E) - \sum_{l=m}^{2m} \tau_1 \varphi_l(E)\right| + \left|\mathbb{P}_\infty(E) - \sum_{l=m}^{2m} \tau_1 \varphi_l(E)\right|. \tag{4.33}$$

For the second term on the right-hand side, we use (4.31) and (4.30) to obtain

$$\left|\mathbb{P}_\infty(E) - \sum_{l=m}^{2m} \tau_1 \varphi_l(E)\right| \leq \sum_{l=2m+1}^{\infty} \tau_1 |\varphi_l(E)| \leq K\beta \sum_{l=2m+1}^{\infty} (l-m+1)^{(2-d)/2} \leq K\beta m^{(4-d)/2}. \tag{4.34}$$

For the first term on the right-hand side of (4.33), we use (4.28) to obtain

$$\left|\mathbb{P}_{2m}(E) - \sum_{l=m}^{2m} \tau_1 \varphi_l(E)\right| \leq \sum_{l=m}^{2m-1} \tau_1 |\varphi_l(E)| \left|\frac{\tau_{2m-l-1}}{\tau_{2m}} - 1\right| + |\varphi_{2m}(E)| \left|\frac{1}{\tau_{2m}} - \tau_1\right|. \tag{4.35}$$

By (4.30), the last term is bounded by $K\beta m^{(2-d)/2}$. To bound the sum, we split it into the cases $m \leq l < 3m/2$ and $3m/2 \leq l \leq 2m-1$. In the first case, we use (4.3) to obtain $|(\tau_{2m-l-1}/\tau_{2m}) - 1| \leq K\beta m^{(4-d)/2}$. Then inserting the bound (4.30) and summing over l, we obtain a bound $K\beta m^{(4-d)/2}$ for the first case. In the second case, we bound $|\tau_{2m-l-1}/\tau_{2m} - 1| \leq K$. Inserting the bound on φ_l, and summing over l, we obtain a bound $K\beta m^{(4-d)/2}$ for the second case. Thus, in either case, (4.35) is bounded by $K\beta m^{(4-d)/2}$. For $L \geq L_1$ this bound is at most $\bar{K} m^{(4-d)/2}$. With (4.33)–(4.34), this proves (4.32). □

4.2.2 Proof of Proposition 3.2

In this section, we prove Proposition 3.2. Recall that $\mathbb{P}_\infty = \mathbb{Q}_\infty$. It is enough to show that we can find constants $R_0(d), c_1(d), c_2(d), c_3(d)$ such that for $R \geq R_0$ and $\lambda \leq c_3$ we have

$$\mathbb{P}_\infty(V(R)R^{-2} < \lambda) \leq c_1 \exp\{-c_2 \lambda^{-1/2}\}. \tag{4.36}$$

Indeed, the restrictions on λ and R can be removed by adjusting the constant c_1 as follows. First, for $\lambda > c_3$, if $c_1 > \exp\{c_2(c_3)^{-1/2}\}$, the right hand side of (4.36) is larger than 1. As for $R < R_0$, due to the (deterministic) inequality $V(R) \geq R$, we have $V(R)R^{-2} \geq R\,R^{-2} > R_0^{-1}$. Therefore, if $\lambda < R_0^{-1}$, the left hand side of (4.36) is 0. For $\lambda \geq R_0^{-1}$, it is enough to require that $c_1 > \exp\{c_2 R_0^{1/2}\}$. Finally, note that if initially R_0, c_1, c_2, c_3 are independent of d, then so is the adjusted c_1.

We begin with a simple consequence of Proposition 3.1.

Corollary 4.4. *Given $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon, d)$, such that*

$$\mathbb{Q}_\infty(V(R)R^{-2} < \lambda_0) < \varepsilon, \qquad R \geq 1. \tag{4.37}$$

For $L \geq L_1$, λ_0 can be chosen independent of d.

Proof. This follows from Proposition 3.1 and the fact that Z is strictly positive. $\square$

Let $c = c(d) = \sup_{m\geq 1} \tau_m$. According to (4.37), there is a constant $c_3 = c_3(d)$ such that

$$\mathbb{P}_\infty(V(R) < 4c_3(R+1)^2) < \frac{1}{3c}, \qquad R \geq 1. \tag{4.38}$$

We fix $m_0 = m_0(d)$ such that for $m \geq m_0$ the error term on the right-hand side of (4.32) is at most $(3c)^{-1}$. Let $R_0 = 16c_3m_0^2$. Fix $\lambda \leq c_3$ and $R \geq R_0$. We will prove that (4.36) holds for λ and R with the choice of c_3 made and with $c_1 = 1$ and $c_2 = \frac{1}{2}\log(3/2)c_3^{1/2}$.

There is nothing to prove if $\lambda < R_0/R^2$, since, in this case

$$\mathbb{P}_\infty(V(R)R^{-2} < \lambda) \leq \mathbb{P}_\infty(V(R) < R_0) \leq \mathbb{P}_\infty(V(R) < R) = 0 \tag{4.39}$$

and (4.36) holds trivially. Hence, without loss of generality, we assume that

$$\frac{16c_3m_0^2}{R^2} = \frac{R_0}{R^2} \leq \lambda \leq c_3. \tag{4.40}$$

To estimate $\mathbb{P}_\infty(V(R) < \lambda R^2)$, we subdivide the time interval $[0, R]$ into blocks that provide roughly independent contributions to the volume, and apply (4.38) in each block. The number of blocks is $S = \lfloor (c_3/\lambda)^{1/2} \rfloor$, which is at least 1 by (4.40). The length of a block is $2m$, with $m = \lfloor R/2S \rfloor$. Note that $m \geq m_0$, since

$$\frac{R}{2S} \geq \frac{R}{2(c_3/\lambda)^{1/2}} \geq 2m_0 > 1, \tag{4.41}$$

and hence

$$m = \left\lfloor \frac{R}{2S} \right\rfloor \geq \frac{R}{4S} \geq \frac{R}{4(c_3/\lambda)^{1/2}} \geq m_0. \tag{4.42}$$

Set $n_i = i(2m)$, $i = 0, \ldots, S$, so that the i-th block starts at level n_{i-1} and ends at level n_i.

By (4.27),

$$\mathbb{P}_\infty(V(R) < \lambda R^2) = \lim_{N\to\infty} \frac{1}{\tau_N} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_{p_c}(V(R) < \lambda R^2,\, (0,0) \longrightarrow (x,N)). \tag{4.43}$$

The path $(0,0) \longrightarrow (x,N)$ on the right-hand side passes through the levels $n_1, \ldots, n_S$, and hence there exist $0 = x_0, x_1, \ldots, x_S \in \mathbb{Z}^d$ such that

$$(0,0) \longrightarrow (x_1, n_1) \longrightarrow \cdots \longrightarrow (x_S, n_S) \longrightarrow (x, N).$$

We write $\boldsymbol{x}_i = (x_i, n_i)$ for $i = 0, \ldots, S$, and write $\boldsymbol{x} = (x, N)$. It follows that

$$\begin{aligned}
&\mathbb{P}_{p_c}(V(R) < \lambda R^2,\, (0,0) \longrightarrow (x,N)) \\
&\quad = \mathbb{P}_{p_c}\Bigg(\bigcup_{x_1,\ldots,x_S\in\mathbb{Z}^d} \{V(R) < \lambda R^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i,\, i = 1,\ldots,S\} \cap \{\boldsymbol{x}_S \longrightarrow \boldsymbol{x}\} \Bigg) \\
&\quad \leq \sum_{x_1,\ldots,x_S\in\mathbb{Z}^d} \mathbb{P}_{p_c}(V(R) < \lambda R^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i,\, i = 1,\ldots,S,\, \boldsymbol{x}_S \longrightarrow \boldsymbol{x}).
\end{aligned} \tag{4.44}$$

Let

$$\mathcal{C}(\boldsymbol{y};n) = C(\boldsymbol{y}) \cap (\mathbb{Z}^d \times \{0,1,\dots,n\}). \tag{4.45}$$

On the event on the right-hand side of (4.44), $\boldsymbol{x}_{i-1}$ is contained in $B(R)$, and hence $\mathcal{C}(\boldsymbol{x}_{i-1};n_{i-1}+m) \subset B(R)$. Denote $V_i = \mu(\mathcal{C}(\boldsymbol{x}_{i-1};n_{i-1}+m))$. Then on the event in the right-hand side of (4.44), since $\lambda \le c_3/S^2$ by the choice of S, we have

$$V_i \le V(R) < \lambda R^2 \le \frac{c_3}{S^2} R^2 = 4c_3 \left(\frac{R}{2S}\right)^2 \le 4c_3(m+1)^2. \tag{4.46}$$

Hence, the right-hand side of (4.44) is at most

$$\sum_{x_1,\dots,x_S \in \mathbb{Z}^d} \mathbb{P}_{p_c}\left(\bigcap_{i=1}^S \{V_i < 4c_3(m+1)^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i\} \cap \{\boldsymbol{x}_S \longrightarrow \boldsymbol{x}\}\right). \tag{4.47}$$

The $S+1$ events in (4.47) depend on disjoint sets of bonds, so the probability factors as

$$\sum_{x_1,\dots,x_S \in \mathbb{Z}^d} \mathbb{P}_{p_c}(\boldsymbol{x}_S \longrightarrow \boldsymbol{x}) \prod_{i=1}^S \mathbb{P}_{p_c}(V_i < 4c_3(m+1)^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i). \tag{4.48}$$

We insert this into (4.44), and use (4.43), (4.3) and (4.26) to obtain

$$\begin{aligned} \mathbb{P}_\infty(V(R) < \lambda R^2) &\le \prod_{i=1}^S \left(\sum_{x_i \in \mathbb{Z}^d} \mathbb{P}_{p_c}(V_i < 4c_3(m+1)^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i)\right) \limsup_{N\to\infty} \frac{\tau_{N-n_S}}{\tau_N} \\ &= \Big[\tau_{2m}\mathbb{P}_{2m}(V(m) < 4c_3(m+1)^2)\Big]^S. \end{aligned} \tag{4.49}$$

By Lemma 4.3, the right-hand side equals

$$\tau_{2m}^S \Big[\mathbb{P}_\infty(V(m) < 4c_3(m+1)^2) + \mathcal{O}((m+1)^{(4-d)/2})\Big]^S. \tag{4.50}$$

By the choice of m_0 and (4.38), both terms inside the square brackets are at most $(3c)^{-1}$. Since

$$S = \left\lfloor \frac{c_3}{\lambda} \right\rfloor^{1/2} \ge \frac{1}{2}\left(\frac{c_3}{\lambda}\right)^{1/2},$$

it follows from our choice of c that

$$\mathbb{P}_\infty(V(R) < \lambda R^2) \le \tau_{2m}^S \left(\frac{2}{3c}\right)^S \le \left(\frac{2}{3}\right)^S \le \exp\{-\tfrac{1}{2}\log(3/2)c_3^{1/2}\lambda^{-1/2}\}. \tag{4.51}$$

The choice $c_2 = \frac{1}{2}\log(3/2)c_3^{1/2}$ gives (4.36). Noting that for $L \ge L_1$, c, c_3 and m_0 (and hence all further constants chosen) are independent of d, this completes the proof of Proposition 3.2.

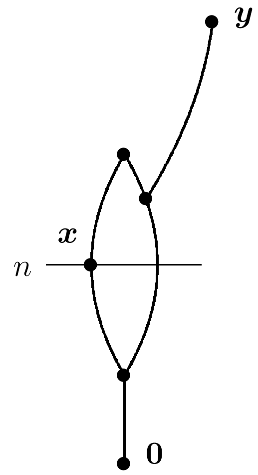

Figure 2: The configuration bounded in (5.4). The vertices $\boldsymbol{x} = (x, n)$ and $\boldsymbol{y} = (y, N)$ are summed over $x, y \in \mathbb{Z}^d$, and the three unlabelled vertices are summed over space and time.

5 IIC resistance estimates: Proof of Proposition 3.3

In this section we prove Proposition 3.3. According to (3.3),

$$D(n) = \left\{ e = (\boldsymbol{w}, \boldsymbol{x}) \subset \mathcal{C} : \begin{array}{l} |\boldsymbol{x}| = n, \ \boldsymbol{x} \text{ is RW-connected to} \\ \text{level } R \text{ by a path in } \mathcal{C} \cap U(n) \end{array} \right\}, \quad 0 < n \le R. \tag{5.1}$$

Our goal is to prove that

$$\mathbb{E}_\infty(|D(n)|) \le c_1(a, d), \quad 0 < n < aR, \quad 0 < a < 1. \tag{5.2}$$

In the rest of this section, we denote the spatial component of a vertex $\boldsymbol{x}, \boldsymbol{y}, \dots \in \mathbb{Z}^d \times \mathbb{Z}_+$ by $x, y, \dots$. Writing $\boldsymbol{y} = (y, N)$, by (4.27) and (4.3) we have

$$\begin{aligned} \mathbb{E}_\infty |D(n)| &= \sum_{w,x \in \mathbb{Z}^d} \mathbb{P}_\infty \left[(\boldsymbol{w}, \boldsymbol{x}) \in D(n) \right] \\ &= \frac{1}{A} \lim_{N \to \infty} \sum_{w,x,y \in \mathbb{Z}^d} \mathbb{P}_{p_c} \left[(\boldsymbol{w}, \boldsymbol{x}) \in D(n), \, \mathbf{0} \longrightarrow \boldsymbol{y} \right]. \end{aligned} \tag{5.3}$$

Hence we will focus on the event $\{(\boldsymbol{w}, \boldsymbol{x}) \in D(n), \, \mathbf{0} \longrightarrow \boldsymbol{y}\}$, for fixed n, $\boldsymbol{w} = (w, n-1)$, $\boldsymbol{x} = (x, n)$ and $\boldsymbol{y} = (y, N)$.

Remark. For a quick indication of why we need to assume $d > 6$, consider the configuration in Figure 2, which contributes to the right-hand side of (5.3). Using the fact that τ_n^{-1} and τ_n are both bounded by a constant by (4.4), and using (4.2) (see also (5.32) below), the configuration in Figure 2 can be bounded above by

$$c \sum_{l=n}^{\infty} \sum_{k=n}^{l} \sum_{j=0}^{n} (l-j+1)^{-d/2} \le c \sum_{l=n}^{\infty} \sum_{k=n}^{l} (l-n+1)^{(2-d)/2} \le c \sum_{l=n}^{\infty} (l-n+1)^{(4-d)/2} = c \sum_{m=1}^{\infty} m^{(4-d)/2}, \tag{5.4}$$

where j, k, l are the time coordinates of the unlabelled vertices, from bottom to top. The right-hand side is bounded only for $d > 6$. Our complete proof of (5.2) is more involved since we must estimate the contributions to (5.3) due also to more complex zigzag random walk paths.

In Section 5.1, we prove Lemma 5.1, which shows that the occurrence of $\{(\boldsymbol{w},\boldsymbol{x}) \in D(n),\ \mathbf{0} \longrightarrow \boldsymbol{y}\}$ implies the occurrence of various intersections. Then, in Section 5.2, we apply Lemma 5.1 to construct events $A_J = A_J(n,\boldsymbol{w},\boldsymbol{x},\boldsymbol{y})$, $J \geq 0$ such that

$$\{(\boldsymbol{w},\boldsymbol{x}) \in D(n),\ \mathbf{0} \longrightarrow \boldsymbol{y}\} \subset \bigcup_{J=0}^{\infty} A_J(n,\boldsymbol{w},\boldsymbol{x},\boldsymbol{y}). \tag{5.5}$$

In Section 5.3, the BK inequality [8] is used to obtain a diagrammatic bound for the probability of the event $A_J(n,\boldsymbol{w},\boldsymbol{x},\boldsymbol{y})$. Finally, in Section 5.4, we estimate the diagrams in this diagrammatic bound, to prove (5.2) and hence Proposition 3.3.

5.1 An intersection lemma

We will need the existence of certain intersections within the cluster $\mathcal{C}$ that are implied by the presence of a random walk path from $\boldsymbol{x}$ to R. These intersections are isolated in the following lemma. The following notation will be convenient:

$$\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})} = \{\boldsymbol{v} : \mathbf{0} \longrightarrow \boldsymbol{v} \text{ disjointly from the edge } (\boldsymbol{p},\boldsymbol{q})\}, \quad (\boldsymbol{p},\boldsymbol{q}) \subset \mathcal{C}.$$

Also, we write $\overline{\boldsymbol{y}_1\boldsymbol{y}_2}$ for an occupied oriented path $\boldsymbol{y}_1 \longrightarrow \boldsymbol{y}_2$. Such paths are in general not unique, but context will often identify a unique path for consideration.

Lemma 5.1. *Assume the event $\{(\boldsymbol{w},\boldsymbol{x}) \in D(n),\ \mathbf{0} \longrightarrow \boldsymbol{y}\}$. In addition, assume the following:*

(i) $(\boldsymbol{p},\boldsymbol{q}) \subset \mathcal{C}$ and either $\boldsymbol{q} \longrightarrow \boldsymbol{w}$ or $(\boldsymbol{p},\boldsymbol{q}) = (\boldsymbol{w},\boldsymbol{x})$;

(ii) $\boldsymbol{q} \not\longrightarrow R$;

(iii) A and B are subgraphs of $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$ with $\mathbf{0} \in A \cup B$, and such that $(A \cup B) \cap \mathcal{C}(\boldsymbol{q}) = \varnothing$;

(iv) every occupied oriented path from B to $\mathcal{C}(\boldsymbol{q})$ passes through a vertex of A.

Then there exist $\boldsymbol{p}' \in \overline{\boldsymbol{q}\boldsymbol{x}}$, $\boldsymbol{r} \in A$ and $\boldsymbol{z}$ with $|\boldsymbol{p}| < |\boldsymbol{z}| < R$, such that

$$\boldsymbol{p}' \longrightarrow \boldsymbol{z} \text{ and } \boldsymbol{r} \longrightarrow \boldsymbol{z} \text{ edge-disjointly, and edge-disjointly from } \overline{\boldsymbol{p}\boldsymbol{x}} \cup A \cup B.$$

Here $\boldsymbol{z}$ may coincide with $\boldsymbol{p}'$ or $\boldsymbol{r}$.

Proof. We first show that $\mathcal{C}(\boldsymbol{q})$ and $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$ must have a common vertex $\boldsymbol{v}$. Fix a random walk path Γ from $\boldsymbol{x}$ to R in $U(n)$, showing that $(\boldsymbol{w},\boldsymbol{x}) \in D(n)$. Note that $\mathcal{C}$ (as a set of vertices) is the union $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})} \cup \mathcal{C}(\boldsymbol{q})$. Since Γ starts at $\boldsymbol{x} \in \mathcal{C}(\boldsymbol{q})$, but $\boldsymbol{q} \not\longrightarrow R$, there is an edge $(\boldsymbol{v},\boldsymbol{v}') \subset \Gamma$ such that $\boldsymbol{v} \in \mathcal{C}(\boldsymbol{q})$ but $\boldsymbol{v}' \notin \mathcal{C}(\boldsymbol{q})$, and therefore $\boldsymbol{v}' \in \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$. We need to have $|\boldsymbol{v}'| = |\boldsymbol{v}| - 1$ (otherwise $\boldsymbol{v}' \in \mathcal{C}(\boldsymbol{q})$). We can rule out $(\boldsymbol{v}',\boldsymbol{v}) = (\boldsymbol{p},\boldsymbol{q})$, since Γ stays in $U(n)$, and $|\boldsymbol{p}| \leq n-1$. It follows that $\boldsymbol{v} \in \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$, and hence is in the intersection $\mathcal{C}(\boldsymbol{q}) \cap \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$.

Choose $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{q}) \cap \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$ with $|\boldsymbol{z}|$ minimal. Since $\boldsymbol{q} \not\longrightarrow R$, $|\boldsymbol{p}| < |\boldsymbol{q}| \leq |\boldsymbol{z}| < R$.

We can find occupied oriented paths $\overline{\boldsymbol{q}\boldsymbol{z}} \subset \mathcal{C}(\boldsymbol{q})$ and $\overline{\mathbf{0}\boldsymbol{z}} \subset \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$. These two paths must be edge-disjoint by minimality of $|\boldsymbol{z}|$. Let $\boldsymbol{p}'$ be the last visit of $\overline{\boldsymbol{q}\boldsymbol{z}}$ to $\overline{\boldsymbol{q}\boldsymbol{x}}$, and let $\boldsymbol{r}$ be the last visit of $\overline{\mathbf{0}\boldsymbol{z}}$ to $A \cup B$. Such a last visit exists, since we assumed $\mathbf{0} \in A \cup B$. Since $\boldsymbol{z} \notin A \cup B$, due to $(A \cup B) \cap \mathcal{C}(\boldsymbol{q}) = \varnothing$, the last visit has to be in A by assumption (iv).

The path $\overline{\boldsymbol{p}'\boldsymbol{z}}$ is edge-disjoint from $\overline{\boldsymbol{p}\boldsymbol{x}}$, by the definition of $\boldsymbol{p}'$. It is also edge-disjoint from $A \cup B$, by minimality of $|\boldsymbol{z}|$. Likewise, the path $\overline{\boldsymbol{r}\boldsymbol{z}}$ is edge-disjoint from $A \cup B$ by definition of $\boldsymbol{r}$. It is also edge-disjoint from $\overline{\boldsymbol{p}'\boldsymbol{z}}$, by minimality of $|\boldsymbol{z}|$. $\square$

Remark. Note that in the proof, we have first found a vertex $\boldsymbol{r} \in A \cup B$, and assumption (iv) was only used to show that we must have $\boldsymbol{r} \in A$. In fact, without assumption (iv), we would get the statement of the Lemma with $\boldsymbol{r} \in A \cup B$. The significance of being able to ensure that $\boldsymbol{r}$ is in the smaller set A, as well as the roles played by A and B will become apparent in Section 5.2.

5.2 The event $A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$

In this section, we define the event $A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$ and prove (5.5). The following lemma is key.

Lemma 5.2. *Let $e = (\boldsymbol{w}, \boldsymbol{x})$, and assume the event $\{e \in D(n), \mathbf{0} \longrightarrow \boldsymbol{y}\}$. Then there exists $J \geq 0$, such that the following vertices and paths (all edge-disjoint) exist:*

(i) vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_J = \boldsymbol{x}$ such that $0 \leq |\boldsymbol{v}_0| \leq |\boldsymbol{v}_1| \leq \dots \leq |\boldsymbol{v}_J| = n$;

(ii) vertices $\boldsymbol{u}_0, \boldsymbol{u}_1, \dots, \boldsymbol{u}_J = \boldsymbol{w}$, and, if $J \geq 1$, vertices $\boldsymbol{z}_1, \dots, \boldsymbol{z}_J$ such that

$$|\boldsymbol{u}_{i-1}| \leq |\boldsymbol{v}_{i-1}| \leq |\boldsymbol{z}_i|, \quad 1 \leq i \leq J; \tag{5.6}$$

$$|\boldsymbol{u}_{i-1}| < |\boldsymbol{z}_i| < R, \quad 1 \leq i \leq J; \tag{5.7}$$

(iii) $\mathbf{0} \longrightarrow \boldsymbol{u}_0$ and $\boldsymbol{u}_{i-1} \longrightarrow \boldsymbol{u}_i$, $1 \leq i \leq J$;

(iv) $\boldsymbol{u}_{i-1} \longrightarrow \boldsymbol{z}_i$, $1 \leq i \leq J$;

(v) $\boldsymbol{v}_{i-1}$ lies either on $\overline{\boldsymbol{u}_{i-1}\boldsymbol{u}_i}$ or $\overline{\boldsymbol{u}_{i-1}\boldsymbol{z}_i}$, and $\boldsymbol{v}_i \longrightarrow \boldsymbol{z}_i$, $1 \leq i \leq J$.

In addition, at least one of the following holds: Case (a) $\boldsymbol{v}_0 \longrightarrow \boldsymbol{y}$; Case (b) $\boldsymbol{v}_0 \longrightarrow R$ and there exists $\boldsymbol{v}_*$ on $\overline{\mathbf{0}\boldsymbol{u}_0}$ such that $\boldsymbol{v}_* \longrightarrow \boldsymbol{y}$.

Definition 5.3. We denote by $A_J = A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$ the event that the vertices and disjoint paths listed in Lemma 5.2 exist, and $(\boldsymbol{w}, \boldsymbol{x})$ is occupied. See Figure 3.

The inclusion (5.5) then follows immediately from Lemma 5.2.

Proof of Lemma 5.2. Throughout the proof, we assume the event $\{e = (\boldsymbol{w}, \boldsymbol{x}) \in D(n), \mathbf{0} \longrightarrow \boldsymbol{y}\}$.

We first show that if $\boldsymbol{x} \longrightarrow R$ then the lemma holds with $J = 0$. Indeed, take $\boldsymbol{u}_0 = \boldsymbol{w}$ and $\boldsymbol{v}_0 = \boldsymbol{x}$. Then $\mathbf{0} \longrightarrow \boldsymbol{u}_0$, since $\boldsymbol{u}_0 \in \mathcal{C}$. Hence it is left to show that at least one of Cases (a) and (b) holds. If $\boldsymbol{v}_0 = \boldsymbol{x} \longrightarrow \boldsymbol{y}$, then Case (a) holds. If not, then since $\mathbf{0} \longrightarrow \boldsymbol{y}$ we can find $\boldsymbol{v}_* \in \overline{\mathbf{0}\boldsymbol{u}_0}$ such that $\boldsymbol{v}_* \longrightarrow \boldsymbol{y}$ edge-disjointly from $\overline{\mathbf{0}\boldsymbol{u}_0}$. The connection $\overline{\boldsymbol{v}_*\boldsymbol{y}}$ has to be edge-disjoint from $\overline{\boldsymbol{w}\boldsymbol{x}R}$, otherwise we are in Case (a). Hence Case (b) holds.

For the rest of the proof, we assume $\boldsymbol{x} \not\longrightarrow R$.

We construct the paths claimed in the lemma recursively. Hence our proof will be based on a recursion hypothesis whose statement involves an integer $I \geq 0$, and which says that a subset of the paths claimed in the lemma (depending on I) have already been constructed. In order to advance the recursion, the hypothesis also specifies graphs A_I and B_I such that Lemma 5.1 can be applied with $A = A_I$ and $B = B_I$.

The outline of the proof is the following. Since the statement of the hypothesis for $I = 0$ is slightly different than for $I \geq 1$, we state and verify the hypothesis for $I = 0$ separately. This will show that the recursion can be started. Since the general step of the recursion is complex, we

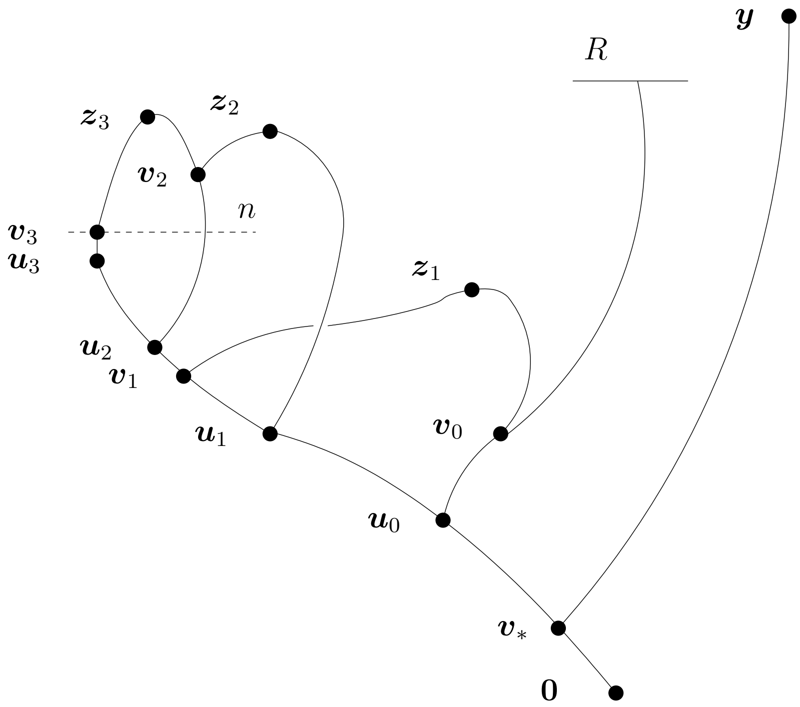

Figure 3: The vertices and disjoint paths of $A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$ for $J = 3$. Here $\boldsymbol{x} = \boldsymbol{v}_3$ and $\boldsymbol{w} = \boldsymbol{u}_3$.

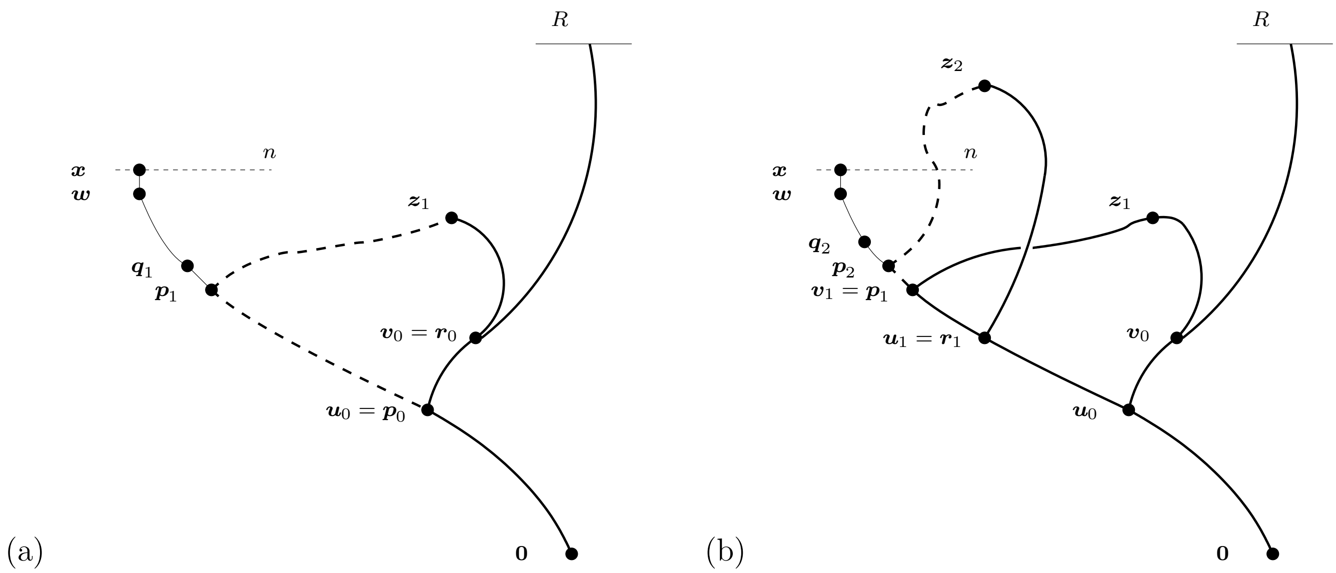

Figure 4: Assumptions of the recursion hypothesis for (a) $I = 1$; (b) $I = 2$. The thick solid lines indicate the sets (a) B_1 and (b) B_2, and the thick dashed lines the sets (a) A_1 and (b) A_2. The intersection lemma is used to produce paths that join the thick dashed lines to the thin solid lines.

explain the first two steps of the recursion ($I = 1$ and $I = 2$) in some detail, before formulating the recursion hypothesis precisely in the general case $I \geq 1$. The recursion will lead to the proof of the lemma by the following steps. We prove that if the hypothesis holds for some value of $I \geq 0$, then either the conclusion of Lemma 5.2 follows with $J = I + 1$, or else the hypothesis also holds for $I + 1$. If, for some $i > 0$, the hypothesis holds for $I = 0, 1, \ldots, i$, then its statement will guarantee the existence of vertices $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_i$ with

$$|\boldsymbol{p}_0| < |\boldsymbol{p}_1| < \cdots < |\boldsymbol{p}_i| < n. \tag{5.8}$$

Consequently the hypothesis cannot hold for all $I = 0, 1, \ldots, n$, and the implications just mentioned provide a proof of Lemma 5.2. We now carry out the details.

(R) Recursion hypothesis for $I = 0$**.** *There exists* $\boldsymbol{p}_0, \boldsymbol{q}_0$ *such that*

$$\mathbf{0} \longrightarrow \boldsymbol{p}_0, \qquad \boldsymbol{p}_0 \longrightarrow R, \tag{5.9}$$
$$\boldsymbol{p}_0 \longrightarrow \boldsymbol{w} \longrightarrow \boldsymbol{x}, \qquad \boldsymbol{q}_0 \not\longrightarrow R, \tag{5.10}$$

where $(\boldsymbol{p}_0, \boldsymbol{q}_0)$ *is the first edge in the path* $\overline{\boldsymbol{p}_0 \boldsymbol{x}}$*. All paths stated are edge-disjoint. Letting*

$$A_0 = \{\overline{\mathbf{0}\boldsymbol{p}_0}, \overline{\boldsymbol{p}_0 R}\} = \{\textit{paths in } (5.9)\},$$
$$B_0 = \varnothing,$$

the hypotheses of Lemma 5.1 are satisfied with $\boldsymbol{p} = \boldsymbol{p}_0$*,* $\boldsymbol{q} = \boldsymbol{q}_0$*,* $A = A_0$ *and* $B = B_0$*.*

Verification of (R) for $I = 0$**.** Since $\mathbf{0} \longrightarrow \boldsymbol{w}$ and $\mathbf{0} \longrightarrow R$, there exists $\boldsymbol{p}_0$ such that

$$\mathbf{0} \longrightarrow \boldsymbol{p}_0, \qquad \boldsymbol{p}_0 \longrightarrow \boldsymbol{w} \quad \text{and} \quad \boldsymbol{p}_0 \longrightarrow R \quad \text{disjointly.}$$

Fix the paths $\overline{\mathbf{0}\boldsymbol{p}_0}$, $\overline{\boldsymbol{p}_0 \boldsymbol{w}}$ and $\overline{\boldsymbol{p}_0 R}$, and let $(\boldsymbol{p}_0, \boldsymbol{q}_0)$ be the first step of the path $\overline{\boldsymbol{p}_0 \boldsymbol{x}}$. If we select $\boldsymbol{p}_0$ so that $|\boldsymbol{p}_0|$ is maximal, then we have $\boldsymbol{q}_0 \not\longrightarrow R$. We verify the hypotheses of Lemma 5.1 with these choices. First, (i), (ii) and $\mathbf{0} \in A_0 \cup B_0$ are immediate. Also, $\mathcal{C}(\boldsymbol{q}_0) \cap (A_0 \cup B_0) = \mathcal{C}(\boldsymbol{q}_0) \cap A_0 = \varnothing$, since otherwise $\boldsymbol{q}_0 \longrightarrow R$. Finally, (iv) is vacuous, since B_0 is empty.

Next, to illustrate the main idea of the proof, we explain the first two steps of the recursion.

Since we have verified **(R)** in the case $I = 0$, we can apply Lemma 5.1 with $\boldsymbol{p} = \boldsymbol{p}_0$, $\boldsymbol{q} = \boldsymbol{q}_0$, $A = A_0$ and $B = B_0$. Lemma 5.1 shows that there exist $\boldsymbol{p}' \in \overline{\boldsymbol{q}_0 \boldsymbol{x}}$ and $\boldsymbol{r} \in A_0 = \overline{\mathbf{0}\boldsymbol{p}_0} \cup \overline{\boldsymbol{p}_0 R}$ and a vertex $\boldsymbol{z}$ such that $\boldsymbol{p}' \longrightarrow \boldsymbol{z}$ and $\boldsymbol{r} \longrightarrow \boldsymbol{z}$. For reasons that will be explained later, we select $\boldsymbol{p}'$ with $|\boldsymbol{p}'|$ maximal such that the conclusions of Lemma 5.1 hold. With this choice of $\boldsymbol{p}'$, we set $\boldsymbol{p}_1 = \boldsymbol{p}'$, $\boldsymbol{z}_1 = \boldsymbol{z}$ and $\boldsymbol{r}_0 = \boldsymbol{r}$. Note that $|\boldsymbol{p}_1| > |\boldsymbol{p}_0|$. We define the vertices $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ as follows. Note that $\boldsymbol{r}_0 \in A_0$, which is the union of the paths $\overline{\mathbf{0}\boldsymbol{p}_0}$ and $\overline{\boldsymbol{p}_0 R}$. If $\boldsymbol{r}_0 \in \overline{\boldsymbol{p}_0 R}$ then we set $\boldsymbol{v}_0 = \boldsymbol{r}_0$ and $\boldsymbol{u}_0 = \boldsymbol{p}_0$, and if $\boldsymbol{r}_0 \in \overline{\mathbf{0}\boldsymbol{p}_0}$ then we set $\boldsymbol{v}_0 = \boldsymbol{p}_0$, $\boldsymbol{u}_0 = \boldsymbol{r}_0$. In either case, we have $|\boldsymbol{u}_0| \leq |\boldsymbol{p}_0| < |\boldsymbol{z}_1| < R$, and hence (5.7) holds for $i = 1$.

The paths constructed so far are depicted in Figure 4 (a). For the moment, the reader should disregard $\boldsymbol{q}_1$, and the distinction between thin, thick and dashed paths in the figure. We either have $|\boldsymbol{p}_1| < |\boldsymbol{x}| = n$, as depicted in Figure 4(a), or $\boldsymbol{p}_1 = \boldsymbol{x}$.

We first argue that in the case $\boldsymbol{p}_1 = \boldsymbol{x}$, Lemma 5.2 holds with $J = 1$. Indeed, if $\boldsymbol{p}_1 = \boldsymbol{x}$, we set $\boldsymbol{u}_1 = \boldsymbol{w}$ and $\boldsymbol{v}_1 = \boldsymbol{x}$. Then apart from the claim regarding Cases (a) and (b), the vertices and paths required by Lemma 5.2 for $J = 1$ have been constructed. (Note that the conclusion of

Lemma 5.1 guarantees that the newly constructed paths are edge-disjoint from the old ones.) It is not difficult to also show that either Case (a) or (b) holds, and we leave the details of this to when we deal with the general recursion step.

Next we explain how to continue the construction if $|\boldsymbol{p}_1| < |\boldsymbol{x}| = n$. Let $\boldsymbol{q}_1$ denote the first vertex on the path $\overline{\boldsymbol{p}_1\boldsymbol{x}}$ following $\boldsymbol{p}_1$. Let B_1 denote the union of the thick solid lines in Figure 4(a), that is, $B_1 = \overline{\boldsymbol{0}\boldsymbol{p}_0} \cup \overline{\boldsymbol{p}_0 R} \cup \overline{\boldsymbol{r}_0\boldsymbol{z}_1} = A_0 \cup \overline{\boldsymbol{r}_0\boldsymbol{z}_1}$. Let A_1 denote the union of the dashed lines in Figure 4(a), that is, $A_1 = \overline{\boldsymbol{p}_0\boldsymbol{p}_1} \cup \overline{\boldsymbol{p}_1\boldsymbol{z}_1}$. We want to apply Lemma 5.1 with $A = A_1$, $B = B_1$, etc. It is easy to verify conditions (i)–(iii) of the lemma. The crucial condition here is (iv), which allows us to conclude that $\boldsymbol{r} \in A_1$, and hence the two new paths produced by Lemma 5.1 will connect the dashed lines to the thin solid lines in Figure 4(a). The reason condition (iv) is satisfied is that we chose $|\boldsymbol{p}_1|$ to be maximal. Indeed, a glance at Figure 4(a) suggests that if we had paths from $\overline{\boldsymbol{q}_1\boldsymbol{x}}$ and $B_1 \setminus A_1$ to a vertex $\boldsymbol{z}$ that are edge-disjoint from $A_1 \cup B_1$, then that would contradict the maximality of $|\boldsymbol{p}_1|$. (Recall the earlier application of Lemma 5.1 with $A = A_0$, $B = B_0$, etc., and the choice of $\boldsymbol{p}_1$.) We will verify the details of this when we deal with the general case $I \geq 1$.

We can summarize the above discussion by saying that Hypothesis **(R)** for $I = 0$ should imply the in the case $\boldsymbol{p}_1 \neq \boldsymbol{x}$ the following statement holds.

(R) Recursion hypothesis for $I = 1$. *Vertices and paths (all edge-disjoint) with the following properties exist:*

(i) $\boldsymbol{p}_1$ *and* $\boldsymbol{q}_1$ *such that*

$$\boldsymbol{p}_1 \longrightarrow \boldsymbol{w} \longrightarrow \boldsymbol{x}, \qquad \boldsymbol{q}_1 \not\longrightarrow R, \tag{5.11}$$

where $(\boldsymbol{p}_1, \boldsymbol{q}_1)$ *is the first edge of the path* $\overline{\boldsymbol{p}_1\boldsymbol{x}}$, *and* $|\boldsymbol{p}_1| > |\boldsymbol{p}_0|$;

(ii) $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, $\boldsymbol{z}_1$, *such that*

$$\boldsymbol{0} \longrightarrow \boldsymbol{u}_0,\ \boldsymbol{u}_0 \longrightarrow \boldsymbol{z}_1,\ \boldsymbol{v}_0 \longrightarrow R; \tag{5.12}$$

(iii) $\boldsymbol{u}_0 \longrightarrow \boldsymbol{p}_1$;

(iv) $\boldsymbol{v}_0$ *lies either on* $\overline{\boldsymbol{u}_0\boldsymbol{p}_1}$, *in which case* $\boldsymbol{p}_0 = \boldsymbol{v}_0$, *or on* $\overline{\boldsymbol{u}_0\boldsymbol{z}_1}$, *in which case* $\boldsymbol{p}_0 = \boldsymbol{u}_0$;

(v) $\boldsymbol{p}_0 \longrightarrow \boldsymbol{p}_1 \longrightarrow \boldsymbol{z}_1$.

Letting

$$\begin{aligned} A_1 &= \{\overline{\boldsymbol{p}_0\boldsymbol{p}_1}, \overline{\boldsymbol{p}_1\boldsymbol{z}_1}\}, \\ B_1 &= A_0 \cup \{\overline{\boldsymbol{r}_0\boldsymbol{z}_1}\} = \{\textit{paths in } (5.12)\} \cup \{\overline{\boldsymbol{u}_0\boldsymbol{p}_0}\}, \end{aligned}$$

the hypotheses of Lemma 5.1 are satisfied with $\boldsymbol{p} = \boldsymbol{p}_1$, $\boldsymbol{q} = \boldsymbol{q}_1$, $A = A_1$ *and* $B = B_1$.

The next step of the construction is carried out similarly. An application of Lemma 5.1 gives the paths shown in Figure 4(b). Again, we chose $\boldsymbol{p}'$ so that $|\boldsymbol{p}'|$ is maximal, and set $\boldsymbol{p}_2 = \boldsymbol{p}'$, $\boldsymbol{z}_2 = \boldsymbol{z}$ and $\boldsymbol{r}_1 = \boldsymbol{r}$ for this choice of $\boldsymbol{p}'$. We define $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ depending on the location of $\boldsymbol{r}_1$, similarly to the previous step.

If $\boldsymbol{p}_2 = \boldsymbol{x}$, we can conclude similarly to the previous step that the lemma holds with $J = 2$. If $\boldsymbol{p}_2 \neq \boldsymbol{x}$, as in Figure 4(b), we advance the induction similarly to the previous step. This time, we use both the choice of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ to conclude the necessary statement about A_2 and B_2.

Now we state the recursion hypothesis in general for $I \geq 1$.

(R) Recursion hypothesis for $I \geq 1$. *Vertices and paths (all edge-disjoint) with the following properties exist:*

(i) $\boldsymbol{p}_I$ *and* $\boldsymbol{q}_I$ *such that*

$$\boldsymbol{p}_I \longrightarrow \boldsymbol{w} \longrightarrow \boldsymbol{x}, \qquad \boldsymbol{q}_I \not\longrightarrow R, \tag{5.13}$$

where $(\boldsymbol{p}_I, \boldsymbol{q}_I)$ *is the first edge of the path* $\overline{\boldsymbol{p}_I\boldsymbol{x}}$*, and* $|\boldsymbol{p}_I| > |\boldsymbol{p}_{I-1}|$*;*

(ii) $\boldsymbol{u}_i$, $0 \leq i < I$*;* $\boldsymbol{v}_i$, $0 \leq i < I$*;* $\boldsymbol{z}_i$, $1 \leq i \leq I$*, such that*

$$\text{Lemma 5.2 (iii) holds with } i \text{ restricted to } 1 \leq i < I, \tag{5.14}$$
$$\text{Lemma 5.2 (iv) holds with } i \text{ restricted to } 1 \leq i \leq I, \tag{5.15}$$
$$\text{Lemma 5.2 (v) holds with } i \text{ restricted to } 1 \leq i < I, \tag{5.16}$$
$$\boldsymbol{v}_0 \longrightarrow R; \tag{5.17}$$

(iii) $\boldsymbol{u}_{I-1} \longrightarrow \boldsymbol{p}_I$*;*

(iv) $\boldsymbol{v}_{I-1}$ *lies either on* $\overline{\boldsymbol{u}_{I-1}\boldsymbol{p}_I}$*, in which case* $\boldsymbol{p}_{I-1} = \boldsymbol{v}_{I-1}$*, or on* $\overline{\boldsymbol{u}_{I-1}\boldsymbol{z}_I}$*, in which case* $\boldsymbol{p}_{I-1} = \boldsymbol{u}_{I-1}$*;*

(v) $\boldsymbol{p}_{I-1} \longrightarrow \boldsymbol{p}_I \longrightarrow \boldsymbol{z}_I$*.*

Letting

$$A_I = \{\overline{\boldsymbol{p}_{I-1}\boldsymbol{p}_I}, \overline{\boldsymbol{p}_I\boldsymbol{z}_I}\},$$
$$B_I = B_{I-1} \cup A_{I-1} \cup \{\overline{\boldsymbol{r}_{I-1}\boldsymbol{z}_I}\} = \{\text{paths in } (5.14)\text{–}(5.17)\} \cup \{\overline{\boldsymbol{u}_{I-1}\boldsymbol{p}_{I-1}}\},$$

the hypotheses of Lemma 5.1 are satisfied with $\boldsymbol{p} = \boldsymbol{p}_I$*,* $\boldsymbol{q} = \boldsymbol{q}_I$*,* $A = A_I$ *and* $B = B_I$*.*

Figure 4 illustrates those paths of Figure 3 that have been constructed at the stages $I = 1$ and $I = 2$. Note that $\boldsymbol{p}_I$ receives either the label $\boldsymbol{u}_I$ or $\boldsymbol{v}_I$. Hence $\boldsymbol{p}_i$ will always equal either $\boldsymbol{u}_i$ or $\boldsymbol{v}_i$, depending on the location of $\boldsymbol{v}_i$ (by part (iv) of the hypothesis). Note also that (5.8) holds if **(R)** holds for all $I = 0, 1, \ldots, i$.

Consequence of (R): definition of $\boldsymbol{p}_{I+1}$, $\boldsymbol{u}_I$, $\boldsymbol{v}_I$ and $\boldsymbol{z}_{I+1}$. We now assume that **(R)** holds for some $I \geq 0$. An application of Lemma 5.1 with the data given in the hypothesis shows the existence of vertices $\boldsymbol{p}'$, $\boldsymbol{r}$ and $\boldsymbol{z}$ with certain properties. We now choose $\boldsymbol{p}'$ so that $|\boldsymbol{p}'|$ be maximal, and such that the properties claimed in Lemma 5.1 hold. We set $\boldsymbol{p}_{I+1} = \boldsymbol{p}'$, $\boldsymbol{z}_{I+1} = \boldsymbol{z}$ and $\boldsymbol{r}_I = \boldsymbol{r}$ for this choice.

Note that $\boldsymbol{r}_I \in A_I$, which is a union of two paths in both cases $I = 0$ and $I \geq 1$. In the case $I = 0$, if $\boldsymbol{r}_0 \in \overline{\boldsymbol{p}_0 R}$ then we set $\boldsymbol{v}_0 = \boldsymbol{r}_0$ and $\boldsymbol{u}_0 = \boldsymbol{p}_0$, and if $\boldsymbol{r}_0 \in \overline{\boldsymbol{0}\boldsymbol{p}_0}$ then we set $\boldsymbol{v}_0 = \boldsymbol{p}_0$, $\boldsymbol{u}_0 = \boldsymbol{r}_0$. Similarly, in the case $I \geq 1$, we set $\boldsymbol{v}_I = \boldsymbol{r}_I$ and $\boldsymbol{u}_I = \boldsymbol{p}_I$ if $\boldsymbol{r}_I \in \overline{\boldsymbol{p}_I\boldsymbol{z}_I}$, and we set $\boldsymbol{v}_I = \boldsymbol{p}_I$, $\boldsymbol{u}_I = \boldsymbol{r}_I$ if $\boldsymbol{r}_I \in \overline{\boldsymbol{p}_{I-1}\boldsymbol{p}_I}$. In both cases, it is clear that $|\boldsymbol{u}_I| \leq |\boldsymbol{p}_I| < |\boldsymbol{z}_{I+1}| < R$, and hence (5.7) holds for $i = I + 1$.

It follows immediately from these definitions, and from the disjointness properties ensured by Lemma 5.1, that assumptions (ii)–(v) of **(R)** now hold with I replaced by $I + 1$.

Verification of Lemma 5.2 if $\boldsymbol{p}_{I+1} = \boldsymbol{x}$. We show that if $\boldsymbol{p}_{I+1} = \boldsymbol{x}$, then Lemma 5.2 holds with $J = I + 1$. For this, we define $\boldsymbol{u}_{I+1} = \boldsymbol{w}$ and $\boldsymbol{v}_{I+1} = \boldsymbol{x}$. It is immediate from these definitions, from the disjointness properties ensured by Lemma 5.1, and from the already established properties (ii)–(v) of hypothesis **(R)** for $I + 1 = J$, that (i)–(v) of Lemma 5.2 hold.

It remains to show that either Case (a) or Case (b) holds. Since $\mathbf{0} \longrightarrow \boldsymbol{y}$, there exists $\boldsymbol{v}_* \in \overline{\mathbf{0}\boldsymbol{u}_0}$, such that $\boldsymbol{v}_* \longrightarrow \boldsymbol{y}$ disjointly from $\overline{\mathbf{0}\boldsymbol{u}_0}$. If $\overline{\boldsymbol{v}_*\boldsymbol{y}}$ is not disjoint from $\overline{\boldsymbol{v}_0 R}$, we are in Case (a), and we can ignore $\boldsymbol{v}_*$. If $\overline{\boldsymbol{v}_*\boldsymbol{y}}$ intersects $\overline{\boldsymbol{u}_0\boldsymbol{p}_0}$ or $\overline{\boldsymbol{u}_0\boldsymbol{z}_1}$, let $\boldsymbol{v}_0'$ be the last such intersection. Note that $\overline{\boldsymbol{v}_*\boldsymbol{y}}$ must be disjoint from all other paths constructed, since those are subsets of $\mathcal{C}(\boldsymbol{q}_0)$, and $\boldsymbol{q}_0 \not\longrightarrow R$. Hence if the intersection $\boldsymbol{v}_0'$ exists, we can replace $\boldsymbol{v}_0$ by $\boldsymbol{v}_0'$ and we are in Case (a). If the intersection $\boldsymbol{v}_0'$ does not exist, we are in Case (b). This verifies the claims of Lemma 5.2.

We are left to show that if $\boldsymbol{p}_{I+1} \neq \boldsymbol{x}$, then **(R)** must hold for $I + 1$.

Advancing the recursion $I \Longrightarrow I + 1$ if $\boldsymbol{p}_{I+1} \neq \boldsymbol{x}$. Since $\boldsymbol{p}_{I+1} \in \overline{\boldsymbol{q}_I\boldsymbol{x}}$, but $\boldsymbol{p}_{I+1} \neq \boldsymbol{x}$, we have $|\boldsymbol{p}_{I+1}| > |\boldsymbol{p}_I|$, and $\boldsymbol{p}_{I+1} \longrightarrow \boldsymbol{w}$, showing (i) of hypothesis **(R)**. We have already seen that (ii)–(v) are guaranteed to hold.

We are left to show that the hypotheses of Lemma 5.1 hold with the data given. (i), (ii) and $\mathbf{0} \in A_{I+1} \cup B_{I+1}$ are clear from the definitions. By the definition of $\boldsymbol{q}_{I+1}$, $A_{I+1} \cup B_{I+1}$ is a subgraph of $\tilde{\mathcal{C}}^{(\boldsymbol{q}_{I+1})}$.

Assume, for a contradiction, that we have $\boldsymbol{z}_* \in \mathcal{C}(\boldsymbol{q}_{I+1}) \cap (A_{I+1} \cup B_{I+1})$. Without loss of generality, assume that $\boldsymbol{z}_*$ is the first visit of an occupied path $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ to $A_{I+1} \cup B_{I+1}$. In particular, $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ is edge-disjoint from $A_{I+1} \cup B_{I+1}$. Observe that

$$A_{I+1} \cup B_{I+1} = A_{I+1} \cup A_I \cup B_I \cup \{\overline{\boldsymbol{r}_I\boldsymbol{z}_{I+1}}\}.$$

If we had $\boldsymbol{z}_* \in A_{I+1}$, then the disjoint paths $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*\boldsymbol{z}_{I+1}}$ and $\overline{\boldsymbol{r}_I\boldsymbol{z}_{I+1}}$ would satisfy the conclusions of Lemma 5.1 for $\boldsymbol{p} = \boldsymbol{p}_I$, $\boldsymbol{q} = \boldsymbol{q}_I$, etc. This contradicts the choice of $\boldsymbol{p}_{I+1}$ (the maximality of $|\boldsymbol{p}_{I+1}|$), since $|\boldsymbol{q}_{I+1}| > |\boldsymbol{p}_{I+1}|$. If we had $\boldsymbol{z}_* \in \overline{\boldsymbol{r}_I\boldsymbol{z}_{I+1}}$, we get a similar contradiction due to the paths $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ and $\overline{\boldsymbol{r}_I\boldsymbol{z}_*}$. Finally, we can rule out $\boldsymbol{z}_* \in A_I \cup B_I$, since $\mathcal{C}(\boldsymbol{q}_{I+1}) \subset \mathcal{C}(\boldsymbol{q}_I)$, and the latter is disjoint from $A_I \cup B_I$.

We are left to show that every occupied path from B_{I+1} to $\mathcal{C}(\boldsymbol{q}_{I+1})$ has to pass through A_{I+1}. Assume, for a contradiction, that there exists $\boldsymbol{z}_* \in \mathcal{C}(\boldsymbol{q}_{I+1})$, and $\boldsymbol{z}_*' \in B_{I+1}$ such that $\boldsymbol{z}_*' \longrightarrow \boldsymbol{z}_*$ disjointly from A_{I+1}. By considering the last visit, we may also assume that $\boldsymbol{z}_*'$ is the only vertex of $\overline{\boldsymbol{z}_*'\boldsymbol{z}_*}$ in $A_{I+1} \cup B_{I+1}$. We may also assume that $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ and $\overline{\boldsymbol{z}_*'\boldsymbol{z}_*}$ are edge-disjoint. We already saw $\mathcal{C}(\boldsymbol{q}_{I+1}) \cap (A_{I+1} \cup B_{I+1}) = \varnothing$, in particular, $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ is edge-disjoint from $A_{I+1} \cup B_{I+1}$. Observe that

$$B_{I+1} = A_I \cup B_I \cup \{\overline{\boldsymbol{r}_I\boldsymbol{z}_{I+1}}\} = \bigcup_{i=0}^{I} (A_i \cup \{\overline{\boldsymbol{r}_i\boldsymbol{z}_{i+1}}\}. \tag{5.18}$$

If we had $\boldsymbol{z}_*' \in \overline{\boldsymbol{r}_i\boldsymbol{z}_{i+1}}$, then the paths $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ and $\overline{\boldsymbol{r}_i\boldsymbol{z}_*'\boldsymbol{z}_*}$ would contradict the choice of $\boldsymbol{p}_{i+1}$. Finally, if we had $\boldsymbol{z}_*' \in A_i$, then the paths $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ and $\overline{\boldsymbol{z}_*'\boldsymbol{z}_*}$ would contradict the choice of $\boldsymbol{p}_{i+1}$. This completes the verification of hypothesis **(R)** for $I + 1$.

This completes the proof of Lemma 5.2. □

5.3 A diagrammatic bound

In this section, we use Lemma 5.2 and the BK inequality [8] to bound $\mathbb{P}_{p_c}[A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})]$. For this, we need the following preliminaries.

The *critical survival probability* is defined by

$$\theta_N = \mathbb{P}_{p_c}(\mathbf{0} \longrightarrow N). \tag{5.19}$$

The combined results of [21, 22] show that $\theta_N \sim cN^{-1}$ as $N \to \infty$, for $d > 4$ and $L \geq L_0(d)$, and for some $c = c(d, L) = 2 + \mathcal{O}(L^{-d})$. Moreover,

$$\theta_N \leq \frac{K'}{N}, \qquad N \geq 0, \quad L \geq L_0, \tag{5.20}$$

with the constant $K' = 5$ which is of course independent of both d and L (see [21, Eqn. (1.11)]).

To abbreviate the notation, when $\boldsymbol{y}_1 = (y_1, m_1)$ and $\boldsymbol{y}_2 = (y_2, m_2)$ we write $\tau(\boldsymbol{y}_1, \boldsymbol{y}_2) = \tau_{m_2 - m_1}(y_2 - y_1)$. We also introduce

$$\begin{aligned} U_1(\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{z}_1) &= \tau(\boldsymbol{v}_0, \boldsymbol{u}_1)\, \tau(\boldsymbol{u}_1, \boldsymbol{v}_1)\, \tau(\boldsymbol{v}_1, \boldsymbol{z}_1)\, \tau(\boldsymbol{u}_0, \boldsymbol{z}_1) \\ U_2(\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{z}_1) &= \tau(\boldsymbol{u}_0, \boldsymbol{u}_1)\, \tau(\boldsymbol{u}_1, \boldsymbol{v}_1)\, \tau(\boldsymbol{v}_1, \boldsymbol{z}_1)\, \tau(\boldsymbol{v}_0, \boldsymbol{z}_1) \\ U &= U_1 + U_2. \end{aligned} \tag{5.21}$$

For $0 \leq |\boldsymbol{u}_0| < n$ and $|\boldsymbol{u}_0| \leq |\boldsymbol{v}_0| < R$ and $\boldsymbol{y} = (y, N)$, let

$$\begin{aligned} \varphi(\boldsymbol{u}_0, \boldsymbol{v}_0) &= \sum_{y \in \mathbb{Z}^d} \tau(\mathbf{0}, \boldsymbol{u}_0)\, \tau(\boldsymbol{u}_0, \boldsymbol{v}_0)\, \tau(\boldsymbol{v}_0, \boldsymbol{y}), \\ \varphi_R(\boldsymbol{u}_0, \boldsymbol{v}_0) &= \sum_{y \in \mathbb{Z}^d} \sum_{\boldsymbol{v}_* \in \mathbb{Z}^d \times \mathbb{Z}_+} \tau(\mathbf{0}, \boldsymbol{v}_*)\, \tau(\boldsymbol{v}_*, \boldsymbol{u}_0)\, \tau(\boldsymbol{u}_0, \boldsymbol{v}_0)\, \theta_{R - |\boldsymbol{v}_0|}\, \tau(\boldsymbol{v}_*, \boldsymbol{y}) \\ \psi^{(0)}(\boldsymbol{u}_0, \boldsymbol{v}_0) &= \varphi(\boldsymbol{u}_0, \boldsymbol{v}_0) + \varphi_R(\boldsymbol{u}_0, \boldsymbol{v}_0). \end{aligned} \tag{5.22}$$

For $I \geq 1$, $0 \leq |\boldsymbol{u}_I| < n$ and $|\boldsymbol{u}_I| \leq |\boldsymbol{v}_0| < R$, let

$$\begin{aligned} \psi^{(I)}(\boldsymbol{u}_I, \boldsymbol{v}_I) = \sum_{\substack{\boldsymbol{u}_{I-1} \in \mathbb{Z}^d \times Z_+ \\ 0 \leq |\boldsymbol{u}_{I-1}| \leq |\boldsymbol{u}_I|}} \sum_{\substack{\boldsymbol{z}_I \in \mathbb{Z}^d \times Z_+ \\ |\boldsymbol{u}_I| < |\boldsymbol{z}_I| < R}} \sum_{\substack{\boldsymbol{v}_{I-1} \in \mathbb{Z}^d \times Z_+ \\ |\boldsymbol{u}_{I-1}| \leq |\boldsymbol{v}_{I-1}| \leq |\boldsymbol{z}_I|}} & U(\boldsymbol{u}_{I-1}, \boldsymbol{v}_{I-1}, \boldsymbol{u}_I, \boldsymbol{v}_I, \boldsymbol{z}_I) \\ & \times \psi^{(I-1)}(\boldsymbol{u}_{I-1}, \boldsymbol{v}_{I-1}). \end{aligned} \tag{5.23}$$

Lemma 5.4. *For $J \geq 0$,*

$$\sum_{y \in \mathbb{Z}^d} \mathbb{P}_{p_c}\left[A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})\right] \leq \psi^{(J)}(\boldsymbol{w}, \boldsymbol{x}). \tag{5.24}$$

Proof. Definition 5.3 guarantees that on the event $A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$ certain disjoint paths exist. If we fix the vertices $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_J$, $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_J$ and $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_J$, then the probability of the existence of the disjoint paths is bounded by the product of the probabilities of the existence of the individual paths, by the BK inequality [8]. An individual path $\overline{\boldsymbol{y}_1 \boldsymbol{y}_2}$ contributes a factor $\tau(\boldsymbol{y}_1, \boldsymbol{y}_2)$. Now summing the bound over all the vertices but $\boldsymbol{u}_J$ and $\boldsymbol{v}_J$, gives an upper bound on $\mathbb{P}_{p_c}[A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})]$. Further summing over $y \in \mathbb{Z}^d$ gives an upper bound on the left-hand side of (5.24).

Now it is merely a matter of bookkeeping to check that we get the expressions $\psi^{(J)}$. The terms φ and φ_R correspond to Cases (a) and (b) of Lemma 5.2, respectively, and their sum $\psi^{(0)}$ bounds the contribution of the paths constructed when we initialized the recursion, together with the path leading to $\boldsymbol{y}$. When $J = 0$, and we take $\boldsymbol{u}_0 = \boldsymbol{w}$ and $\boldsymbol{v}_0 = \boldsymbol{x}$, we get the bound in (5.24), with $J = 0$.

When $J \geq 1$, the recursive definition of $\psi^{(I)}$ reflects the recursion of Lemma 5.2. The factor $U = U_1 + U_2$ gives the contribution of the paths added in the I-th step: for U_1 these are $\overline{\boldsymbol{v}_{I-1}\boldsymbol{u}_I}$, $\overline{\boldsymbol{u}_I\boldsymbol{v}_I}$, $\overline{\boldsymbol{v}_I\boldsymbol{z}_I}$ and $\overline{\boldsymbol{u}_{I-1}\boldsymbol{z}_I}$ (when $\boldsymbol{v}_{I-1}$ lies on $\overline{\boldsymbol{u}_{I-1}\boldsymbol{u}_I}$), and for U_2 they are $\overline{\boldsymbol{u}_{I-1}\boldsymbol{u}_I}$, $\overline{\boldsymbol{u}_I\boldsymbol{v}_I}$, $\overline{\boldsymbol{v}_I\boldsymbol{z}_I}$ and $\overline{\boldsymbol{v}_{I-1}\boldsymbol{z}_I}$ (when $\boldsymbol{v}_{I-1}$ lies on $\overline{\boldsymbol{u}_{I-1}\boldsymbol{z}_I}$). Note that the path $\overline{\boldsymbol{u}_{I-1}\boldsymbol{v}_{I-1}}$ is not present in U, since it is taken care of inside $\psi^{(I-1)}$. □

5.4 Estimation of diagrams

It follows from (5.3), (5.5) and Lemma 5.4 that

$$\begin{aligned}\mathbb{E}_\infty |D(n)| &\leq \frac{1}{A} \limsup_{N\to\infty} \sum_{w,x,y\in\mathbb{Z}^d} \sum_{J=0}^{\infty} \mathbb{P}_{p_c}\left[A_J(n,\boldsymbol{w},\boldsymbol{x},\boldsymbol{y})\right] \\ &\leq \frac{1}{A} \limsup_{N\to\infty} \left[\sum_{J=0}^{\infty} \sum_{w,x\in\mathbb{Z}^d} \psi^{(J)}(\boldsymbol{w},\boldsymbol{x})\right]. \end{aligned} \tag{5.25}$$

In this section, we prove two lemmas which together imply that the right-hand side of (5.25) is bounded. This will prove (5.2) and hence Proposition 3.3. It is in Lemma 5.6, and only there, that we need to assume $d > 6$ rather than $d > 4$.

It suffices to prove that there are constants $c_2 = c_2(a, d)$ and $0 < c_3 < 1$, such that

$$\limsup_{N\to\infty} \sum_{w,x\in\mathbb{Z}^d} \psi^{(J)}(\boldsymbol{w},\boldsymbol{x}) \leq c_2 c_3^J, \quad J \geq 0, \tag{5.26}$$

since (4.3) and (5.25)–(5.26) then imply that

$$\mathbb{E}_\infty |D(n)| \leq \bar{K} c_2 \sum_{J=0}^{\infty} c_3^J = \bar{K} \frac{c_2}{1-c_3} = c_1(a,d).$$

We now state and prove two lemmas which imply (5.26). Their proofs use the bound

$$\tau_n \leq \bar{K}, \quad n \geq 0, \tag{5.27}$$

of (4.3), as well as (5.20). The first lemma gives a bound on $\psi^{(0)}$.

Lemma 5.5. *Let $d > 4$. Let $\boldsymbol{w} = (w, n-1)$ and $\boldsymbol{x} = (x, n)$. Then*

$$\limsup_{N\to\infty} \sum_{w,x\in\mathbb{Z}^d} \psi^{(0)}(\boldsymbol{w},\boldsymbol{x}) \leq (\bar{K}^3 + \bar{K}^4 K' a/(1-a)). \tag{5.28}$$

Proof. By definition and (5.27),

$$\sum_{w,x\in\mathbb{Z}^d} \varphi(\boldsymbol{w},\boldsymbol{x}) = \tau_{n-1}\tau_1\tau_{N-n} \leq \bar{K}^3.$$

Similarly, writing $\boldsymbol{v}_* = (v_*, l_*)$,

$$\sum_{w,x\in\mathbb{Z}^d} \varphi_R(\boldsymbol{w},\boldsymbol{x}) = \sum_{l_*=0}^{n-1} \tau_{l_*}\,\tau_{n-l_*-1}\,\tau_1\,\theta_{R-n}\,\tau_{N-l_*} \leq \frac{\bar{K}^4 K' n}{R-n}.$$

Since $n/(R-n) \leq a/(1-a)$ because $n < aR$, this gives (5.28). □

For $J \geq 1$, we use a somewhat stronger formulation of the bound, in which $|\boldsymbol{u}_J|$ and $|\boldsymbol{v}_J|$ are not restricted to the values $n-1$ and n. This will allow us to prove a bound on $\psi^{(J)}$ by induction.

Lemma 5.6. *Let $d > 6$. Suppose that $0 \leq k_J < n$, $k_J \leq l_J < R$, $\boldsymbol{u}_J = (u_J, k_J)$ and $\boldsymbol{v}_J = (v_J, l_J)$. Then*

$$\limsup_{N\to\infty} \sum_{u_J, v_J \in \mathbb{Z}^d} \psi^{(J)}(\boldsymbol{u}_J, \boldsymbol{v}_J) \leq (2\bar{K}^3 K^3 \beta)^J (\bar{K}^3 + 3\bar{K}^5 K' a/(1-a)), \quad J \geq 1. \tag{5.29}$$

Proof. We start by inserting the definition of $\psi^{(J)}$ into the left-hand side of (5.29). With $\boldsymbol{z}_J = (z_J, s_J)$, $\boldsymbol{u}_{J-1} = (u_{J-1}, k_{J-1})$ and $\boldsymbol{v}_{J-1} = (v_{J-1}, l_{J-1})$, the left-hand side of (5.29) equals

$$\begin{aligned} \limsup_{N\to\infty} \sum_{u_J, v_J \in \mathbb{Z}^d} \sum_{z_J, u_{J-1}, v_{J-1} \in \mathbb{Z}^d} \sum_{k_{J-1}=0}^{k_J} \sum_{s_J = l_J}^{R-1} \sum_{l_{J-1}=k_{J-1}}^{s_J} U(\boldsymbol{u}_{J-1}, \boldsymbol{v}_{J-1}, \boldsymbol{u}_J, \boldsymbol{v}_J, \boldsymbol{z}_J) \\ \times \psi^{(J-1)}(\boldsymbol{u}_{J-1}, \boldsymbol{v}_{J-1}). \end{aligned} \tag{5.30}$$

The vertices $\boldsymbol{u}_J$, $\boldsymbol{v}_J$ and $\boldsymbol{z}_J$ only appear in the factor U. We claim that

$$\sum_{u_J, v_J, z_J \in \mathbb{Z}^d} U(\boldsymbol{u}_{J-1}, \boldsymbol{v}_{J-1}, \boldsymbol{u}_J, \boldsymbol{v}_J, \boldsymbol{z}_J) \leq 2\bar{K}^3 K \beta (s_J - k_{J-1} + 1)^{-d/2}. \tag{5.31}$$

To see this, note that $s_J = |\boldsymbol{z}_J| > |\boldsymbol{u}_{J-1}| = k_{J-1}$, by (5.7). For the U_1 term, we use (4.2) to bound $\tau(\boldsymbol{u}_{J-1}, \boldsymbol{z}_J)$ by $K\beta(s_J - k_{J-1} + 1)^{-d/2}$. Then the sums over z_J, v_J and u_J contribute the factor $\bar{K}^3$, by using (5.27) for the other three factors in U_1. For the U_2 term, we apply (4.2) and $\tau_n \leq \bar{K}$ to see that

$$\sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \tau_n(y) \tau_m(x-y) \leq K\beta(n+m+1)^{-d/2}, \quad n+m \geq 1. \tag{5.32}$$

An application of (5.32) to the convolution of $\tau(\boldsymbol{u}_{J-1}, \boldsymbol{u}_J)$, $\tau(\boldsymbol{u}_J, \boldsymbol{v}_J)$ and $\tau(\boldsymbol{v}_J, \boldsymbol{z}_J)$, together with (5.27), yields an upper bound of the same form. This proves (5.31). Inserting (5.31) into (5.30) and rearranging, we get

$$\begin{aligned} (5.30) \leq 2\bar{K}^3 K\beta \sum_{k_{J-1}=0}^{k_J} \sum_{s_J = l_J}^{R-1} (s_J - k_{J-1} + 1)^{-d/2} \\ \times \sum_{l_{J-1}=k_{J-1}}^{s_J} \limsup_{N\to\infty} \sum_{u_{J-1}, v_{J-1} \in \mathbb{Z}^d} \psi^{(J-1)}(\boldsymbol{u}_{J-1}, \boldsymbol{v}_{J-1}). \end{aligned} \tag{5.33}$$

Now we prove (5.29) by induction on J. To start the induction, we verify (5.29) for $J = 1$. This is most of the work; advancing the induction is easy. When $J = 1$, the $\limsup$ in (5.33) consists of two terms, corresponding to φ and φ_R. The φ-term is bounded by

$$\limsup_{N\to\infty} \sum_{u_0, v_0 \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \tau(\boldsymbol{0}, \boldsymbol{u}_0)\, \tau(\boldsymbol{u}_0, \boldsymbol{v}_0)\, \tau(\boldsymbol{v}_0, \boldsymbol{y}) = \limsup_{N\to\infty} \tau_{k_0} \tau_{l_0 - k_0} \tau_{N - l_0} \leq \bar{K}^3. \tag{5.34}$$

Inserting this into (5.33), and assuming $d > 6$, we see that the φ contribution to (5.33) is bounded by

$$2\bar{K}^3 K\beta \bar{K}^3 \sum_{k_0=0}^{k_1} \sum_{s_1 = l_1}^{R-1} (s_1 - k_0 + 1)^{(2-d)/2} \leq (2\bar{K}^3 K^2 \beta)(\bar{K}^3). \tag{5.35}$$

The φ_R term is bounded as follows. First, the lim sup is bounded by

$$\begin{aligned}\limsup_{N\to\infty} \sum_{u_0,v_0\in\mathbb{Z}^d} \sum_{l_*=0}^{k_0} \sum_{y,v_*\in\mathbb{Z}^d} \tau(\mathbf{0},\boldsymbol{v}_*)\,\tau(\boldsymbol{v}_*,\boldsymbol{u}_0)\,\tau(\boldsymbol{u}_0,\boldsymbol{v}_0)\,\theta_{R-l_0}\,\tau(\boldsymbol{v}_*,\boldsymbol{y}) \\ \le \bar{K}^4 \sum_{l_*=0}^{k_0} \theta_{R-l_0} \le \bar{K}^4(k_0+1)\frac{K'}{R-l_0} \le \bar{K}^4 K' n \frac{1}{R-l_0}.\end{aligned} \tag{5.36}$$

We insert this bound into (5.33) to obtain

$$(2\bar{K}^3 K\beta)(\bar{K}^4K')n \sum_{k_0=0}^{k_1} \sum_{s_1=l_1}^{R-1} (s_1-k_0+1)^{-d/2} \sum_{l_0=k_0}^{s_1} \frac{1}{R-l_0}. \tag{5.37}$$

We split the sum over s_1 into the cases: (1) $s_1 < n+(R-n)/2$; (2) $s_1 \ge n+(R-n)/2$. In case (1), we have

$$\frac{1}{R-l_0} \le \frac{1}{R-s_1} \le \frac{2}{R-n}.$$

Inserting this into (5.37), the contribution of case (1) to the expression in (5.37) is bounded by

$$\begin{aligned}(2\bar{K}^3 K\beta)(2\bar{K}^4K')\frac{n}{R-n} \sum_{k_0=0}^{k_1} \sum_{s_1=l_1}^{n+(R-n)/2} (s_1-k_0+1)^{(2-d)/2} \\ \le (2\bar{K}^3K^2\beta)(2\bar{K}^4K')\frac{n}{R-n} \le (2\bar{K}^3K^2\beta)(2\bar{K}^4K')\frac{a}{1-a}.\end{aligned} \tag{5.38}$$

In case (2), since $n \ge k_1 \ge k_0$ we have

$$(s_1-k_0+1)^{-d/2} \le K(R-k_0+1)^{-d/2},$$

and the sum over l_0 in (5.37) is bounded by $\log(R-k_0+1) \le \bar{K}(R-k_0+1)^{\delta}$ for some fixed exponent δ (e.g., $\delta = 1/4$ suffices). Therefore the contribution of case (2) to the expression in (5.37) is bounded by

$$\begin{aligned}(2\bar{K}^3K^2\beta)(\bar{K}^5K')n\frac{R-n}{2} \sum_{k_0=0}^{k_1} (R-k_0+1)^{(2\delta-d)/2} \\ \le (2\bar{K}^3K^3\beta)(\bar{K}^5K')n\frac{R-n}{2}(R-n)^{(2\delta+2-d)/2} \\ \le (2\bar{K}^3K^3\beta)(\bar{K}^5K')\frac{n}{R-n}(R-n)^{(2\delta+6-d)/2} \\ \le (2\bar{K}^3K^3\beta)(\bar{K}^5K')\frac{a}{1-a}.\end{aligned} \tag{5.39}$$

Putting (5.38) and (5.39) together, we get that (5.37) is bounded by $(2\bar{K}^3K^3\beta)(3\bar{K}^5K'a/(1-a))$. Together with (5.35) this proves the $J=1$ case of (5.29).

To advance the induction, we assume now that (5.29) holds for an integer $J = M-1 \ge 1$, and prove that it holds for $J = M$. Using $d > 6$, we insert the bound (5.29) into (5.33) to get that the right-hand side of (5.33) is bounded by

$$(2\bar{K}^3 K\beta)(2\bar{K}^3 K^3\beta)^{M-1}(\bar{K}^3 + 3\bar{K}^5 K'a/(1-a)) \sum_{k_{M-1}=0}^{k_M} \sum_{s_M=l_M}^{R-1} (s_M - k_{M-1} + 1)^{(2-d)/2}$$
$$\leq (2\bar{K}^3 K^3\beta)^M (\bar{K}^3 + 3\bar{K}^5 K'a/(1-a)). \qquad (5.40)$$

This completes the proof of (5.29). □

Proof of (5.26). It follows immediately from Lemmas 5.5–5.6 that (5.26) holds with $c_2 = (\bar{K}^3 + 3\bar{K}^5 K'a/(1-a))$ and $c_3 = 2\bar{K}^3 K^3\beta$. Recall that the constant K' of (5.20) is independent of d and L. Choosing β small ensures that $0 < c_3 < 1$. This proves (5.26), and thus completes the proof of Proposition 3.3. □

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