

Random walks on supercritical percolation clusters

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Version 2.0, 8 August 2003

Abstract. We obtain Gaussian upper and lower bounds on the transition density $q_t(x, y)$ of the continuous time simple random walk on a supercritical percolation cluster \mathcal{C}_∞ in the Euclidean lattice. The bounds, analogous to Aronsen's bounds for uniformly elliptic divergence form diffusions, hold with constants c_i depending only on p (the percolation probability) and d . The irregular nature of the medium means that the bound for $q_t(x, \cdot)$ only holds for $t \geq S_x(\omega)$, where the constant $S_x(\omega)$ depends on the percolation configuration ω .

0. Introduction. In this paper we study the simple random walk on the infinite component of supercritical bond percolation in the lattice \mathbb{Z}^d . We recall the definition of percolation (see [Gm]): for edges $e = \{x, y\} \in \mathbb{E}_d = \{\{x, y\} : |x - y| = 1\}$, we have i.i.d. Bernoulli r.v. η_e , with $\mathbb{P}_p(\eta_e = 1) = p \in [0, 1]$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$. Edges e with $\eta_e = 1$ are called *open* and the open cluster $\mathcal{C}(x)$ containing x is the set of y such that x and y are connected by an open path. It is well known that there exists $p_c \in (0, 1)$ such that when $p > p_c$ there is a unique infinite open cluster, which we denote $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$.

For each ω let $Y = (Y_t, t \geq 0, P_\omega^x, x \in \mathcal{C}_\infty)$ be the continuous time simple random walk (CTSRW) on \mathcal{C}_∞ . Y is the process that waits an exponential mean 1 time at each vertex x , and then jumps along one of the open edges e containing x , with each edge chosen with equal probability. If we write $\nu_{xy}(\omega) = 1$ if $\{x, y\}$ is an open edge and 0 otherwise, and set $\mu(x) = \sum_y \nu_{xy}$, then Y is the Markov process with generator

$$\mathcal{L}_\omega f(x) = \mu(x)^{-1} \sum_y \nu_{xy} (f(y) - f(x)), \quad x \in \mathcal{C}_\infty. \quad (0.1)$$

A number of papers have studied this process, or the closely related discrete time random walk $X = (X_n, n \geq 0, \hat{P}_\omega^x, x \in \mathcal{C}_\infty)$. De Gennes [DG] discussed the link between the behaviour of X and resistance properties of \mathcal{C}_∞ , and coined the term ‘the ant in the

¹ Research partially supported by a NSERC (Canada) grant, and by CNRS (France).
AMS 2000 subject classification: 60K35

Key words and phrases: Percolation, random walks, heat kernel

labyrinth' to describe its motion. It is believed that, in all dimensions, there is no infinite cluster if $p = p_c$ (this is known to be true if $d = 2$ or $d \geq 19$). Kesten [Ke1] defined the *incipient infinite cluster* when $d = 2$, and proved in [Ke2] that the random walk on this set has subdiffusive behaviour.

For the supercritical case $p > p_c$ one expects Y to have similar long time behaviour to the random walk on \mathbb{Z}^d , and this has been confirmed in several ways. [DFGW] proved an invariance principle ($d = 2$), while [GKZ] proved that Y is recurrent if $d = 2$ and transient when $d \geq 3$. Let

$$q_t(x, y) = q_t^\omega(x, y) = P_\omega^x(Y_t = y)\mu(y)^{-1} \quad (0.2)$$

be the transition density of Y with respect to μ . More recently [HH] and [MR] have obtained estimates on $q_t(x, y)$. Theorem 1.2 of [MR] gives the (quenched) bound

$$q_t^\omega(0, y) \leq c_1(p, d)t^{-d/2}, \quad t \geq T_0(\omega), y \in \mathcal{C}_\infty,$$

\mathbb{P}_p -a.s. on the set $\{\omega : 0 \in \mathcal{C}_\infty\}$. A similar (annealed) estimate is given in [HH], but with an extra logarithmic factor.

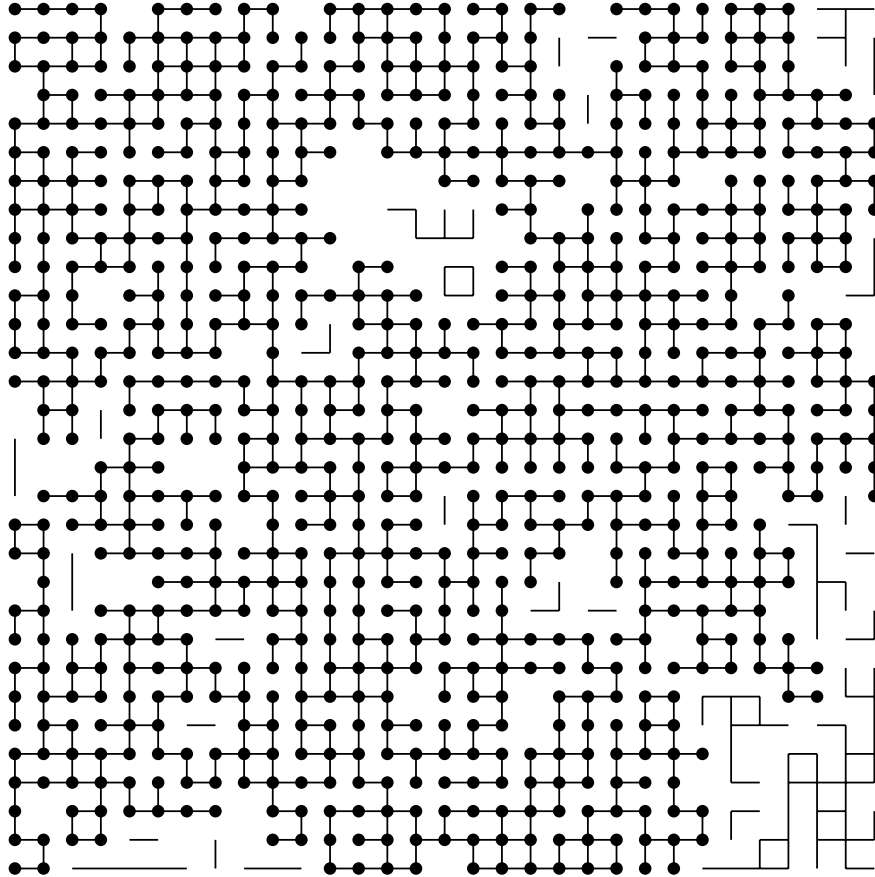


Figure 1: Bond percolation with $p = 0.53$. The vertices in the largest open cluster are marked with black circles.

The main result of this paper is the following two-sided bound on q_t .

Theorem 1. *Let $p > p_c$. There exists $\Omega_1 \subseteq \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$, and r.v. $S_x, x \in \mathbb{Z}^d$, such that $S_x(\omega) < \infty$ for each $\omega \in \Omega_1, x \in \mathcal{C}_\infty(\omega)$. There exist constants $c_i = c_i(d, p)$ such that for $x, y \in \mathcal{C}_\infty(\omega), t \geq 1$ with*

$$S_x(\omega) \vee |x - y|_1 \leq t, \quad (0.3)$$

the transition density $q_t^\omega(x, y)$ of Y satisfies:

$$c_1 t^{-d/2} e^{-c_2 |x-y|_1^2/t} \leq q_t^\omega(x, y) \leq c_3 t^{-d/2} e^{-c_4 |x-y|_1^2/t}. \quad (0.4)$$

Remarks. 1. $|x - y|_1 = \sum_{i=1}^d |x_i - y_i|$ is the usual graph distance on \mathbb{Z}^d .

2. The CTSRW on \mathbb{Z}^d (i.e. $p = 1$) satisfies these bounds with $S_x \equiv 1$. For $|x - y|_1 > t$ one has bounds which, up to constants, depend only on the tail of the Poisson distribution and not on the geometry of \mathcal{C}_∞ – see Lemma 1.1.

3. If G is any finite graph which can be embedded in \mathbb{Z}^d then \mathcal{C}_∞ contains infinitely many copies of G (attached at one point to the rest of \mathcal{C}_∞). Since (0.4) does not hold uniformly for all such graphs, it is clear that one cannot expect (0.4) for all x, y, t with $|x - y|_1 \leq t$. This irregularity of \mathcal{C}_∞ is taken care of by the random variable S_x ; after an initial period of possible bad behaviour $q_t(x, \cdot)$ settles down to a distribution with Gaussian tails.

4. In (0.4) one can replace $|x - y|_1$ by the graph distance in \mathcal{C}_∞ , and in fact this is the result that we prove first (Proposition 6.1).

5. The constants c_i are non-random, and depend only on p and d . For p sufficiently close to 1 it would in principle be possible to estimate them by careful tracking of the various constants in this paper. However, for general $p \in (p_c, 1)$ the constants arise in a ‘non effective’ fashion – if $p > p_c$ then one knows that certain kinds of good behaviour occur in cubes of side $k \geq k_0(p, d)$, but has no control on k_0 . The constants c_i then depend on k_0 .

6. The tail of the random variable S_x satisfies

$$\mathbb{P}_p(x \in \mathcal{C}_\infty, S_x \geq n) \leq c \exp(-c' n^{\varepsilon_d}); \quad (0.5)$$

see Lemma 2.24 and Section 6.

7. Similar bounds hold for the discrete time r.w. X . The proofs (which are not given here) run along the same lines, but with some extra (mainly minor) difficulties due to the discrete time. As \mathbb{Z}^d and \mathcal{C}_∞ are bipartite one needs to replace $q_t(x, y)$ in (0.4) by $p_{n-1}(x, y) + p_n(x, y)$.

8. It seems very likely that this theorem holds for other lattices in \mathbb{Z}^d , and the ‘random walk’ part of the proofs below transfers easily to other situations. But many of the percolation estimates that we use have only been proved for the Euclidean lattice.

Similar estimates hold in the annealed case.

Theorem 2. *Let $p > p_c$. There exist constants $c_i = c_i(d, p)$ such that for $x, y \in \mathbb{Z}^d$, $t \geq 1$ with $|x - y|_1 \leq t$,*

$$c_1 t^{-d/2} e^{-c_2 |x-y|_1^2/t} \leq \mathbb{E}_p(q_t^\omega(x, y) | x, y \in \mathcal{C}_\infty) \leq c_3 t^{-d/2} e^{-c_4 |x-y|_1^2/t}. \quad (0.6)$$

An immediate consequence of Theorem 1 is that Y is transient if and only if $d \geq 3$, but of course this was already known from [GKZ]. As an example of a new application, the ‘off-diagonal’ bounds in Theorem 1 enable one to control harmonic functions on \mathcal{C}_∞ . Write $d_\omega(x, y)$ for the graph distance on \mathcal{C}_∞ , and $B_\omega(x, R) = \{y : d_\omega(x, y) < R\}$ for balls. A function $h : B_\omega(x_0, R+1) \rightarrow \mathbb{R}$ is *harmonic* on $B_\omega(x, R)$ if $\mathcal{L}h(x') = 0$, $x' \in B_\omega(x, R)$. We have the following Harnack inequality.

Theorem 3. *Let $p > p_c$. There exists $c_1 = c_1(p, d)$ and $\Omega_1 \subseteq \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$, and $R_0(x, \omega)$ such that $R_0(x, \omega) < \infty$ for each $\omega \in \Omega_1$, $x \in \mathcal{C}_\infty$. If $R \geq R_0(x, \omega)$ and $h : B_\omega(x, 2R+1) \rightarrow (0, \infty)$ is a positive harmonic function on $B_\omega(x, 2R)$ then, writing $B = B_\omega(x, R)$,*

$$\sup_B h \leq c_1 \inf_B h. \quad (0.7)$$

This leads immediately to the Liouville property for positive harmonic functions.

Theorem 4. (a) *Let $h : \mathcal{C}_\infty \rightarrow \mathbb{R}$ be positive and harmonic on \mathcal{C}_∞ . Then h is constant.*
(b) *Let \mathcal{T} denote the tail σ -field of Y . There exists Ω_2 with $\mathbb{P}_p(\Omega_2) = 1$ such that for each $\omega \in \Omega_2$, and for each $F \in \mathcal{T}$ either $P_\omega^x(F) = 0$ for all $x \in \mathcal{C}_\infty(\omega)$ or else $P_\omega^x(F) = 1$ for all $x \in \mathcal{C}_\infty(\omega)$.*

Remark. The Liouville property for bounded harmonic functions on \mathcal{C}_∞ is already known – see Lemma 4.6 of [BLS] and [Ka].

We can also use Theorem 1 to estimate $E_\omega^x |Y_t - x|^2$.

Theorem 5. *Let $p > p_c$. There exists $\Omega_1 \subseteq \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$, and r.v. $S'_x, x \in \mathbb{Z}^d$, such that $S'_x(\omega) < \infty$ for each $\omega \in \Omega_1$, $x \in \mathcal{C}_\infty(\omega)$. There exist constants $0 < c_5 \leq c_6$ such that for $x \in \mathcal{C}_\infty(\omega)$, $t \geq S'_x$,*

$$c_5 t \leq E_\omega^x |Y_t - x|^2 \leq c_6 t. \quad (0.8)$$

This result implies that the (annealed) invariance principle, proved in [DFGW] for $d = 2$, can be extended to $d \geq 3$. See [SS] for a discussion of this, and for a much more delicate quenched invariance principle when $d \geq 4$.

The proof of Theorem 1 breaks into two fairly distinct parts. First (in section 2) we prove suitable geometric and analytic properties of \mathcal{C}_∞ . Then (sections 3–5) we use ‘heat kernel’ techniques to obtain the estimate (0.4). These techniques originate in the work of De Giorgi, Moser and Nash on divergence form elliptic equations; more recently

they have been employed to study random walks on graphs. While they have been very successful in a wide variety of algebraic and geometric contexts, this has almost always been in circumstances in which the same regularity condition holds for all balls of a given size r .

A guide to the kind of properties one needs is given by the following theorem of Delmotte, which is a translation to graphs of results of Grigor'yan [Gg] and Saloff-Coste [SC1] on manifolds. (The version given here has been adapted to the CTSRW on a positive density subgraph \mathcal{G} of \mathbb{Z}^d .)

Theorem A. (See [D1]). Let $\mathcal{G} = (G, E)$ be a subgraph of \mathbb{Z}^d with graph distance d . Let c_0, c_1, c_2 be positive constants. Suppose that \mathcal{G} satisfies the following two conditions:

(a) For all balls $B(x, R)$ in G :

$$c_0 R^d \leq \mu(B(x, R)) \leq c_1 R^d. \quad (V_d)$$

(b) For any ball $B = B(x, R)$ and function $f : B \rightarrow \mathbb{R}$, writing $\bar{f}_B = \sum_{x \in B} f(x) \mu(x) / \mu(B)$,

$$\sum_{x \in B} (f(x) - \bar{f}_B)^2 \mu(x) \leq c_2 R^2 \sum_{x \in B} \sum_{\substack{y \in B \\ y \sim x}} (f(x) - f(y))^2. \quad (PI)$$

Then the transition density $q_t(x, y)$ of Y satisfies, for $t \geq d(x, y) \vee 1$,

$$c_3 t^{-d/2} e^{-c_4 d(x, y)^2 / t} \leq q_t(x, y) \leq c_5 t^{-d/2} e^{-c_6 d(x, y)^2 / t}. \quad (0.9)$$

The first condition is of regular volume growth of balls in \mathcal{G} , and can be replaced by a more general ‘volume doubling’ condition. The second is a family of Poincaré or spectral gap inequalities for \mathcal{G} .

This theorem suggests that to prove Theorem 1 one should obtain volume growth and Poincaré inequalities for \mathcal{C}_∞ . Some results of this kind are in the literature – see [P], [DP] for volume growth estimates, and [BM] for a Poincaré inequality. These results show that for fixed $x \in \mathcal{C}_\infty$ the probability that (V_d) and (PI) hold for a ball $B_\omega(x, R)$ increases to 1 as $R \rightarrow \infty$.

However, on its own this is not enough to give Theorem 1. There are now several proofs of Theorem A ([D1] uses Moser iteration to prove a parabolic Harnack inequality), but all involve iterative methods or differential inequalities which use (V_d) and (PI) for many balls of different sizes. The exact definition of a ‘good’ and ‘very good’ balls is given in Section 1.7 below, but roughly speaking we say a ball $B_\omega(y, r)$ is good if (V_d) and (PI) hold, and a ball $B_\omega(x, R)$ is very good if all sub-balls $B_\omega(y, r) \subseteq B_\omega(x, R)$ are good for $r \geq R^{1/(d+2)}$. We need to prove that all sufficiently large balls $B_\omega(x_0, R)$ (centered at a fixed x_0) are very good, and to do this we have to extend some of the estimates in the

literature. This is done in Section 2 below. The estimates in [P], [DP] are enough for the volume growth bounds, but more work is needed for the Poincaré inequality. As in [BM] we prove this from an isoperimetric inequality, which is obtained in [BM] and [MR]. We will use the methods of those papers, but the need for better control of the probabilities means that we have to rework some of these arguments to identify more precisely the set of percolation configurations ω for which a ball is good or very good. If p is sufficiently close to 1 then a fairly direct counting argument (see [BM], [MR]) is all that is needed, but for general $p > p_c$ one has to use a renormalization argument, as in [AP]. In this paper we follow quite closely the approach in [MR] – there is a gap in the renormalization argument in [BM].

While percolation arguments generally use cubes in \mathbb{Z}^d , the heat kernel estimates work most naturally if we use the ‘chemical’ or graph distance $d_\omega(x, y)$ on \mathcal{C}_∞ . We can compare these two metrics using the main theorem of [AP].

Many of the methods used to derive (0.9) from (V_d) and (PI) are not suitable for the percolation context. For example, it is proved in [SC1] that (PI) and (V_d) imply a ‘Nash estimate’: for $f : G \rightarrow \mathbb{R}$,

$$\int |\nabla f|^2 \geq c \|f\|_2^{2+4/d} \|f\|_1^{-4/d}, \quad (N)$$

and it is proved in [CKS] that (N) is equivalent to:

$$q_t(x, y) \leq c' t^{-d/2} \text{ for all } x, y \in G, t \geq 1. \quad (0.10)$$

However, since \mathcal{C}_∞ contains copies of $\{0, \dots, n\}$ for all n , (0.10) is clearly false for \mathcal{C}_∞ . More generally, one cannot use any method which relies on global Sobolev inequalities; it is necessary to use ‘local’ methods. (One approach here might be that of rooted or anchored isoperimetric inequalities, as in [T] or [BLS]. However, the correct extension to Nash or Sobolev inequalities has not yet been made.)

In Section 3 we obtain an initial global upper bound on q_t using the Poincaré inequality directly, following the approach of Kusuoka and Zhou [KZ]. One can then obtain the off-diagonal upper bound in (0.4) using a method of Nash [N], Bass [Bas] and [BB1].

We use the method of Fabes and Stroock [FS], also based on ideas of Nash, to obtain a local lower bound – i.e. for $q_t(x, y)$ if $d(x, y) \leq t^{1/2}$. However, a difficulty arises in extending this to prove (0.4) for points x, y with $d_\omega(x, y) \approx t^{1-\varepsilon}$. The standard technique is ‘chaining’: using a sequence of small balls $B(z_i, r)$ connecting x and y , and the Chapman-Kolmogorov equations. It turns out that one needs to take $r \approx t/d_\omega(x, y)$, so that one may need balls so small that one cannot be sure that they are very good. This problem is resolved by an additional percolation argument: for some fixed $r_1 = r_1(p, d) \gg 1$ one can show that the collection of good balls of size $r \geq r_1$ is large enough so that a suitable chain $(B(z_i, r), 0 \leq i \leq m)$ of very good balls exists with z_0 close to x and z_m close to y .

(See Theorems 2.23 and 5.4). This argument needs renormalization techniques, even if p is close to 1.

Section 1 contains a brief account of various known facts on random walks on graphs which will be used in the rest of the paper. The percolation arguments are given in Section 2. Sections 3–5 have been written for a general graph \mathcal{G} satisfying appropriate volume growth and Poincaré inequalities, and can be read independently of Section 2. Upper bounds on q_t are obtained in Section 3. Section 4 proves a weighted Poincaré inequality from the unweighted (weak) Poincaré inequalities derived in Section 2, using methods of [SCS] and [J]. This is then used in section 5 to obtain lower bounds on q_t . Section 6 then ties these results together, and gives the proofs of Theorems 1–4.

We use c_i to denote constants whose values are fixed within each argument, and C to denote constants fixed within each section. $c_{2.3.1}$ denotes the constant c_1 of Lemma 2.3. c and c' are constants whose values may change on each appearance. The constants C_i , c_i , c , c' are always strictly positive. The notation $k = k(p, d)$ means that the constant k depends only on p and d .

1. Graphs and random walks.

In this section we review some well known facts concerning graphs, random walks, and isoperimetric, Cheeger and Poincaré inequalities. Let $\mathcal{G} = (G, E)$ be an infinite, locally finite, connected graph. We define weights ν_{xy} by

$$\nu_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{otherwise,} \end{cases}$$

and set $\mu(x) = \sum_y \nu_{xy}$. We extend μ to a measure on G , and ν to a measure on E . Given a function $f : G \rightarrow \mathbb{R}$, we define $|\nabla f| : E \rightarrow \mathbb{R}$ by $|\nabla f|(e) = |f(x) - f(y)|$ if $e = \{x, y\}$. We write

$$\int f = \int f d\mu = \sum_{x \in V} f(x) \mu(x), \quad \int |\nabla f|^p = \int |\nabla f|^p d\nu = \sum_{e \in E} (|\nabla f|(e))^p \nu_e.$$

Let $d(x, y)$ be the graph distance on G , and define $B(x, r) = \{y : d(x, y) < r\}$. We assume we have a global upper bound on the size of balls: for any $x \in G$, $r \geq 1$:

$$\mu(B(x, r)) \leq C_0 r^d. \tag{1.1}$$

Note that this implies that for each $x \in G$,

$$1 \leq \mu(x) \leq C_0. \tag{1.2}$$

Let $Y = (Y_t, t \geq 0, P^x, x \in G)$ be the continuous time random walk on G ; this is the Markov process with generator

$$\mathcal{L}f(x) = \mu(x)^{-1} \sum_y \nu_{xy} (f(y) - f(x)). \tag{1.3}$$

Thus Y waits at x for an exponential mean 1 random time and then moves to a neighbour of x at random. We define the transition density of Y with respect to μ (or heat kernel density) by

$$q_t(x, y) = \mu(y)^{-1} P^x(Y_t = y).$$

Note that by (1.2) we have $q_t(x, y) \leq 1$ for all x, y, t . Given any points $x_0, \dots, x_k \in G$ and times t_1, \dots, t_k then by the Markov property, if $t = \sum_k t_k$,

$$\begin{aligned} q_t(x_0, x_k) &\geq \mu(x_k)^{-1} \prod_{i=1}^k P^{x_{i-1}}(Y_{t_i} = x_i) \\ &= \mu(x_k)^{-1} \prod_{i=1}^k q_{t_i}(x_{i-1}, x_i) \mu(x_i) \geq \prod_{i=1}^k q_{t_i}(x_{i-1}, x_i). \end{aligned} \quad (1.4)$$

We begin by recalling some general bounds on q_t . These are not given in full generality, but just as they apply to the situation here.

Lemma 1.1. *Let \mathcal{G} satisfy (1.1).*

(a) *There exist constants $c_i = c_i(d, C_0)$ such that, writing $D = d(x, y)$,*

$$q_t(x, y) \leq c_1 e^{-c_2 D^2/t}, \quad D \leq t, \quad (1.5)$$

$$c \exp(-c_3 D(1 + \log \frac{D}{t})) \leq q_t(x, y) \leq c' \exp(-c_4 D(1 + \log \frac{D}{t})), \quad D \geq t \geq 1, \quad (1.6)$$

and

$$\frac{c_3}{(t \log t)^{d/2}} \leq q_t(x, x) \leq \frac{c_4}{t^{1/2}}, \quad t \geq 1. \quad (1.7)$$

(b) *If $d(x, y) \geq R \geq 2$ and $t \leq c_5 R^2 / \log R$ then*

$$q_t(x, y) \leq c_6 t^{-d}. \quad (1.8)$$

Proof. (a) See Corollaries 11 and 12 of [Da] for (1.5) and (1.6). For the discrete time random walk X on \mathcal{G} the lower bound in (1.7) is immediate from [CG, Theorem 2.2] and (1.1), while the upper bound follows from [CG, Theorem 2.3] and the fact that $\mu(B(x, r)) \geq r$ for all $r \geq 1$. These bounds then transfer to q_t by integration.

(b) If $t < c_5 R^2 / \log R$ then $t \log t \leq 2c_5 R^2$ provided $R \geq c_7$. Hence $t^{-d} \geq \exp(-2dc_5 R^2/t)$ and, taking c_5 sufficiently small, the bound (1.8) is an easy consequence of (1.5). If $R < c_8$ then $t \leq c_9$ and (1.8) still holds on adjusting the constant c_6 . \square

We now review some geometric and analytic inequalities on \mathcal{G} . Let $H \subseteq G$ be finite, write

$$E(H) = \{e = \{x, y\} : x, y \in H\},$$

and call $\mathcal{H} = (H, E(H))$ the *induced subgraph* on H . We define the measures μ_0 on H and ν_0 on $E(H)$ in the same way as μ and ν are defined for \mathcal{G} . Note that while ν_0 and ν agree on $E(H)$, we only have

$$\mu_0(A) \leq \mu(A) \leq C_0 \mu_0(A), \quad A \subseteq H. \quad (1.9)$$

We now assume that \mathcal{H} is connected. For $A_1, A_2 \subseteq H$ let

$$\partial_E(A_1, A_2) = \{e = \{x, y\} : x \in A_1, y \in A_2\}. \quad (1.10)$$

Let

$$i(A) = \frac{\nu(\partial_E(A, H - A))}{\mu_0(A)}, \quad A \subseteq H,$$

and define the isoperimetric constant

$$I_H = \min_{0 < \mu_0(A) \leq \frac{1}{2}\mu_0(H)} i(A).$$

Closely related is the Cheeger constant; let

$$\chi(A) = \frac{\mu_0(H)\nu(\partial_E(A, H - A))}{\mu_0(A)\mu_0(H - A)},$$

and define

$$J_H = \min_{A \neq \emptyset, H} \chi(A). \quad (1.11)$$

Lemma 1.2. *The minimum in (1.11) is attained by a set A such that A and $H - A$ are connected.*

This is quite well known; for a proof see for example [MR, Section 3.1].

Lemma 1.3. (See [MR, Section 3.1], [BM].) *Let H be finite and connected.*

(a) $I_H \geq 2/\mu_0(H)$.

(b) If $I_H^* = \min\{i(A) : 0 < \mu(A) \leq \frac{1}{2}\mu(H), A \text{ and } H - A \text{ are connected}\}$ then

$$I_H \leq I_H^* \leq 2I_H. \quad (1.12)$$

Proof. (a) Let $0 < \mu_0(A) \leq \frac{1}{2}\mu_0(H)$. Then since H is connected $\partial_E(A, H - A)$ is non-empty, so $i(A) \geq 1/\mu_0(A) \geq 2/\mu_0(H)$.

(b) The left-hand bound in (1.12) is obvious. If $0 < \mu_0(A) \leq \frac{1}{2}\mu_0(H)$ then since $1 \leq \mu_0(H)/\mu_0(H - A) \leq 2$ we have $i(A) \leq \chi(A) \leq 2i(A)$. This immediately implies that $I_H \leq J_H \leq 2I_H$. Let A be a minimal set for J_H ; by Lemma 1.2 we can assume that A and $H - A$ are connected. We can also take $\mu_0(A) \leq \frac{1}{2}\mu_0(H)$. Then $I_H^* \leq i(A) \leq \chi(A) = J_H \leq 2I_H$, proving the right-hand bound in (1.12). \square

Proposition 1.4. *Let $H \subseteq G$ be finite. Suppose that $I_H \geq \alpha^{-1}$.*

(a) *If $f : H \rightarrow \mathbb{R}$ then*

$$\min_a \int_H |f - a|^2 d\mu_0 \leq c_1 \alpha^2 \int_{E(H)} |\nabla f|^2 d\nu.$$

(b) *If $f : H \rightarrow \mathbb{R}$ then*

$$\min_a \int_H |f - a|^2 d\mu \leq c_1 C_0 \alpha^2 \int_{E(H)} |\nabla f|^2 d\nu. \quad (1.13)$$

Proof. (a) This result is well known; for a recent proof see [SC2, Lemma 3.3.7].

(b) Since one can use (1.9) to replace μ_0 by μ this follows immediately from (a). \square

The inequality (1.13) is a *Poincaré* or *spectral gap* inequality for H . The minimum in (1.13) is of course attained by the value $a = \bar{f}_H = \int_H f d\mu / \mu(H)$. The following result is immediate from Lemma 1.3 and Proposition 1.4.

Corollary 1.5. *Let $H \subseteq G$ be finite and connected, and P_H be the best constant in the Poincaré inequality (1.13).*

(a) $P_H \leq c\mu(H)^2$.

(b) *If $i(A) \geq \alpha^{-1}$ for all $A \subseteq H$ such that A and $H - A$ are connected then $P_H \leq c\alpha^2$.*

We note the following discrete version of the Gauss-Green lemma.

Lemma 1.6. *Let $f, g \in L^2(G, \mu)$. Then*

$$\sum_{x \in G} g(x) \mathcal{L}f(x) \mu(x) = \frac{1}{2} \sum_x \sum_y (f(x) - f(y))(g(x) - g(y)) \nu_{xy}. \quad (1.14)$$

In the sequel we will need the following definitions.

Definition 1.7. Let C_V, C_P , and $C_W \geq 1$ be fixed constants. We say $B(x, r)$ is (C_V, C_P, C_W) -good if:

$$C_V r^d \leq \mu(B(x, r)), \quad (1.15)$$

and the weak Poincaré inequality

$$\int_{B(x, r)} (f - \bar{f}_{B(x, r)})^2 d\mu \leq C_P r^2 \int_{E(B(C_W x, r))} |\nabla f|^2 d\nu \quad (1.16)$$

holds for every $f : B(x, C_W r) \rightarrow \mathbb{R}$.

We say $B(x, R)$ is (C_V, C_P, C_W) -very good if there exists $N_B = N_{B(x, R)} \leq R^{1/(d+2)}$ such that $B(y, r)$ is good whenever $B(y, r) \subseteq B(x, R)$, and $N_B \leq r \leq R$. We can always assume that $N_B \geq 1$. Usually the values of C_V, C_P, C_W will be clear from the context and we will just use the terms ‘good’ and ‘very good’.

2. Percolation estimates

We will work with both bond and site percolation in \mathbb{Z}^d . We regard \mathbb{Z}^d as a graph, with edge set $\mathbb{E}_d = \{\{x, y\} : |x - y| = 1\}$, and write $x \sim y$ to mean $\{x, y\} \in \mathbb{E}_d$. Given $Q \subseteq \mathbb{Z}^d$ define the internal and external boundaries of $A \subseteq Q$ by

$$\begin{aligned}\partial_i(A|Q) &= \{y \in A : y \sim x \text{ for some } x \in A^c \cap Q\}, \\ \partial_e(A|Q) &= \{y \in A^c \cap Q : y \sim x \text{ for some } x \in A\} = \partial_i(Q - A|Q).\end{aligned}$$

We begin with the notation for site percolation. Let $q \in (0, 1)$, and $\Omega_s = \{0, 1\}^{\mathbb{Z}^d}$, and define the coordinate maps $\zeta_x(\omega) = \omega(x)$. Let \mathbb{Q}_q be the probability measure on Ω_s which makes the ζ_x i.i.d. Bernoulli r.v. with $\mathbb{Q}_q(\zeta_x = 1) = q$. We call those x such that $\zeta_x = 1$ the *open sites*, and write $\mathcal{O} = \mathcal{O}(\omega) = \{x : \zeta_x = 1\}$.

For $A \subseteq \mathbb{Z}^d$ we define the graph distance $d_A(x, y)$ to be the smallest k such that there exists a path $\gamma = \{x_0, x_1, \dots, x_k\} \subseteq A$ with $x = x_0$, $x_k = y$, and $x_{i-1} \sim x_i$, $1 \leq i \leq k$. If there is no such k then $d_A(x, y) = \infty$. (We have $d_A(x, x) = \infty$ if $x \notin A$.) We write $d_\omega(x, y) = d_{\mathcal{O}(\omega)}(x, y)$, and will refer to a path $x = x_0, x_1, \dots, x_k = y$ such that each x_i is open as an *open path*. We say A is *connected* if $d_A(x, y) < \infty$ for all $x, y \in A$.

Now write

$$\mathcal{C}(x) = \{y : d_\omega(x, y) < \infty\}$$

for the connected open cluster containing x . Write also, given $Q \subseteq \mathbb{Z}^d$,

$$\mathcal{C}_Q(x) = \{y : d_{Q \cap \mathcal{O}(\omega)}(x, y) < \infty\}.$$

This is the set of points connected to x by an open path within Q . We call sets of the form $\mathcal{C}(x)$ *open clusters*, and the sets $\mathcal{C}_Q(x)$ *open Q -clusters*. It is well known that there exists $q_c = q_c(d) \in (0, 1)$ such that if $q > q_c$ then \mathbb{Q}_q -a.s. there is a unique infinite open cluster \mathcal{C}_∞ . However, for site percolation we will only be interested in the case when q is either close to 1 or close to 0. Given a cube $Q \subseteq \mathbb{Z}^d$ the set $Q \cap \mathcal{C}_\infty$ will in general not be connected. We write

$$\mathcal{C}^\vee(Q) \text{ for the largest open } Q\text{-cluster.}$$

(We adopt some procedure for breaking ties.)

Write $|x - y|_\infty = \max_i |x_i - y_i|$, let $\mathbb{E}_d^* = \{\{x, y\} : |x - y|_\infty = 1\}$, and write $x \overset{*}{\sim} y$ if $\{x, y\} \in \mathbb{E}_d^*$. We will also need to consider site percolation in the graph $(\mathbb{Z}^d, \mathbb{E}_d^*)$. We say that $A \subseteq \mathbb{Z}^d$ is **-connected* if A is connected in the graph $(\mathbb{Z}^d, \mathbb{E}_d^*)$, and define the clusters $\mathcal{C}^*(x)$ analogously.

Definition 2.1.

1. Let Q be a cube of side n in \mathbb{Z}^d . We write $s(Q) = n$ for the side length of Q . Let $Q^+ = A_1 \cap \mathbb{Z}^d$, and $Q^\oplus = A_2 \cap \mathbb{Z}^d$, where A_1 and A_2 are the cubes in \mathbb{R}^d with the same center as Q and sides $\frac{3}{2}n$ and $\frac{6}{5}n$ respectively.

2. A cluster \mathcal{C} in a cube Q is *crossing* for a cube $Q' \subseteq Q$ if for all d directions there exists an open path in $\mathcal{C} \cap Q'$ connecting the two opposing faces of Q' .
3. The *diameter* of a set A is defined by $\text{diam}(A) = \max\{|x - y|_\infty : x, y \in A\}$.
4. Given a set A we write $|A|$ for the number of elements in A .

Remark. In the arguments in this section, we will frequently need to assume that a cube Q is sufficiently large. More precisely, we will need that $s(Q) \geq n_1$, where $n_1 = n_1(d, p)$ is a constant depending only on d and p . We will make this assumption in our proofs whenever necessary without stating it explicitly each time. Unless otherwise indicated the statements of the results will be true for all n – this can be ensured by adjusting the constants so that the result is automatic for small cubes.

Let Q be a cube in \mathbb{Z}^d . Define the event

$$K(Q, \lambda) = \{\omega : \mathcal{C}^\vee(Q) \text{ is crossing for } Q \text{ and } \frac{|\mathcal{C}^\vee(Q)|}{|Q|} > \lambda\}. \quad (2.1)$$

The following estimate is proved in [DP, Theorem 1.1].

Lemma 2.2. *Let Q be a cube of side n and $\lambda < 1$. Then there exists $q_0 = q_0(d, \lambda) < 1$ and $c_i = c_i(\lambda, d)$ such that if $q \in [q_0, 1)$ then*

$$\mathbb{Q}_q(K(Q, \lambda)^c) \leq c_1 e^{-c_2 n^{d-1}}.$$

Let $\Sigma = \{x \in \mathbb{Z}^d : x \sim 0\} \cup \{0\}$. For $\sigma \in \Sigma$ define the shifted set of open sites $\mathcal{O}_\sigma = \{x - \sigma : x \in \mathcal{O}\}$. Let

$$\beta = 1 - 2(1 + d)^{-1} < (d - 1)/d. \quad (2.2)$$

Set for $r \in \mathbb{N}$, $\varepsilon > 0$,

$$\begin{aligned} F(Q, r, \sigma, \varepsilon) &= \{\text{any } *\text{-connected set } A \subseteq Q \text{ with } |A| = r \\ &\quad \text{satisfies } |A \cap \mathcal{O}_\sigma| \geq (1 - \varepsilon)|A|\}, \\ F(Q, \varepsilon) &= \bigcap_{r \geq s(Q)^\beta} \bigcap_{\sigma \in \Sigma} F(Q, r, \sigma, \varepsilon). \end{aligned}$$

Note that the event $F(Q, r, \sigma, \varepsilon)$ is increasing, and that while $K(Q, \lambda)$ is not in general increasing, it is if $\lambda > \frac{1}{2}$.

Recall the following bounds on the tail of the binomial.

Lemma 2.3. *Let $X \sim \text{Binomial}(n, p)$ and $\lambda \in (0, 1)$. Then*

$$\mathbb{P}(X < \lambda n) \leq e^{-nb(\lambda, p)},$$

where $b(\lambda, p) \rightarrow \infty$ as $p \rightarrow 1$ for each fixed $\lambda \in (0, 1)$.

Lemma 2.4. (See [Gm, Sect. 4.2]). The number of $*$ -connected sets A with $|A| = r$ containing a fixed point $x_0 \in \mathbb{Z}^d$ is bounded by $e^{c_1 r}$.

Lemma 2.5. Let $\varepsilon \in (0, 1)$, and Q be a cube of side n . There exists $q_1 = q_1(\varepsilon, d) > q_c$ such that if $q \geq q_1$ then

$$\mathbb{Q}_q(F(Q, \varepsilon)^c) \leq c \exp(-n^\beta).$$

Proof. If A is a fixed connected set in Q with $|A| = r$ then by Lemma 2.3

$$\mathbb{Q}_q(|A \cap \mathcal{O}_\sigma| < (1 - \varepsilon)|A|) \leq e^{-rb(1-\varepsilon, q)}.$$

Given ε , choose q_1 large enough so that $b(\lambda, p) > 2 + c_{2.4.1}$ for $\lambda \geq 1 - \varepsilon$, $q \geq q_1$. Then since there are at most $(n + 1)^d e^{c_{2.4.1} r}$ $*$ -connected sets in Q of size r ,

$$\mathbb{Q}_q(F(Q, r, \sigma, \varepsilon)^c) \leq (n + 1)^d e^{c_{2.4.1} r - rb(1-\varepsilon, q)} \leq n^d e^{-2r},$$

and, as $|\Sigma| = 2d + 1$,

$$\mathbb{Q}_q(F(Q, \varepsilon)^c) \leq \sum_{r=n_0}^{\infty} (2d + 1)(n + 1)^d e^{-2r} \leq cn^d e^{-2n^\beta} \leq c' e^{-n^\beta}. \quad \square$$

We collect from [DP] the following results on the boundaries of discrete sets contained in cubes.

Lemma 2.6. Let Q be a cube in \mathbb{Z}^d .

(a) Let $A \subsetneq Q$ be $*$ -connected. Let Λ_j , $1 \leq j \leq k$ be the connected components of $Q - A$. Then $\partial_i(\Lambda_j|Q)$, $\partial_e(\Lambda_j|Q)$, $1 \leq j \leq k$ are $*$ -connected.

(b) Let $A \subseteq Q$ with $|A| \leq (15/16)|Q|$. Then

$$|A| \leq c_1 |\partial_i(A|Q)|^{d/(d-1)} \text{ and } |A| \leq c_1 |\partial_e(A|Q)|^{d/(d-1)}. \quad (2.3)$$

Proof. (a) is Lemma 2.1(ii) of [DP], while the discrete isoperimetric inequality (2.3) is the assertion (A.3) on p. 480 of [DP]. \square

The following result is based on ideas in [MR]. Recall from (1.10) the definition of $\partial_E(A_1, A_2)$.

Proposition 2.7. Let $\varepsilon < 1/(4d + 2)$ and $\lambda \geq \frac{7}{8}$. Suppose that both $F(Q, \varepsilon)$ and $K(Q, \lambda)$ occur for a cube Q with side n . Let $A \subseteq Q$ be connected.

(a) If $A \cap \mathcal{C}^\vee(Q) = \emptyset$ then $|A| < cn^{\beta d/(d-1)} < n$.

(b) If $|A| \leq \frac{3}{4}|Q|$ and $A \cap \mathcal{C}^\vee(Q) \neq \emptyset$ then there exists c_1 such that

$$|\partial_E(A \cap \mathcal{C}^\vee(Q), (Q - A) \cap \mathcal{C}^\vee(Q))| \geq c_1 n^{-1} |A|. \quad (2.4)$$

Proof. We write $\mathcal{C}^\vee = \mathcal{C}^\vee(Q)$.

(a) Let A_0 be the connected component of $Q - \mathcal{C}^\vee$ containing A . By Lemma 2.6 $\partial_i(A_0|Q)$ is $*$ -connected. The definition of A_0 implies that $\partial_i(A_0|Q) \cap \mathcal{O} = \emptyset$. Since $F(Q, \varepsilon)$ occurs, this implies that $|\partial_i(A_0|Q)| < n^\beta$ (since otherwise $|\partial_i(A_0|Q) \cap \mathcal{O}| > 0$). Hence by the discrete isoperimetric inequality (2.3)

$$|A| \leq |A_0| \leq c|\partial_i(A_0|Q)| \leq cn^{\beta d/(d-1)} < n.$$

(b) Now let $A \cap \mathcal{C}^\vee \neq \emptyset$, and $|A| \leq n$. So there exists $x \in A \cap \mathcal{C}^\vee$. Since $|\mathcal{C}^\vee| \geq \frac{7}{8}n^d$, there exists $y \in (Q - A) \cap \mathcal{C}^\vee$, and as \mathcal{C}^\vee is connected there therefore exists $\{x', y'\} \in \partial_E(A \cap \mathcal{C}^\vee(Q), (Q - A) \cap \mathcal{C}^\vee(Q))$. So

$$|\partial_E(A \cap \mathcal{C}^\vee(Q), (Q - A) \cap \mathcal{C}^\vee(Q))| \geq 1 \geq n^{-1}|A|.$$

It remains to consider the case $A \cap \mathcal{C}^\vee \neq \emptyset$, $|A| > n$. Let

$$\Lambda_1 = \bigcup \{\mathcal{C}_Q(y) : y \in \partial_e(A|Q) \cap \mathcal{O} - \mathcal{C}^\vee\}, \quad A_1 = A \cup \Lambda_1.$$

Let C_i , $1 \leq i \leq k$ be the connected components of $Q - A_1$, and

$$\Lambda_2 = \bigcup \{C_i : C_i \cap \mathcal{C}^\vee = \emptyset\}, \quad A_2 = A_1 \cup \Lambda_2.$$

It is clear from the construction of A_1 and A_2 as a union of connected sets each connected to A or A_1 that A_2 is connected. Note that since $\Lambda_1 \cup \Lambda_2 \subseteq Q - \mathcal{C}^\vee$, and $K(Q, \lambda)$ holds, we have $|A_2| \leq |A| + |Q - \mathcal{C}^\vee| \leq \frac{7}{8}|Q|$.

We now show that

$$\partial_e(A|Q) \cap \mathcal{C}^\vee = \partial_e(A_2|Q) \cap \mathcal{O}. \quad (2.5)$$

First let $y \in \partial_e(A|Q) \cap \mathcal{C}^\vee$, so that $y \sim x$ with $x \in A$. As $y \in \mathcal{C}^\vee$ we cannot have $y \in \Lambda_1 \cup \Lambda_2$, so $y \notin A_2$, and thus $y \in \partial_e(A_2|Q) \cap \mathcal{O}$.

Now let $y \in \partial_e(A_2|Q) \cap \mathcal{O}$, so that $y \sim z$ for some $z \in A_2$. If $z \in \Lambda_2$ then $z \in C_i$ for some i . But as $y \notin A_1$, and $y \sim z$ we would have $y \in C_i$, a contradiction. So $z \in A_1$. If $z \in \Lambda_1$ then $z \in \mathcal{C}(x)$ for some $x \in \partial_e(A|Q) \cap \mathcal{O} - \mathcal{C}^\vee$. Again we have $y \in \mathcal{C}(x)$, giving a contradiction. So we must have $z \in A$, and therefore $y \in \partial_e(A|Q)$. If $y \notin \mathcal{C}^\vee$ then, as $y \in \mathcal{O}$, $\mathcal{C}(y)$ would be included in Λ_1 , and y would be in A_1 . Hence we have $y \in \mathcal{C}^\vee$. This completes the proof of (2.5).

Let $\Gamma_1, \dots, \Gamma_l$ be the connected components of $Q - A_2$, arranged in decreasing order of size.

Case 1. Suppose that $|\Gamma_1| > \frac{1}{2}|Q|$. Let $A_3 = A_2 \cup (\bigcup_{i=2}^l \Gamma_i)$. By Lemma 2.6 $\partial_e(A_3|Q) = \partial_i(\Gamma_1|Q)$ is $*$ -connected. We also have $\partial_e(A_3|Q) \subseteq \partial_e(A_2|Q)$. Note that since $n < |A| \leq |A_3| \leq \frac{1}{2}|Q|$, by the isoperimetric inequality

$$|\partial_e(A_3|Q)| \geq c_2|A_3|^{1-1/d} \geq c_2n^{1-1/d} > n^\beta.$$

Since $F(Q, \varepsilon)$ holds, writing $\mathcal{O}' = \cap_{\sigma \in \Sigma} \mathcal{O}_\sigma$,

$$|\partial_e(A_3|Q) \cap (\mathcal{O}')^c| \leq \sum_{\sigma} |\partial_e(A_3|Q) \cap (\mathcal{O}_\sigma)^c| \leq (2d+1)\varepsilon |\partial_e(A_3|Q)|.$$

We deduce

$$|\partial_e(A_3|Q) \cap \mathcal{O}'| \geq (1 - (2d+1)\varepsilon) |\partial_e(A_3|Q)| \geq \frac{1}{2} |\partial_e(A_3|Q)|.$$

If $y \in \partial_e(A_3|Q) \cap \mathcal{O}'$ then $y \in \partial_e(A_2|Q) \cap \mathcal{O}$ and therefore by (2.5) $y \in \partial_e(A|Q) \cap \mathcal{C}^\vee$. So there exists $x \in A$ with $y \sim x$. As $y \in \mathcal{O}'$ we have $x \in \mathcal{O}$, and thus $x \in \mathcal{C}^\vee$. Hence $\{x, y\} \in \partial_E(A \cap \mathcal{C}^\vee, (Q - A) \cap \mathcal{C}^\vee)$. Thus

$$\begin{aligned} \partial_E(A \cap \mathcal{C}^\vee, (Q - A) \cap \mathcal{C}^\vee) &\geq |\partial_e(A_3|Q) \cap \mathcal{O}'| \\ &\geq \frac{1}{2} |\partial_e(A_3|Q)| \\ &\geq \frac{1}{2} c_2 |A_3|^{1-1/d} \geq \frac{1}{2} c_2 n^{-1} |A_3| \geq \frac{1}{2} c_2 n^{-1} |A|, \end{aligned}$$

proving (2.4) in this case.

Case 2. Suppose $|\Gamma_1| \leq \frac{1}{2} |Q|$. Note that

$$|\partial_E(A \cap \mathcal{C}^\vee, (Q - A) \cap \mathcal{C}^\vee)| = \sum_j |\partial_E(\Gamma_j \cap \mathcal{C}^\vee, (Q - \Gamma_j) \cap \mathcal{C}^\vee)|.$$

We have

$$|\partial_i(\Gamma_j|Q) \cap \mathcal{O}'| \leq |\partial_E(\Gamma_j \cap \mathcal{C}^\vee, (Q - \Gamma_j) \cap \mathcal{C}^\vee)|. \quad (2.6)$$

For if $y \in \partial_i(\Gamma_j|Q) \cap \mathcal{O}'$ then $y \in \Gamma_j$ and there exists $x \in A_2$ with $x \sim y$. So $y \in \partial_e(A_2|Q) \cap \mathcal{O}$ and hence $y \in \partial_e(A|Q) \cap \mathcal{C}^\vee$ by (2.5). As $y \in \mathcal{O}'$ then $x \in \mathcal{O}$, so that $x \in \mathcal{C}^\vee$, and hence $x \in A$ since x cannot be in $\Lambda_1 \cup \Lambda_2$. Therefore $\{y, x\} \in \partial_E(\Gamma_j \cap \mathcal{C}^\vee, (Q - \Gamma_j) \cap \mathcal{C}^\vee)$.

Next, each Γ_j intersects \mathcal{C}^\vee by the construction of A_2 . If $|\partial_i(\Gamma_j|Q)| \geq n^\beta$ then

$$|\partial_i(\Gamma_j|Q) \cap \mathcal{O}'| \geq \frac{1}{2} |\partial_i(\Gamma_j|Q)| \geq c |\Gamma_j|^{1-1/d} \geq c' n^{-1} |\Gamma_j|.$$

If $|\partial_i(\Gamma_j|Q)| < n^\beta$ then $|\Gamma_j| \leq c n^{\beta d/(d-1)} < n$ and

$$|\partial_E(\Gamma_j \cap \mathcal{C}^\vee, (Q - \Gamma_j) \cap \mathcal{C}^\vee)| \geq 1 \geq n^{-1} |\Gamma_j|.$$

Combining these estimates we obtain

$$\begin{aligned} |\partial_E(A \cap \mathcal{C}^\vee, (Q - A) \cap \mathcal{C}^\vee)| &= \sum_j |\partial_E(\Gamma_j \cap \mathcal{C}^\vee, (Q - \Gamma_j) \cap \mathcal{C}^\vee)| \\ &\geq c n^{-1} \sum_j |\Gamma_j| \geq c n^{-1} (|Q - A_2|). \end{aligned}$$

As $|Q - A_2| \geq \frac{1}{8}n^d \geq c|A|$ this proves (2.4). \square

We now follow the renormalization argument in [MR], which uses techniques introduced in [AP]. We introduce a second percolation process, which is bond percolation on $(\mathbb{Z}^d, \mathbb{E}_d)$. Let $p \in (0, 1)$: we set $\Omega_b = \{0, 1\}^{\mathbb{E}_d}$, let η_e , $e \in \mathbb{E}_d$ be the coordinate maps and let \mathbb{P}_p on Ω_b be the probability measure which makes η_e i.i.d. Bernoulli r.v. with $\mathbb{P}_p(\eta_e = 1) = p$. We call the edges e such that $\eta_e = 1$ *open edges*, and write $\mathcal{O}_E = \{e : \eta_e = 1\}$. An *open path* is any path $\gamma = \{x_0, \dots, x_k\}$ such that each $\{x_{i-1}, x_i\} \in \mathcal{O}_E$. As for site percolation we write $d_\omega(x, y)$ for the length of the shortest open path connecting x and y , and set $d_\omega(x, y) = \infty$ if there is no such open path. For a set A we write $d_A(x, y)$ for the length of the shortest open path contained in A connecting x and y . Set $\mathcal{C}(x) = \{y : d_\omega(x, y) < \infty\}$. Let $\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty)$ and $p_c = \inf\{p : \theta(p) > 0\}$; then if $p > p_c$ there exists \mathbb{P}_p -a.s. a unique infinite cluster. We will always assume that $p > p_c$. We define \mathcal{C}_∞ , $\mathcal{C}_Q(x)$ and $\mathcal{C}^\vee(Q)$ in the same way as for site percolation.

Let $n \geq 16$ be fixed, and Q be a cube of side n . Recall the definition of Q^+ and crossing clusters from Definition 2.1. We define the event $R_0(Q)$ in a similar fashion to the event $R_i^{(n)}$ in [AP], and set

$$\begin{aligned} R_0(Q) = \{ & \text{there exists a unique crossing cluster } \mathcal{C} \text{ in } Q^+ \text{ for } Q^+, \text{ all open paths} \\ & \text{contained in } Q^+ \text{ of diameter greater than } \frac{1}{8}n \text{ are connected to } \mathcal{C} \text{ in } Q^+ \\ & \text{and } \mathcal{C} \text{ is crossing for each cube } Q' \subseteq Q \text{ with } s(Q') \geq n/8\}, \end{aligned}$$

$$R(Q) = R_0(Q) \cap \{\mathcal{C}^\vee(Q) \text{ is crossing for } Q\} \cap \{\mathcal{C}^\vee(Q^+) \text{ is crossing for } Q^+\}.$$

Note that if $\omega \in R(Q)$ then this forces $\mathcal{C}^\vee(Q) \subset \mathcal{C}^\vee(Q^+)$, and that $\mathcal{C}^\vee(Q^+)$ is the unique crossing cluster given by the event $R_0(Q)$.

Now let $k \geq 17$ and consider a tiling of \mathbb{Z}^d by disjoint cubes

$$T(x) = \{y \in \mathbb{Z}^d : x_i \leq y_i < x_i + k, 1 \leq i \leq d\} \quad (2.7)$$

with side $k - 1$. Let

$$\varphi(x) = 1_{R(T(x))}. \quad (2.8)$$

Lemma 2.8. (a) *Let Q be a cube of side $k - 1$, and $p > p_c$. There exists $c_1 = c_1(p, d)$ such that*

$$\mathbb{P}_p(R(Q)^c) \leq ce^{-c_1 k}. \quad (2.9)$$

(b) *The process $(\varphi_x, x \in \mathbb{Z}^d)$ dominates Bernoulli site percolation with parameter $q^*(k)$, where $q^*(k) \rightarrow 1$ as $k \rightarrow \infty$.*

Proof. (a) The bound $\mathbb{P}_p(R_0(Q)^c) \leq c \exp(-c'n)$ follows from Theorem 3.1 of [P] ($d \geq 3$) and Theorem 5 of [PP] for $d = 2$. The estimate

$$\mathbb{P}_p(\mathcal{C}^\vee(Q) \text{ is not crossing for } Q) \leq ce^{-c'n^{d-1}}$$

follows from Theorem 1.2 of [P] ($d \geq 3$) and Theorem 1 of [CM] ($d = 2$).

(b) By (a) we have $\mathbb{P}_p(\varphi(x) = 1) \rightarrow 1$ as $k \rightarrow \infty$. The r.v. $\varphi(x)$ and $\varphi(y)$ are independent if $|x - y|_\infty \geq 3$, so the result is immediate from [LSS, Theorem 0.0]. (We remark that using [LSS] means that the events $S_i^{(N)}$ defined in (2.5) of [AP] are no longer needed.) \square

Recall the definition of $q_1(\varepsilon, d)$ from Lemma 2.5, and choose $k_0 = k_0(p, d)$ large enough so that $q^*(k) \geq q_1(\varepsilon, d)$ for all $k \geq k_0$. We fix $k = k_0$, and refer to the process φ as the *macroscopic percolation* process. We write $\tilde{\mathcal{O}}, \tilde{\mathcal{C}}(x)$ etc. to denote the open sites, open clusters for the macroscopic process, and \tilde{F} for associated events.

Let \tilde{Q} be a macroscopic cube of side m , and associate with \tilde{Q} the microscopic cube $Q = \cup\{T^+(x), x \in \tilde{Q}\}$. Any (microscopic) cube which can be obtained in this way will be called a *special cube*. Define $T'(x)$ to be $T(x)$ if x is in the interior of \tilde{Q} , (so that $T^+(x)$ is in the interior of Q). Otherwise, if $x \in \partial_i(\tilde{Q}|\mathbb{Z}^d)$, let $T'(x)$ be $T(x)$ together with all points in $T^+(x)$ which are closer to $T(x)$ than any $T(y)$, $y \neq x$, $y \in \tilde{Q}$. Thus $T'(x)$ is a d -dimensional rectangle, and Q is the disjoint union of the $T'(x)$, $x \in \tilde{Q}$. Note that each side of $T'(x)$ ($x \in \tilde{Q}$) is less than $\frac{3}{2}(k_0 - 1)$.

Fix $\varepsilon_0 = (4d + 2)^{-1}$.

Lemma 2.9. *Let Q be a special cube with $s(Q) = n$, and \tilde{Q} be the associated macroscopic cube. Suppose that $m = s(\tilde{Q}) \geq m_0(k_0)$ and that the event $\tilde{K}(\tilde{Q}, \frac{7}{8}) \cap \tilde{F}(\tilde{Q}, \varepsilon_0)$ holds for $(\varphi_x, x \in \mathbb{Z}^d)$. Then the cluster $\mathcal{C}^\vee(Q)$ satisfies*

$$|\mathcal{C}^\vee(Q)| \geq c_1 n^d, \quad \text{diam}(\mathcal{C}^\vee(Q)) = n.$$

Proof. Write $\mathcal{C}_x = \mathcal{C}^\vee(T^+(x))$. If $x, y \in \tilde{\mathcal{O}} \cap \tilde{Q}$ and $x \sim y$, then the events $R(T(x))$ and $R(T(y))$ force the clusters \mathcal{C}_x and \mathcal{C}_y to be connected. Thus there exists a Q -cluster \mathcal{C}' with $\cup\{\mathcal{C}_x, x \in \tilde{\mathcal{C}}^\vee(\tilde{Q})\} \subseteq \mathcal{C}'$. It follows immediately that

$$|\mathcal{C}'| \geq |\tilde{\mathcal{C}}^\vee(\tilde{Q})| \geq \frac{7}{8}m^d \geq c_1 n^d.$$

Also, since $\tilde{\mathcal{C}}^\vee$ is crossing for \tilde{Q} we deduce that \mathcal{C}' is crossing for Q and $\text{diam}(\mathcal{C}') = n$.

It remains to prove that $\mathcal{C}' = \mathcal{C}^\vee(Q)$; suppose not. Choose m_0 so that $(\frac{3}{2}k_0)^d < \frac{7}{8}m^d$, and therefore the cluster $\mathcal{C}^\vee(Q)$ is not contained in any one cube $T^+(x)$. Let $x \in \tilde{\mathcal{C}}^\vee(\tilde{Q})$. Then $\mathcal{C}^\vee(Q) \cap \mathcal{C}_x = \emptyset$, and so $\mathcal{C}^\vee(Q) \cap T^+(x)$ consists of clusters which have diameter less than $k_0/8$. Since $\mathcal{C}^\vee(Q)$ contains points outside $T^+(x)$ it follows that $\mathcal{C}^\vee(Q) \cap T(x) = \emptyset$.

So, if $\Gamma_1, \dots, \Gamma_k$ are the connected components of $\tilde{Q} - \tilde{\mathcal{C}}^\vee(\tilde{Q})$, we deduce that for some j , $\mathcal{C}^\vee(Q) \subseteq \cup\{T'(x), x \in \Gamma_j\}$. By Proposition 2.7(a) we deduce that $|\Gamma_j| \leq c_2 m^{d\beta/(d-1)}$, and so if m_0 is large enough

$$|\mathcal{C}^\vee| \leq c_2 (\frac{3}{2}k_0)^d m^{d\beta/(d-1)} < \frac{7}{8}m^d,$$

giving a contradiction. Thus $\mathcal{C}' = \mathcal{C}^\vee(Q)$. \square

Let Q be a special cube with $s(Q) = n$, and \tilde{Q} be the associated macroscopic cube. For sets $A_1, A_2 \subseteq Q$ let

$$\partial_E(A_1, A_2 | \mathcal{O}_E) = \{\{x, y\} : \{x, y\} \in \mathcal{O}_E, x \in A_1, y \in A_2\}.$$

Let $A \subseteq \mathcal{C}^\vee = \mathcal{C}^\vee(Q)$, and $\Gamma = \mathcal{C}^\vee - A$ both be connected. Set

$$\begin{aligned}\tilde{A} &= \{x \in \tilde{Q} : A \cap T'(x) \neq \emptyset\}, \\ \tilde{\Gamma} &= \{x \in \tilde{Q} : \Gamma \cap T'(x) \neq \emptyset\}.\end{aligned}$$

Note that

$$|A| \geq |\tilde{A}| \geq (\frac{3}{2}k)^{-d}|A|. \quad (2.10)$$

As A and Γ are connected it is clear that \tilde{A} and $\tilde{\Gamma}$ are connected. Let $\tilde{\mathcal{C}}^\vee = \tilde{\mathcal{C}}^\vee(\tilde{Q})$ be the largest open (macroscopic) cluster in \tilde{Q} .

Lemma 2.10. *Let Q be a special cube with $s(Q) = n$, and \tilde{Q} be the associated macroscopic cube. Suppose that $m = s(\tilde{Q}) \geq m_0(k_0)$ and that the event $\tilde{K}(\tilde{Q}, \frac{7}{8}) \cap \tilde{F}(\tilde{Q}, \varepsilon_0)$ holds for $(\varphi_x, x \in \mathbb{Z}^d)$. Let $A \subseteq \mathcal{C}^\vee$ and $\Gamma = \mathcal{C}^\vee - A$ be connected.*

(a) *Let $x \sim y$, $x \in \tilde{A}$, $y \in \tilde{Q} - \tilde{A}$ with $x, y \in \tilde{\mathcal{O}}$. Then the set $T^+(x) \cap Q$ contains at least one edge in $\partial_E(A, \Gamma | \mathcal{O}_E)$.*

(b) *Suppose $x \in \tilde{A} \cap \tilde{\Gamma} \cap \tilde{\mathcal{O}}$. Then the set $T^+(x) \cap Q$ contains at least one edge in $\partial_E(A, \Gamma | \mathcal{O}_E)$.*

Proof. As in Lemma 2.9 we have that \mathcal{C}^\vee is not contained in any one $T^+(x)$. Let $\mathcal{C}_x = \mathcal{C}^\vee(T^+(x))$ be as in Lemma 2.9.

(a) Since $x \in \tilde{A}$ there exists $x' \in T(x) \cap A$, so since \mathcal{C}^\vee is connected there exists an open path $\gamma \subseteq Q$ from x' to $Q \cap T^+(x)^c$. This path must have diameter greater than $k_0/3$, so is connected within $T^+(x)$ to \mathcal{C}_x . Hence x' is connected within $T^+(x)$ to \mathcal{C}_y . Choose $y' \in \mathcal{C}_y \cap T(y)$; then $y' \notin A$ but $y \in \mathcal{C}^\vee$. There exists an open path γ' from x' to y' , and this must contain at least one edge in $\partial_E(A, \Gamma | \mathcal{O}_E)$.

(b) Let $x' \in A \cap T'(x)$, $y' \in \Gamma \cap T'(x)$. Since $x \in \tilde{\mathcal{O}}$ both x' and y' are in \mathcal{C}_x and there therefore exists an open path in $T^+(x)$ between x' and y' . As in (a) this path must contain at least one edge in $\partial_E(A, \Gamma | \mathcal{O}_E)$. \square

Proposition 2.11. *Let Q, \tilde{Q} be as above with $m = s(\tilde{Q}) \geq m_0$. Suppose that the event $\tilde{K}(\tilde{Q}, \frac{7}{8}) \cap \tilde{F}(\tilde{Q}, \varepsilon_0)$ holds for $(\varphi_x, x \in \mathbb{Z}^d)$. Let A be a connected open subset of $\mathcal{C}^\vee(Q)$ with $|A| \leq \frac{1}{2}|\mathcal{C}^\vee(Q)|$, and such that $\Gamma = \mathcal{C}^\vee(Q) - A$ is also connected. Then*

$$|\partial_E(A, \Gamma | \mathcal{O}_E)| \geq c_1 n^{-1} |A|. \quad (2.11)$$

Proof. We write $\mathcal{C}^\vee = \mathcal{C}^\vee(Q)$, $\tilde{\mathcal{C}}^\vee = \tilde{\mathcal{C}}^\vee(\tilde{Q})$. Since $A \neq \mathcal{C}^\vee(Q)$ and \mathcal{C}^\vee is connected we have $|\partial_E(A, \Gamma | \mathcal{O}_E)| \geq 1$, so (2.11) is immediate if $|A| \leq c_1^{-1}n$. Also, using Lemma 2.10(a) we have

$$|\partial_E(A, \Gamma | \mathcal{O}_E)| \geq c_2 |\partial_E(\tilde{A} \cap \tilde{\mathcal{O}}, (\tilde{Q} - \tilde{A}) \cap \tilde{\mathcal{O}})|. \quad (2.12)$$

We consider four cases.

Case 1: $\tilde{A} \cap \tilde{\mathcal{C}}^\vee = \emptyset$. Let $\tilde{\Lambda}$ be the connected component of $\tilde{Q} - \tilde{\mathcal{C}}^\vee$ which contains \tilde{A} . By Proposition 2.7(a) $|\tilde{\Lambda}| \leq cm^{d\beta/(d-1)}$, so that

$$|A| \leq (\tfrac{3}{2}k_0)^d |\tilde{\Lambda}| \leq ck_0^d m^{d\beta/(d-1)} \leq n,$$

provided m_0 is large enough. Hence (2.11) holds for A .

Case 2: $\tilde{A} \cap \tilde{\mathcal{C}}^\vee \neq \emptyset$, $|\tilde{A}| \leq \frac{3}{4}|\tilde{Q}|$. We apply Proposition 2.7(b) to see that

$$\partial_E(\tilde{A} \cap \tilde{\mathcal{C}}^\vee, (\tilde{Q} - \tilde{A}) \cap \tilde{\mathcal{C}}^\vee) \geq cm^{-1}|\tilde{A}| \geq cn^{-1}|A|,$$

and combining this with (2.12) proves (2.11).

Case 3: $\tilde{A} \cap \tilde{\mathcal{C}}^\vee \neq \emptyset$, $|\tilde{A}| \geq \frac{3}{4}|\tilde{Q}|$, $|\tilde{\Gamma}| \leq \frac{3}{4}|\tilde{Q}|$. Using Lemma 2.9 we have $|\tilde{\Gamma}| \geq (\frac{3}{2}k_0)^{-d}|\Gamma| \geq \frac{1}{2}(\frac{3}{2}k_0)^{-d}|\mathcal{C}^\vee| \geq cn^d$. So by Proposition 2.7(a) $\tilde{\Gamma} \cap \tilde{\mathcal{C}}^\vee \neq \emptyset$, and we can therefore apply Proposition 2.7(b) to $\tilde{\Gamma}$ to deduce

$$\partial_E(\tilde{\Gamma} \cap \tilde{\mathcal{C}}^\vee, (\tilde{Q} - \tilde{\Gamma}) \cap \tilde{\mathcal{C}}^\vee) \geq cm^{-1}|\tilde{\Gamma}| \geq cm^{-1}n^d \geq cn^{-1}|A|.$$

Using (2.12) (with A and Γ interchanged) we deduce that

$$\begin{aligned} |\partial_E(\Gamma, A|\mathcal{O}_E)| &\geq c_2|\partial_E(\tilde{\Gamma} \cap \tilde{\mathcal{O}}, (\tilde{Q} - \tilde{\Gamma}) \cap \tilde{\mathcal{O}})| \\ &\geq c_2|\partial_E(\tilde{\Gamma} \cap \tilde{\mathcal{C}}^\vee, (\tilde{Q} - \tilde{\Gamma}) \cap \tilde{\mathcal{C}}^\vee)| \geq n^{-1}|A|. \end{aligned}$$

Case 4: $\tilde{A} \cap \tilde{\mathcal{C}}^\vee \neq \emptyset$, $|\tilde{A}| \geq \frac{3}{4}|\tilde{Q}|$, $|\tilde{\Gamma}| \geq \frac{3}{4}|\tilde{Q}|$. Since \tilde{A} and $\tilde{\Gamma}$ are both connected, and $\tilde{F}(\tilde{Q}, \varepsilon_0)$ holds, we have

$$\begin{aligned} |\tilde{A} \cap \tilde{\mathcal{O}}| &\geq (1 - \varepsilon_0)|\tilde{A}| > \tfrac{2}{3}|\tilde{Q}|, \\ |\tilde{\Gamma} \cap \tilde{\mathcal{O}}| &\geq (1 - \varepsilon_0)|\tilde{\Gamma}| > \tfrac{2}{3}|\tilde{Q}|. \end{aligned}$$

Hence $|\tilde{A} \cap \tilde{\Gamma} \cap \tilde{\mathcal{O}}| \geq \frac{1}{3}|\tilde{Q}|$. So by Lemma 2.10(b)

$$|\partial_E(A, \Gamma|\mathcal{O}_E)| \geq c|\tilde{A} \cap \tilde{\Gamma} \cap \tilde{\mathcal{O}}| \geq c'm^d \geq c''|A|,$$

which implies (2.11). \square

As in Section 1 we define

$$\nu_{xy} = \nu_{xy}(\omega) = \begin{cases} 1 & \text{if } \{x, y\} \text{ is an open edge,} \\ 0 & \text{otherwise,} \end{cases}$$

$\mu(x) = \mu(x)(\omega) = \sum_y \nu_{xy}$, $x \in \mathbb{Z}^d$, and extend ν and μ to measures. If $f : A \rightarrow \mathbb{R}$ we write

$$\overline{f}_A = \mu(A)^{-1} \int_A f d\mu.$$

Let Q be a special cube, and \tilde{Q} be the associated macroscopic cube. Define

$$H_0(Q) = \tilde{K}(\tilde{Q}, \tfrac{7}{8}) \cap \tilde{F}(\tilde{Q}, \varepsilon_0),$$

and extend the definition of $H_0(Q)$ to all cubes Q by taking $H_0(Q) = H_0(Q')$, where Q' is the largest special cube contained in Q . (We set $H_0(Q) = \Omega$ if there is no such special cube.)

Recall the definition of β from (2.2).

Proposition 2.12. Consider bond percolation η_e on $(\mathbb{Z}^d, \mathbb{E}_d)$ with $p > p_c$. Let Q be a special cube of side n .

(a)

$$\mathbb{P}_p(H_0(Q)^c) \leq c_1 e^{-c_2 n^\beta}. \quad (2.13)$$

(b) If $\omega \in H_0(Q)$ and $f : \mathcal{C}^\vee(Q)(\omega) \rightarrow \mathbb{R}$ then

$$\int_{\mathcal{C}^\vee(Q)} (f(y) - \bar{f}_{\mathcal{C}^\vee(Q)})^2 d\mu \leq c_3 n^2 \int_{E(\mathcal{C}^\vee(Q))} |\nabla f|^2 d\nu.$$

Proof. (a) To prove (2.13) note that

$$\mathbb{P}_p(H_0(Q)^c) \leq \mathbb{P}_p(\tilde{K}(\tilde{Q}, \frac{7}{8})^c) + \mathbb{P}_p(\tilde{F}(\tilde{Q}, \varepsilon_0)^c). \quad (2.14)$$

As the events $K(\cdot, \frac{7}{8})$ and $F(\cdot, \cdot)$ are increasing, the two probabilities on the right hand side of (2.14) are bounded by the probabilities of these events with respect to a Bernoulli site percolation process with probability $q^* = q^*(k_0)$. Using Lemmas 2.2 and 2.5,

$$\begin{aligned} \mathbb{P}_p(\tilde{K}(\tilde{Q}, \frac{7}{8})^c) + \mathbb{P}_p(\tilde{F}(\tilde{Q}, \varepsilon_0)^c) &\leq \mathbb{Q}_{q^*}(K(\tilde{Q}, \frac{7}{8})^c) + \mathbb{Q}_{q^*}(F(\tilde{Q}, \varepsilon_0)^c) \\ &\leq c \exp(-cm^{d-1}) + c' \exp(-c'm^\beta) \\ &\leq c \exp(-c'n^\beta). \end{aligned}$$

(b) is immediate from Propositions 1.4 and 2.11. \square

Remark. See [MR, Section 3.2] for similar bounds in the case when p is close to 1.

Recall from Definition 2.1 that $Q \subseteq Q^\oplus \subseteq Q^+$. Let $\alpha \in (0, \frac{1}{2})$, Q be a cube with $s(Q) = n$, and set

$$\begin{aligned} H(Q, \alpha) &= R(Q) \cap \{R(Q') \cap H_0(Q') \text{ occurs for every cube } Q' \\ &\quad \text{with } (Q')^+ \subseteq Q^+, Q' \cap Q^\oplus \neq \emptyset \text{ and } n^\alpha \leq s(Q') \leq n\}. \end{aligned} \quad (2.15)$$

Lemma 2.13. (a) For $p > p_c$

$$\mathbb{P}_p(H(Q, \alpha)^c) \leq c e^{-c_1 n^{\alpha\beta}}.$$

(b) If $\omega \in H(Q, \alpha)$ and $Q_0 \subset Q$ satisfies the condition (2.15) then $\mathcal{C}^\vee(Q_0) \subset \mathcal{C}^\vee(Q^+)$.

Proof. (a) By (2.9) and (2.13),

$$\mathbb{P}_p(H(Q, \alpha)^c) \leq \sum_{r=n^\alpha}^n c n^d e^{-c_1 r^\beta} \leq c' e^{-c_2 n^{\alpha\beta}}, \quad (2.16)$$

proving (a).

(b) Define a sequence of cubes Q_i , $0 \leq i \leq k$ by $Q_{i+1} = Q_i^*$, and where we stop at the last cube Q_k with $Q_k^+ \subset Q^+$. The events $R(Q_i)$ then force $\mathcal{C}^\vee(Q_i) \subset \mathcal{C}^\vee(Q_{i+1})$, so that $\mathcal{C}^\vee(Q_0) \subset \mathcal{C}^\vee(Q_k^+)$. Since $\text{diam } \mathcal{C}^\vee(Q_k^+) = \text{diam}(Q_k^+) > n/8$, the event $R(Q)$ implies that $\mathcal{C}^\vee(Q_k^+) \subset \mathcal{C}^\vee(Q^+)$. \square

Proposition 2.12 and Lemma 2.13 complete our results on the Poincaré inequality in sets of the form $\mathcal{C}^\vee(Q)$. However, to be able to obtain bounds (particularly lower bounds) on transition densities on \mathcal{C}_∞ we need to relate $|x - y|_1$ to the shortest path (or ‘chemical’) metric d_ω on \mathcal{C}_∞ . This is done in [AP, Theorem 1.1], but as we need some minor extensions of their results, and it is desired to make this paper as self contained as possible, we will repeat some of their constructions.

Let $k \geq 17$ and recall the site process $\varphi(x) = 1_{R(T(x))}$ introduced in (2.8). Set $\varphi'(x) = 1 - \varphi(x)$; since by Lemma 2.8(b) φ dominates Bernoulli site percolation with parameter $q^*(k)$, φ is dominated by Bernoulli site percolation with parameter $q'(k) = 1 - q^*(k)$. We consider the clusters of the process φ' on the graph $(\mathbb{Z}^d, \mathbb{E}_d^*)$. Given a function $\varphi' : \mathbb{Z}^d \rightarrow \{0, 1\}$, we write $\mathcal{O}(\varphi') = \{x : \varphi'(x) = 1\}$. If $\varphi'(x) = 0$ we set $\mathcal{C}^*(\varphi', x) = \emptyset$, and if $\varphi'(x) = 1$ we let $\mathcal{C}^*(\varphi', x)$ be the $*$ -connected component of $\mathcal{O}(\varphi')$ containing x . Let

$$\mathcal{D}(\varphi', x) = \begin{cases} \partial_e(\mathcal{C}^*(\varphi', x) | \mathbb{Z}^d) & \text{if } \varphi'(x) = 1, \\ \{x\} & \text{if } \varphi'(x) = 0. \end{cases} \quad (2.17)$$

For $x, y \in \mathbb{Z}^d$ let $\gamma(x, y)$ be a shortest path (in $(\mathbb{Z}^d, \mathbb{E}_d^d)$) connecting x and y . Note that if x and y are contained in a cube Q then $\gamma(x, y) \subseteq Q$. Set

$$W(\varphi', x, y) = \bigcup_{z \in \gamma(x, y)} \mathcal{D}(\varphi', z). \quad (2.18)$$

The importance of W comes from the following result, which is Proposition 3.1 of [AP].

Proposition 2.14. *Let $k \geq 17$, $x, y \in \mathbb{Z}^d$ and $\tilde{x}, \tilde{y} \in \mathbb{Z}^d$ be such that $x \in T(\tilde{x})$, $y \in T(\tilde{y})$. Suppose that (for the bond percolation process η_e on (Ω_b, \mathbb{P}_p)) $d_\omega(x, y) < \infty$. Then there exists an open path $\gamma'(x, y)$ connecting x and y contained in*

$$W'(x, y) = \bigcup_{\tilde{z} \in W(\varphi', \tilde{x}, \tilde{y})} T^+(\tilde{z}).$$

In particular $d_\omega(x, y) \leq |W'(x, y)| \leq (3k)^d |W(\varphi', \tilde{x}, \tilde{y})|$.

Proposition 2.15. *Let $p > p_c$. There exists $k_1 = k_1(p, d)$ and C_{AP} such that if $k \geq k_1$ then the following hold.*

(a) *If $\tilde{x}, \tilde{y} \in \mathbb{Z}^d$ and $x \in T(\tilde{x})$, $y \in T(\tilde{y})$ then*

$$\mathbb{P}_p(|W(\varphi', x, y)| \geq C_{AP} |\tilde{x} - \tilde{y}|_1) \leq c_2 e^{-c_3 |\tilde{x} - \tilde{y}|_1}.$$

(b) If $\tilde{x}, \tilde{y} \in \mathbb{Z}^d$, $\lambda \geq 0$, then

$$\mathbb{P}_p(\max_{\tilde{z} \in \gamma(\tilde{x}, \tilde{y})} \text{diam}(\mathcal{D}(\varphi', \tilde{z})) > \lambda) \leq c_4 |\tilde{x} - \tilde{y}|_1 e^{-c_5 \lambda}.$$

Proof. (a) is proved on p. 1047 of [AP].

(b) We choose k_1 large enough so that $q'(k) < q_c$ for all $k \geq k_1$. Since φ' is dominated by Bernoulli site percolation with parameter q' we have

$$\mathbb{P}_p(\text{diam}(\mathcal{D}(\varphi', \tilde{z})) > \lambda) \leq \mathbb{Q}_{q'}(2 + \text{diam}(\mathcal{C}^*(0)) > \lambda) \leq ce^{-c' \lambda}.$$

(For the second estimate above see [Gm, Theorem 5.4].) The bound in (b) is now immediate. \square

Now fix k_1 as in Proposition 2.15, and set

$$C_H = dC_{AP}(3k_1)^d. \quad (2.19)$$

Let Q be a cube with side n . For $x, y \in Q$ let

$$E(Q, x, y) = \{x, y \in \mathcal{C}^\vee(Q^+) : d_{\mathcal{C}^\vee(Q^+)}(x, y) > C_H |x - y|_\infty\}.$$

Lemma 2.16. *Let $p > p_c$, Q be a cube with side n , and $x, y \in Q$. Then*

$$\mathbb{P}_p(E(Q, x, y)) \leq ce^{-c_5 |x - y|_\infty}.$$

Proof. Let $x \in T(\tilde{x})$ and $y \in T(\tilde{y})$. Suppose $x, y \in \mathcal{C}^\vee(Q^+)$. Then by Proposition 2.14 there exists a open path γ' connecting x and y contained in $W'(\varphi', x, y)$. So if $E(Q, x, y)$ holds then either $W'(\varphi', x, y)$ is not contained in Q^+ or

$$C_H |x - y|_\infty < d_{\mathcal{C}^\vee(Q^+)}(x, y) \leq (3k_1)^d |W(\varphi', \tilde{x}, \tilde{y})|.$$

Thus

$$\begin{aligned} \mathbb{P}_p(E(Q, x, y)) &\leq \mathbb{P}_p(|W(\varphi', \tilde{x}, \tilde{y})| \geq C_{AP} |\tilde{x} - \tilde{y}|_1) \\ &\quad + \mathbb{P}_p(\max_{\tilde{z} \in \gamma(\tilde{x}, \tilde{y})} \text{diam}(\mathcal{D}(\varphi', \tilde{z})) > \frac{n}{8k_1}) \\ &\leq ce^{-c|\tilde{x} - \tilde{y}|_1} + c|\tilde{x} - \tilde{y}|_1 e^{-cn/k_1} \leq ce^{-c|x - y|_\infty}. \end{aligned}$$

\square

Let Q be a cube with side n . Set

$$\begin{aligned} D_0(Q) &= R(Q) \cap \{d_{\mathcal{C}^\vee(Q^+)}(x, y) \leq C_H |x - y|_\infty \\ &\quad \text{if } x, y \in \mathcal{C}^\vee(Q^+) \cap Q, |x - y|_\infty \geq n/12\}, \end{aligned} \quad (2.20)$$

and

$$\begin{aligned} D(Q, \alpha) &= \{D_0(Q') \text{ occurs for every cube } Q' \text{ with } (Q')^+ \subseteq Q^+, \\ &\quad Q' \cap Q^\oplus \neq \emptyset \text{ and } n^\alpha \leq s(Q') \leq n\}. \end{aligned} \quad (2.21)$$

Let also

$$B_\omega(y, r) = \{x : d_\omega(x, y) < r\}.$$

Since \mathcal{C}_∞ is embedded in \mathbb{Z}^d we have $\mu(B_\omega(y, r)) \leq C_0 r^d$ for some $C_0 = C_0(d)$.

Proposition 2.17. *Let $p > p_c$.*

- (a) $\mathbb{P}_p(D_0(Q)^c) \leq c_1 e^{-c_2 n}$.
- (b) $\mathbb{P}_p(D(Q, \alpha)^c) \leq c_3 e^{-c_4 n^\alpha}$.
- (c) *Let $\omega \in D_0(Q)$, and $x, y \in Q \cap \mathcal{C}^\vee(Q^+)$. Then $d_\omega(x, y) \leq C_H n$.*
- (d) *Let $\omega \in D(Q, \alpha)$, and $x, y \in Q^\oplus \cap \mathcal{C}^\vee(Q^+)$. Then*

$$|x - y|_\infty \leq d_\omega(x, y) \leq C_H((1 + n^\alpha) \vee |x - y|_\infty).$$

- (e) *Let $\omega \in D(Q, \alpha)$, $x \in Q^\oplus$ and Q' satisfy the conditions in (2.21) with $s(Q') = r$. Then $Q' \cap \mathcal{C}^\vee(Q^+) \subset B_\omega(x, C_H r)$.*

Proof. (a) By Lemma 2.16

$$\mathbb{P}_p(D_0(Q)^c) \leq \mathbb{P}_p(R(Q)^c) + \sum_{x', y'} c e^{-c_5 |x' - y'|_\infty},$$

where the sum is over $x', y' \in Q$ with $|x' - y'|_\infty \geq n/12$. Hence, using (2.9),

$$\mathbb{P}_p(D_0(Q)^c) \leq c(n+1)^{2d} e^{-c_5 n/8} \leq c_1 e^{-c_2 n}. \quad (2.22)$$

(b) is immediate from (a), since

$$\mathbb{P}_p(D(Q, \alpha)^c) \leq \sum_{r=n^\alpha}^n \left(\frac{3}{2}n + 1\right)^d c_1 e^{-c_2 r} \leq c \exp(-c' n^\alpha).$$

(c) This is immediate if $|x - y|_\infty \geq n/12$. If $|x - y|_\infty < n/8$ then choose $z' \in \mathcal{C}^\vee(Q^+) \cap Q$ with $n/12 \leq |z - x|_\infty \leq n/4$, $n/12 \leq |z - y|_\infty \leq n/4$. Since $\mathcal{C}^\vee(Q^+)$ is crossing for Q such a choice of z is possible. Then $d_\omega(x, y) \leq d_\omega(x, z) + d_\omega(z, y) \leq 2C_H(n/4) \leq C_H n$.

(d) Since $D(Q, \alpha) \subseteq D_0(Q)$ this is immediate from (b) if $|x - y|_\infty \geq n/12$. Otherwise choose a smallest possible cube Q' such that $s(Q') \geq n^\alpha \vee |x - y|_\infty$ and $x, y \in Q'$. We have $(Q')^+ \subset Q^+$. As in Lemma 2.13(b) we have $\mathcal{C}^\vee(Q') \subset \mathcal{C}^\vee(Q^+)$, and by (c) $d_{\mathcal{C}^\vee(Q^+)}(x, y) \leq C_H s(Q') \leq C_H(|x - y|_\infty \vee (1 + n^\alpha))$.

(e) Since $D_0(Q')$ occurs, this is immediate from (c).

Recall from Definition 1.7 the definition of good and very good balls.

Theorem 2.18. *Let $\alpha \in (0, \frac{1}{2})$, Q be a cube of side n , $\omega \in H(Q, \alpha) \cap D(Q, \alpha)$, and $C_H n^\alpha \leq r \leq n$. Write $Q(y, s) = \{z \in \mathbb{Z}^d : |z - y|_\infty \leq s\}$. Let $y \in \mathcal{C}^\vee(Q^+) \cap Q^\oplus$ with $Q(y, r + k_0)^+ \subseteq Q^+$.*

(a) *There exists $C_V = C_V(p, d)$ such that*

$$C_V r^d \leq |B_\omega(y, r)| \leq C_0 r^d. \quad (2.23)$$

(b) There exist constants $C_P(p, d)$, $C_W(p, d)$ such that if $f : B_\omega(y, C_W r) \rightarrow \mathbb{R}$, and writing $\bar{f}_B = \bar{f}_{B_\omega(y, r)}$, then

$$\int_{B_\omega(y, r)} (f - \bar{f}_{B_\omega(y, r)})^2 d\mu \leq C_P r^2 \int_{E(B_\omega(y, C_W r))} |\nabla f|^2 d\nu. \quad (2.24)$$

(c) If $(C_H n^\alpha)^{d+2} \leq R \leq n$ and $B_\omega(y, \frac{3}{2}R) \subseteq Q^\oplus$ then $B_\omega(y, R)$ is (C_V, C_P, C_W) -very good with $N_{B_\omega(y, R)} \leq C_H n^\alpha$.

Proof. Recall from Lemma 2.13(b) that $\omega \in H(Q, \alpha)$ implies that $\mathcal{C}^\vee(Q') \subseteq \mathcal{C}^\vee(Q^+)$ for every Q' satisfying (2.15).

(a) Since $B_\omega(y, r) \subseteq Q(y, r)$ the upper bound in (2.23) is clear. Let $s = r/(2C_H)$, so that $2s \geq n^\alpha$. By Proposition 2.17(e) $\mathcal{C}^\vee(Q(y, s)) \subset B_\omega(y, C_H s)$, so that by Lemma 2.9,

$$|B_\omega(y, r)| \geq |\mathcal{C}^\vee(Q(y, s))| \geq c_1 s^d \geq c_2 r^d.$$

(b) Let Q_1 be the smallest special cube containing $Q(y, r)$; we have $Q_1^+ \subseteq Q^+$. Let $r_1 = s(Q_1)$. By Proposition 2.17(e) $\mathcal{C}^\vee(Q_1) \subset B_\omega(y, C_H r_1)$, and so taking $C_W = 2C_H$, $\mathcal{C}^\vee(Q_1) \subseteq B_\omega(y, C_W r)$. By Proposition 2.12(b),

$$\begin{aligned} \int_{B_\omega(y, r)} (f - \bar{f}_B)^2 d\mu &\leq \int_{B_\omega(y, r)} (f - \bar{f}_{\mathcal{C}^\vee})^2 d\mu \\ &\leq \int_{\mathcal{C}^\vee(Q_1)} (f - \bar{f}_{\mathcal{C}^\vee})^2 d\mu \\ &\leq c_3 r_1^2 \int_{E(\mathcal{C}^\vee(Q_1))} |\nabla f|^2 d\nu \leq c_4 r^2 \int_{E(B_\omega(y, C_W r))} |\nabla f|^2 d\nu. \end{aligned}$$

(c) is immediate from (a), (b) and the definition of very good balls. \square

Using the estimates in Lemma 2.13(a), Proposition 2.17(b) and Borel-Cantelli we obtain

Lemma 2.19. *Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exists $M_x(\omega)$ with $\mathbb{P}_p(M_x \geq n) \leq c_1 e^{-c_2 n^{\alpha, \beta}}$ such that whenever $n \geq M_x$ then $H(Q, \alpha) \cap D(Q, \alpha)$ holds for all cubes Q of side n with $x \in Q$.*

Remarks. 1. An inequality of the form (2.24) is called a *weak Poincaré inequality*. In many situations (including this one) it is possible to derive a strong Poincaré inequality (i.e. with $C_W = 1$) from a family of weak ones – see Lemma 4.9.

2. Note that if $x \in Q$ and $s(Q) \geq M_x$ then $\mathcal{C}^\vee(Q) \subseteq \mathcal{C}_\infty$.

Theorem 2.18 and Lemma 2.19 are suitable for most of our needs, but they have the defect that the minimum size of ball inside a cube Q of side n for which the Poincaré inequality is certain to hold increases with n . Since (for a fixed C_P) the cluster \mathcal{C}_∞ contains

arbitrarily large balls in which the Poincaré inequality fails, we cannot do better than this as long as we require it for all balls of some size. However, we can improve Theorem 2.18 if we relax this condition, and in Section 5 we will want to connect points by a chain of very good balls of some fixed size. To do this we need an additional percolation argument.

We consider again Bernoulli site percolation ζ_x on $(\mathbb{Z}^d, \mathbb{E}_d)$ with parameter q , where $q > q_c$ is close to 1. Let Q be a cube of side n . For $x, y \in Q$, $\lambda > 1$, let

$$S(\zeta, Q, \lambda, x, y) = K(Q, \frac{7}{8}) \cap \{ \text{there exist } x', y' \in \mathcal{C}^\vee(Q^+) \text{ with } |x - x'|_\infty \leq n^{1/9}, \\ |y - y'|_\infty \leq n^{1/9}, \text{ such that } d_{\mathcal{C}^\vee(Q^+)}(x', y') \leq \lambda |x - y|_\infty \}. \quad (2.25)$$

Note that this event is increasing.

Lemma 2.20. *Let Q be a cube of side n , and $x, y \in Q$. There exists $q_2 = q_2(d) \in (q_c, 1)$ and $\lambda_0 \geq 1$ so that if $q > q_2$, then*

$$\mathbb{Q}_q(S(\zeta, Q, \lambda_0, x, y)^c) \leq c_1 e^{-c_2 n^{1/9}}. \quad (2.26)$$

Proof. We follow the proof of Theorem 1.1 of [AP] and consider the dual process to ζ given by $\zeta'_x = 1 - \zeta_x$. We view ζ' as site percolation (with parameter $q' = 1 - q$) on the lattice $(\mathbb{Z}^d, \mathbb{E}_d^*)$, and write $\mathcal{C}^*(\zeta', z)$ for the $*$ -connected cluster of the process ζ' containing z . Then by [Gm, Theorem 5.4] we can choose q_2 large enough so that if $q \geq q_2$ then

$$\mathbb{Q}_q(|\mathcal{C}^*(\zeta', z)| \geq k) \leq e^{-c_3 k}, \quad k \geq 1.$$

So, if

$$G = \{|\mathcal{C}^*(\zeta', x)| \leq n^{1/9} \text{ for all } x \in Q\};$$

then (using Lemma 2.2)

$$\mathbb{Q}_q(K(Q, \frac{7}{8})^c \cup G^c) \leq e^{-c_4 n^{1/9}}. \quad (2.27)$$

If $|x - y|_\infty \leq n^{1/9}$ and $\omega \in G$ there exists $x' \in \mathcal{C}^\vee(Q^+)$ with $|x - x'|_\infty \leq n^{1/9}$. In this case we can take $y' = x'$.

So suppose $\omega \in G$ and $|x - y|_\infty > n^{1/9}$. Let $l = |x - y|_1$ and $\gamma = \{x = x_0, x_1, \dots, x_l = y\}$ be a path in $(\mathbb{Z}^d, \mathbb{E}_d)$ of length l connecting x and y – note that $\gamma \subset Q$. As each cluster $\mathcal{C}^*(\zeta', y)$, $y \in Q$, has diameter less than $n^{1/9}$ the path γ must intersect $\mathcal{C}^\vee(Q^+)$. Let V_x and V_y be the first and last (respectively) points in $\gamma \cap \mathcal{C}^\vee(Q^+)$; we have $|x - V_x| \leq n^{1/9}$ and $|y - V_y| \leq n^{1/9}$.

We take $x' = V_x$, $y' = V_y$, and construct a path Γ from x' to y' in $\mathcal{C}^\vee(Q^+)$. This path follows γ whenever possible, and when it encounters a site z with $\zeta_z = 0$ it ‘walks around’ $\mathcal{C}^*(\zeta', z)$ – this requires at most $3^d |\mathcal{C}^*(\zeta', z)|$ steps. Since $\omega \in G$ this path does not leave Q^+ . Hence, recalling from (2.18) the definition of $W(\cdot)$,

$$|\Gamma| \leq l + 3^d \sum_{i=0}^l |\mathcal{C}^*(\zeta', z_i)| \leq 3^d |W(\zeta', x', y')| \leq 3^d |W(\zeta', x, y)|. \quad (2.28)$$

By Proposition 2.15(a) we have

$$\mathbb{Q}_q(|W(\zeta', x, y)| \geq c_5|x - y|_1) \leq c_6 e^{-c_7|x-y|_1} \leq c_6 e^{-c_8 n^{1/9}}. \quad (2.29)$$

Taking $\lambda_0 = dc_5$ and combining the bounds (2.27)–(2.29) completes the proof. \square

Let $m \geq k_0 \vee k_1$, and let $\{T^m(\tilde{x}), \tilde{x} \in \mathbb{Z}^d\}$ be the tiling of \mathbb{Z}^d by disjoint cubes of side $m - 1$ given by (2.7). Let $\alpha_1 = 1/(4 + d)$ and define (on the space (Ω, \mathbb{P}_p) carrying the bond percolation process η_e)

$$\psi_{\tilde{x}}^{(m)} = 1_{H(T^m(\tilde{x}), \alpha_1) \cap D(T^m(\tilde{x}), \alpha_1)}, \quad \tilde{x} \in \mathbb{Z}^d. \quad (2.30)$$

Lemma 2.21. *There exists $C_E = C_E(d, p) \geq 1$ such that for any $k \geq C_E$ the process $\psi^{(m)}$, $\tilde{x} \in \mathbb{Z}^d$, under \mathbb{P}_p dominates Bernoulli site percolation on \mathbb{Z}^d with parameter q_2 .*

Proof. Note that $\psi_{\tilde{x}}^{(m)}$ and $\psi_{\tilde{y}}^{(m)}$ are independent if $|\tilde{x} - \tilde{y}|_\infty \geq 3$. Using this, and the fact that $\mathbb{P}_p(H(Q, \alpha_1) \cap D(Q, \alpha_1)) \rightarrow 1$ as $p \uparrow 1$, this is an immediate consequence of Theorem 0.0 of [LSS]. \square

Let λ_0 be as in Lemma 2.20, and Q be a cube of side n . For $x_0, x_1 \in Q$, and $C_E \leq m \leq n^{1/9}$ set

$$\begin{aligned} L(Q, m, x_0, x_1) = & \\ & \{\text{there exist } x'_0, x'_1 \in \mathcal{C}^\vee(Q^+) \text{ with } |x_j - x'_j| \leq n^{2/9}, j = 0, 1, \\ & k \text{ with } mk < 2\lambda_0|x_0 - x_1|_\infty \text{ and a path } \{\tilde{y}_0, \dots, \tilde{y}_k\} \text{ in } (\mathbb{Z}^d, \mathbb{E}_d) \text{ such that} \\ & T^m(\tilde{y}_i) \subseteq Q^+, 0 \leq i \leq k, x'_0 \in T^m(\tilde{y}_0), x'_1 \in T^m(\tilde{y}_k), \\ & \text{and } H(T^m(\tilde{y}_i), \alpha_1) \cap D(T^m(\tilde{y}_i), \alpha_1) \text{ holds for each } i\}. \end{aligned} \quad (2.31)$$

Lemma 2.22. *Let Q be a cube of side n , and let $C_E \leq m \leq n^{1/9}$. Then if $|x_0 - x_1|_\infty \geq n^{2/9}$*

$$\mathbb{P}_p(L(Q, m, x_0, x_1)^c) \leq c \exp(-c_1 n^{1/11}). \quad (2.32)$$

Proof. As m is fixed in this argument, we write $T(\tilde{x})$ for $T^m(\tilde{x})$. Let n' be such that $mn' \geq n \geq m(n' - 1)$.

Let \tilde{Q} be a (macroscopic) cube of side n' such that $Q \subset \cup\{T(\tilde{x}), \tilde{x} \in \tilde{Q}\}$. Let \tilde{x}_i be such that $x_i \in T(\tilde{x}_i)$, and $s = |\tilde{x}_0 - \tilde{x}_1|_1$, so that $m(s - 1) \leq |x_0 - x_1|_\infty \leq m(s + 1)$ and $s \geq n^{1/9}$. Let $\tilde{S} = \tilde{S}(\psi^{(m)}, \tilde{Q}, \lambda_0, \tilde{x}_0, \tilde{x}_1)$ be the event defined from the process $\psi^{(m)}$ in the same way as $S(\cdot)$ in (2.25) is for ζ . Then, as S is an increasing event, by Lemmas 2.20 and 2.21,

$$\mathbb{P}_p(\tilde{S}) \leq ce^{-c'(n')^{1/9}} \leq ce^{-c_2 n^{1/11}}. \quad (2.33)$$

Let $\omega \in \tilde{S}$, and $\tilde{x}'_0 = \tilde{y}_0, \dots, \tilde{y}_k = \tilde{x}'_1$ be the open path (with respect to $\psi^{(m)}$) given by the event \tilde{S} . Since \tilde{S} occurs we have $|\tilde{x}_i - \tilde{x}'_i|_\infty \leq (n')^{1/9}$ for $i = 0, 1$, and $k \leq \lambda|\tilde{x}_0 - \tilde{x}_1|_\infty$. Choose $x'_i \in T(\tilde{x}'_i) \cap \mathcal{C}^\vee(Q^+)$; then

$$|x_i - x'_i|_\infty \leq m(1 + |\tilde{x}_i - \tilde{x}'_i|_\infty) \leq m(1 + (n')^{1/9}) < n^{2/9}.$$

Also, since $k \leq \lambda_0 s$, $mk \leq ms\lambda_0 \leq \lambda_0(m + |x_0 - x_1|_\infty) \leq 2\lambda_0|x_0 - x_1|_\infty$. Thus $\omega \in L(Q, m, x_0, x_1)$, and using (2.33) this proves the Lemma. \square

Now let $\alpha_2 = (11(d+2))^{-1}$ and let

$$L(Q) = H(Q, \alpha_2) \cap D(Q, \alpha_2) \cap \{L(Q, m, x, y) \text{ holds for every } x, y \in Q, \text{ with } |x - y|_\infty \geq n^{2/9} \text{ and } C_E \leq m \leq n^{1/9}\}.$$

Theorem 2.23. *Let Q be a cube of side n , and $p > p_c$.*

- (a) $\mathbb{P}_p(L(Q)^c) \leq c_1 \exp(-c_2 n^{\alpha_2 \beta})$.
- (b) *Let $\omega \in L(Q)$, and $C_E \leq m \leq n^{1/9}$. Then if $x_0, x_1 \in Q \cap \mathcal{C}^\vee(Q^+)$ with $d_\omega(x_0, x_1) \geq \frac{1}{3}n^{1/4}$ there exist $x'_i \in \mathcal{C}^\vee(Q^+)$ with $d_\omega(x_i, x'_i) \leq \frac{1}{3}n^{1/4}$, and a path $\gamma = (z_0, \dots, z_j)$ in $\mathcal{C}^\vee(Q^+)$ connecting x'_0, x'_1 such that*
 - (i) *For each $0 \leq l \leq j$, the ball $B_l = B_\omega(z_l, m/16)$ is very good, with $N_{B_l} \leq C_H m^{\alpha_1}$.*
 - (ii) *$j \leq c_3|x_0 - x_1|_\infty \leq C_F|x_0 - x_1|_1$.*

Proof. (a) is immediate from the bounds in Lemma 2.13, Proposition 2.17 and Lemma 2.22.

(b) Since $D(Q, \alpha_2)$ occurs, $|x_0 - x_1|_\infty \geq cd_\omega(x_0, x_1) \geq n^{2/9}$ and so $\omega \in L(Q, m, x_0, x_1)$. Let $x'_i, \tilde{y}_0, \dots, \tilde{y}_k$ be as in (2.31). Note that we can choose x'_i to be within a distance $m/8$ of the centre of the cubes $T(\tilde{y}_0)$ and $T(\tilde{y}_k)$. Then, by Proposition 2.17(d), $d_\omega(x_i, x'_i) \leq C_H(1 + n^{\alpha_2}) \vee |x_i - x'_i|_\infty \leq cn^{2/9} \leq \frac{1}{3}n^{1/4}$.

We now show that the clusters $\mathcal{C}^\vee(Q_i)$ are all in $\mathcal{C}^\vee(Q^+)$. Consider first two adjacent cubes $T(y_i)$ and $T(y_{i+1})$. Since the event $R(T(y_i)) \cap R(T(y_{i+1}))$ occurs, the clusters $\mathcal{C}^\vee(T(y_i^+))$ and $\mathcal{C}^\vee(T(y_{i+1}^+))$ are connected. Thus there exists a Q^+ -cluster \mathcal{C} which contains each $\mathcal{C}^\vee(T(y_i^+))$, and so has diameter D with $D \geq |x'_0 - x'_1|_\infty \geq |x_0 - x_1|_\infty - 2n^{1/9} \geq \frac{1}{2}n^{1/4}$. It now follows, as in Lemma 2.13(b), that $\mathcal{C} \subset \mathcal{C}^\vee(Q^+)$.

Since each event $D(Q_i, \alpha_1)$ holds we can find a path $\gamma = \{z_0, \dots, z_j\} \subseteq \cup T(y_i)$ connecting x'_0 and x'_1 with length $j \leq 2C_H mk \leq c|x_0 - x_1|_\infty$. Each point z_i is in a cube Q_i for which $H(Q_i, \alpha_1) \cap D(Q_i, \alpha_1)$ occurs, so using Theorem 2.18(c) $B_i = B_\omega(z_i, m/16)$ is very good with $N_{B_i} \leq C_H m^{\alpha_1}$. \square

Lemma 2.24. *Let $p > p_c$, and for each $x \in \mathbb{Z}^d$ let N_x be the largest n such that $L(Q)$ fails for some Q with $s(Q) = n$ and $x \in Q$. Then*

$$\mathbb{P}_p(N_x \geq n) \leq c_1 \exp(-c_2 n^{\alpha_2 \beta}). \quad (2.34)$$

Proof. This is immediate from Theorem 2.23(a). \square

3. Upper bounds

We now consider a connected graph $\mathcal{G} = (G, E)$ satisfying the conditions (1.1) and (1.2). We use the notation of Section 1, and study the transition density $q_t(x, y)$ of the continuous time r.w. Y_t on \mathcal{G} . Fix constants C_V , C_P and C_W , and recall from Definition 1.7 the definition of good and very good balls, and of N_B . In this section the constants c_i will depend on the constants d , C_0 , C_V , C_P and C_W in (1.1), (1.15) and (1.16).

As in Section 2, we will assume without always stating it explicitly that the radius R of a ball $B(x, R)$ is sufficiently large; that is that $R \geq c_0 = c_0(d, C_0, C_P, C_W)$. All the bounds in this section hold for balls $B(x, R)$ with $R \leq c_0$, with a suitable choice of the constants c_i in the bounds, by elementary arguments.

We begin by investigating the on-diagonal decay of $q_t(x, x)$. We remark that a similar result is proved in [MR], using an isoperimetric inequality directly. We give another proof here because it is quite short, and also our method allows us to estimate the ‘initial time’ T_B directly in terms of N_B .

Proposition 3.1. (See [MR, Theorem 1.2.]). *Let $x_0 \in G$, and $B = B(x_0, R)$ be very good, so that $N_B^{2+d} \leq R$. Then*

$$q_t(x_1, x_1) \leq \frac{c_2}{t^{d/2}}, \quad \text{for } c_1 N_B^{2d} \leq t \leq \frac{R^2}{\log R}, \quad \text{and } x_1 \in B(x_0, \frac{8}{9}R). \quad (3.1)$$

Proof. Let $c_3 < c_{1.1.5}$, $t_2 = c_3 R^2 / \log R$ and suppose that $t \leq t_2$. Then provided $R \geq c$ we have $t \log t \leq c_3 R^2$, and hence that $t \leq \exp(c_3 R^2 / t)$. Fix $x_1 \in B(x_0, \frac{8}{9}R)$, write $f_t(x) = q_t(x_1, x)$, and let $\psi(t) = \int q_t(x_1, y)^2 d\mu = q_{2t}(x_1, x_1)$. Note that by (1.7)

$$\frac{c_4}{(t \log t)^{d/2}} \leq \psi(t) = q_{2t}(x_1, x_1) \leq \frac{1}{c_5 t^{1/2}}, \quad t \geq 1. \quad (3.2)$$

Using the discrete Gauss-Green formula

$$\psi'(t) = 2 \sum_x f_t(x) \frac{\partial f_t(x)}{\partial t} = 2 \sum_x f_t(x) \mathcal{L} f_t(x) = - \sum_{x,y} a_{xy} (f_t(x) - f_t(y))^2,$$

and in particular $\psi(t)$ is decreasing (and continuous).

Define t_1 so that $t_1^{1/2} = c_6 N_B^d$, where c_6 will be chosen later, and choose $r(t)$ so that

$$\frac{2}{c_5 c_6} \geq r(t)^d \psi(t) \geq \frac{1}{c_5 c_6}. \quad (3.3)$$

Then, if $t \geq t_1$, by (3.2) $\psi(t)^{-1} \geq c_5 c_6 N_B^d$ and so $r(t) \geq N_B$. If $t \leq t_2$ then by (3.2) and (3.3), $r(t)^d \leq c(t \log t)^{d/2} \leq c c_3^{d/2} R^2$, so if the constant c_3 is chosen small enough, we have $r(t) \leq R/18$.

Write $B' = B(x_0, 17R/18)$. Let $t \in [t_1, t_2]$, so that $r = r(t) \in [r_0, R/18]$, and let $B(y_i, r/2)$, $i = 1, \dots, m$, be a maximal collection of disjoint balls with centres in B' . Set $B_i = B(y_i, r)$, and $B_i^* = B(y_i, C_W r)$. Note that $B' \subseteq \cup_i B_i \subseteq B$. If $x \in B \cap B_i^*$ we have $B(y_i, r/2) \subseteq B(x, r(1 + C_W))$, and so

$$\begin{aligned} C_0(1 + C_W)^d r^d &\geq \mu(B(x, r(1 + C_W))) \\ &\geq \sum_i 1_{\{x \in B_i^*\}} \mu(B(y_i, r/2)) \geq |\{i : x \in B_i^*\}| C_V 2^{-d} r^d. \end{aligned}$$

Thus any $x \in B$ is in at most c_7 of the B_i^* .

The bounds on r above imply that each $B(y_i, r)$ is good. So, applying the Poincaré inequality (1.16) to each B_i , and writing $\bar{f}_{t,i} = \mu(B_i)^{-1} \int_{B_i} f_t$, we have

$$\begin{aligned} -c_7 \psi'(t) &\geq \sum_i \int_{E(B_i^*)} |\nabla f_t|^2 \\ &\geq C_P^{-1} r^{-2} \sum_i \int_{B_i} |f_t - \bar{f}_{t,i}|^2 \\ &= C_P^{-1} r^{-2} \sum_i \int_{B_i} f_t^2 - C_P^{-1} r^{-2} \sum_i \mu(B_i)^{-1} \left(\int_{B_i} f_t \right)^2. \end{aligned} \quad (3.4)$$

By Lemma 1.1(b)

$$\int_{G-B'} f_t^2 \leq \left(\sup_{G-B'} f_t \right) \int_{G-B'} f_t \leq c t^{-d},$$

while, by (3.2), $\psi(t) \geq c_4(t \log t)^{-d/2}$. So as $t \geq c_1$,

$$\sum_i \int_{B_i} f_t^2 \geq \int_{B'} f_t^2 = \psi(t) - \int_{G-B'} f_t^2 \geq \frac{1}{2} \psi(t).$$

Also, since f_t has total mass 1,

$$\sum_i \mu(B_i)^{-1} \left(\int_{B_i} f_t \right)^2 \leq (C_V r^d)^{-1} \left(\sum_i \int_{B_i} f_t \right)^2 \leq c_7 (C_V r^d)^{-1} = c_8 r(t)^{-d}.$$

Combining these estimates we obtain

$$-c_7 \psi'(t) \geq C_P^{-1} r(t)^{-2} \left(\frac{1}{2} \psi(t) - c_8 r(t)^{-d} \right). \quad (3.5)$$

Now let $c_6 = (4c_5 c_8)^{-1}$ so that by the choice of $r = r(t)$ in (3.3)

$$-\psi'(t) \geq c_9 r(t)^{-2} \psi(t) \geq c_{10} \psi(t)^{1+2/d}.$$

Setting $\varphi(t) = \psi(t)^{-2/d}$ we have $\varphi'(t) \geq 2c_{10}/d$, from which it follows that

$$\varphi(t) \geq \varphi(t_1) + (2c_{10}/d)(t - t_1) \geq c_{11}t, \quad 2t_1 \leq t \leq t_2.$$

Rearranging, this gives $\psi(t) \leq c t^{-d/2}$ for $2t_1 \leq t \leq t_2$. Since ψ is decreasing it follows, adjusting the constant c , that $q_t(x_1, x_1) = \psi(t/2) \leq c_{11} t^{-d/2}$ for $4t_1 \leq t \leq R^2/\log R$. \square

We need a bound for y outside $B(x_0, R)$.

Corollary 3.2. *Let $x_0 \in G$, and $B = B(x_0, R)$ be very good. Then*

$$q_t(x_1, y) \leq \frac{c_1}{t^{d/2}}, \quad \text{for } c_2 N_B^{2d} \leq t \leq \frac{c_3 R^2}{\log R}, \quad x_1 \in B(x_0, \frac{7}{9}R), \quad \text{and } y \in G. \quad (3.6)$$

Proof. If $y \in B(x_0, \frac{8}{9}R)$ then $q_t(x_1, y) \leq q_t(x_1, x_1)^{1/2} q_t(y, y)^{1/2} \leq ct^{-d/2}$, by Proposition 3.1. If $y \notin B(x_1, \frac{8}{9}R)$ then $d(x, y) \geq R/9$ and we use Lemma 1.1(a). \square

For a very good ball B let $T_B = c_{3.2.2} N_B^{2d}$, $T'_B = c_{3.2.3} R^2 / \log R$.

Remark. It is natural to ask if the bounds in (3.1) and (3.6) hold for $t \leq cR^2$ rather than just $t \leq cR^2 / \log R$. However, in this paper this restriction on t will not matter, since we will ultimately apply (3.6) in the situation where $B(x_0, R)$ is very good for all sufficiently large R .

We now use the method of Bass [Bas] to obtain off-diagonal upper bounds. Following [N], [Bas] we introduce the functions, for $x_1 \in G$, $t > 0$,

$$\begin{aligned} M(t) &= M(x_1, t) = \sum_y d(x_1, y) q_t(x_1, y) \mu(y), \\ Q(t) &= Q(x_1, t) = - \sum_y q_t(x_1, y) \log q_t(x_1, y) \mu(y). \end{aligned}$$

We can extend M and Q to $t = 0$ by continuity: $M(0) = 0$, while since $q_t(x_1, x_1) \rightarrow \mu(x_1)^{-1}$ as $t \downarrow 0$, $Q(0) = \log \mu(x_1) > 0$.

Lemma 3.3. *Let $B(x_0, R)$ be very good, and $x_1 \in B(x_0, \frac{7}{9}R)$.*

(a) *We have*

$$Q(x_1, t) \geq c_1 + \frac{1}{2}d \log t, \quad T_B \leq t \leq T'_B.$$

(b)

$$M(x_1, t) \geq c_2 e^{Q(x_1, t)/d}, \quad t \geq c_3.$$

Proof. Fix $x_1 \in B(x_0, \frac{7}{9}R)$. (a) follows directly from the upper bound (3.6).

The proof of (b) is similar to that in [N] or [Bas]. Let $0 < a < 1$ and set $D_0 = \{x_0\}$, and $D_n = B(x_0, 2^n) - B(x_0, 2^{n-1})$ for $n \geq 1$. Then using (1.1) to bound $\mu(D_n)$ we have, for $a \leq 2$,

$$\begin{aligned} \sum_{y \in G} e^{-ad(x_1, y)} \mu(y) &\leq \sum_{n=0}^{\infty} \sum_{y \in D_n} \exp(-a2^n) \mu(y) \\ &\leq \sum_{n=0}^{\infty} C_0 2^{nd} \exp(-a2^n) \leq c_4 a^{-d}. \end{aligned}$$

Now note that $u(\log u + \lambda) \geq -e^{-1-\lambda}$ for $u > 0$. So, setting $\lambda = ad(x_0, y) + b$, where $a \leq 2$,

$$\begin{aligned} -Q(x_1, t) + aM(x_1, t) + b &= \sum_y q_t(x_1, y) (\log q_t(x_1, y) + ad(x_1, y) + b) \mu(y) \\ &\geq - \sum_y e^{-1-ad(x_1, y)-b} \mu(y) \\ &\geq -e^{-1-b} \sum_y e^{-ad(x_1, y)} \mu(y) \geq -c_5 e^{-b} a^{-d}. \end{aligned}$$

Since $M(x_1, t) \geq P^{x_1}(Y_t \neq x_1)$, using (1.7) we have $M(x_1, t) \geq \frac{1}{2}$ when $t \geq c_5$. Setting $a = 1/M(x_1, t)$ and $e^b = M(x_1, t)^d = a^{-d}$ we obtain

$$-Q(x_1, t) + 1 + d \log M(x_1, t) \geq -c_4,$$

and rearranging gives (b). □

Proposition 3.4. *Let $x_0 \in G$, and $B(x_0, R)$ be very good. Then*

$$c_1 t^{1/2} \leq M(x_1, t) \leq c_2 t^{1/2}, \quad \text{for } x \in B(x_0, \frac{7}{9}R), \quad \text{and } T_B \log T_B \leq t \leq T'_B. \quad (3.7)$$

Proof. For the moment we just write $Q(t)$, $M(t)$. Set $f_t(x) = q_t(x_1, x)$, and let $b_t(x, y) = f_t(x) + f_t(y)$. We have

$$\begin{aligned} M'(t) &= \sum_y d(x_1, y) \frac{\partial f_t(y)}{\partial t} \mu(y) = \sum_y d(x_1, y) \mathcal{L} f_t(y) \mu(y) \\ &= -\frac{1}{2} \sum_x \sum_y a_{xy} (d(x_1, y) - d(x_1, x)) (f_t(y) - f_t(x)) \\ &\leq \frac{1}{2} \sum_x \sum_y a_{xy} |f_t(y) - f_t(x)| \\ &\leq \frac{1}{2} \left(\sum_x \sum_y a_{xy} b_t(x, y) \right)^{1/2} \left(\sum_x \sum_y a_{xy} \frac{(f_t(y) - f_t(x))^2}{b_t(x, y)} \right)^{1/2} \\ &\leq c \left(\sum_x \sum_y a_{xy} \frac{(f_t(y) - f_t(x))^2}{f_t(x) + f_t(y)} \right)^{1/2}. \end{aligned} \quad (3.8)$$

In the calculation above the use of discrete Gauss-Green to obtain (3.8) is valid since by (1.5) $q_t(x_1, \cdot)$ decays exponentially. Since we have, for $u, v > 0$,

$$\frac{(u - v)^2}{u + v} \leq (u - v)(\log u - \log v),$$

we deduce

$$M'(t)^2 \leq \sum_x \sum_y a_{xy} (f_t(y) - f_t(x)) (\log f_t(y) - \log f_t(x)).$$

On the other hand (again using (1.5) and discrete Gauss-Green)

$$\begin{aligned} Q'(t) &= - \sum_y (1 + \log f_t(y)) \mathcal{L} f_t(y) \\ &= \frac{1}{2} \sum_x \sum_y a_{xy} (\log f_t(y) - \log f_t(x)) (f_t(y) - f_t(x)) \geq \frac{1}{2} M'(t)^2. \end{aligned} \quad (3.10)$$

The remainder of this proof is similar to that in [N] or [Bas], except that we have to control the growth of M for small t . Set $R(t) = d^{-1}(Q(t) - c_{3.3.1} - \frac{1}{2}d \log t)$, so that $R(t) \geq 0$ if $T_B \leq t \leq T'_B$. Define

$$T_0 = \begin{cases} 1, & \text{if } R(t) \geq 0 \text{ on } [1, T'_B], \\ \sup\{t \leq T'_B : R(t) < 0\}, & \text{otherwise.} \end{cases}$$

If $T_0 > 1$ then $T_0 \leq T_B$, and by (3.10)

$$\begin{aligned} M(T_0) &= \int_0^{T_0} M'(s) ds \leq 2^{1/2} \int_0^{T_0} Q'(s)^{1/2} ds \\ &\leq 2^{1/2} \left(\int_0^{T_0} Q'(s) ds \right)^{1/2} T_0^{1/2} \\ &\leq c_3 T_0^{1/2} (Q(T_0) - Q(0))^{1/2} \\ &\leq c_3 T_0^{1/2} (c_{3.3.1} + \frac{1}{2}d \log T_0)^{1/2} \leq c_4 (T_B \log T_B)^{1/2}. \end{aligned}$$

If $T_0 = 1$ then $M(T_0) = E^x d(x, Y_1) \leq c_5$ by elementary arguments.

By Lemma 3.3(b) and (3.10), if $T_0 < t < T'_B$ then

$$\begin{aligned} ct^{1/2} e^{R(t)} &= e^{Q(t)/d} \leq M(t) \leq M(T_0) + 2^{1/2} \int_{T_0}^t Q'(s)^{1/2} ds \\ &\leq M(T_0) + (2d)^{1/2} \int_{T_0}^t (R'(s) + \frac{1}{2s})^{1/2} ds. \end{aligned}$$

Using the inequality $(a + b)^{1/2} \leq b^{1/2} + a/(2b)^{1/2}$ gives

$$ct^{1/2} e^{R(t)} \leq M(t) \leq M(T_0) + ct^{1/2} + c \int_{T_0}^t s^{1/2} R'(s) ds. \quad (3.11)$$

Integrating by parts, and using the fact that $R \geq 0$ on $[T_0, T'_B]$, the final term in (3.11) is bounded by $c(1 + R(t)t^{1/2})$. Combining these estimates, for $T_B \leq t \leq T'_B$,

$$ct^{1/2} e^{R(t)} \leq M(t) \leq c(1 + R(t))t^{1/2} + c_4 (T_B \log T_B)^{1/2}.$$

So for $T_B \log T_B \leq t \leq T'_B$,

$$ct^{1/2} e^{R(t)} \leq M(t) \leq c(1 + R(t))t^{1/2}.$$

Thus $R(t)$ is bounded and this implies (3.7). \square

As in [Bas] we can use the moment bounds in (3.7) to obtain off-diagonal upper bounds on q_t , by the method of [BB1], [BB2]. We define

$$\tau(x, r) = \inf\{t : Y_t \notin B(x, r)\}, \quad x \in G, r > 0,$$

and begin by controlling the probability that $\tau(x, r)$ is small.

Lemma 3.5. *Let $x_0 \in G$, and $B(x_0, R)$ be very good. Let $c_1 N_B^d (\log N_B)^{1/2} \leq r \leq R$. Then*

$$P^x(\tau(x, r) < t) \leq \frac{1}{2} + \frac{c_2 t}{r^2}, \quad \text{for } x \in B(x_0, \frac{6}{9}R), \quad \text{and } 0 \leq t \leq \frac{1}{2}T'_B. \quad (3.12)$$

Proof. Suppose first that $r < R/9$. Let $x \in B(x_0, \frac{6}{9}R)$, $A = B(x, r) \cup \partial B(x, r)$, and $\tau = \tau(x, r)$. Then if $T_B \log T_B \leq t \leq \frac{1}{2}T'_B$, since $A \subseteq B(x_0, \frac{7}{9}R)$,

$$\begin{aligned} c_3 t^{1/2} &\geq E^x d(x, Y_{2t}) \geq E^x (d(x, Y_{t \wedge \tau}) - d(Y_{t \wedge \tau}, Y_{2t})) \\ &\geq E^x 1_{(\tau < t)} d(x, Y_\tau) - E^x (E^{Y_{t \wedge \tau}} d(Y_{t \wedge \tau}, Y_{2t-t \wedge \tau})) \\ &\geq P^x(\tau < t) r - \sup_{z \in A, s \leq t} E^z d(z, Y_{2t-s}) \\ &\geq P^x(\tau < t) r - c_3 t^{1/2}. \end{aligned}$$

Thus

$$P^x(\tau < t) \leq 2c_3 t^{1/2}/r. \quad (3.13)$$

Since $\lambda \leq \frac{1}{2}(1 + \lambda^2)$, (3.12) is immediate. If $t \leq T_B \log T_B$ then

$$P^x(\tau(x, r) < t) \leq P^x(\tau(x, r) < T_B \log T_B) \leq 2c_3 (T_B \log T_B)^{1/2} r^{-1} \leq \frac{1}{2},$$

provided $r \geq c(T_B \log T_B)^{1/2} = c' N_B^d (\log N_B)^{1/2}$. Finally if $R/9 \leq r \leq R$ we have $\tau(x, r) \geq \tau(x, R/9)$, so (adjusting the constant c_2) we deduce (3.12). \square

Remark. In the end one gains nothing useful by using the stronger bound (3.13).

We will need the following estimate.

Lemma 3.6. *[BB1, Lemma 1.1.] Let $\xi_1, \xi_2, \dots, \xi_n, V$ be non-negative r.v. such that $V \geq \sum_1^n \xi_i$. Suppose that for some $p \in (0, 1)$, $a > 0$,*

$$P(\xi_i \leq t | \sigma(\xi_1, \dots, \xi_{i-1})) \leq p + at, \quad t > 0.$$

Then

$$\log P(V \leq t) \leq 2 \left(\frac{ant}{p} \right)^{1/2} - n \log \frac{1}{p}.$$

Proposition 3.7. *Let $x_0 \in G$, and $B(x_0, R)$ be very good. If $x \in B(x_0, \frac{5}{9}R)$, and $t > 0$, $\rho > 0$ satisfy*

$$\rho \leq R, \quad \text{and } c_1 N_B^d (\log N_B)^{1/2} \rho \leq t, \quad \text{and } t \leq T'_B, \quad (3.14)$$

then

$$P^x(\tau(x, \rho) < t) \leq c_2 e^{-c_3 \rho^2/t}. \quad (3.15)$$

Proof. Let $r_1 = c_{3.5.1} N_B^d (\log N_B)^{1/2}$. Suppose first that in addition $\rho < R/9$. Let $m \geq 1$ be chosen later, and $s = t/m$ and $r = \lfloor \rho/m \rfloor$. Define stopping times

$$S_0 = 0, \quad S_i = \inf\{t \geq S_{i-1} : d(Y_{S_{i-1}}, Y_t) = r\}, \quad i \geq 1.$$

Set $\xi_i = S_i - S_{i-1}$, and write $\mathcal{F}_t = \sigma(Y_s, s \leq t)$ for the filtration of Y . By Lemma 3.5

$$P^x(\xi_i > u | \mathcal{F}_{S_{i-1}}) \leq \frac{1}{2} + \frac{c_4 u}{r^2}, \quad u > 0, \quad (3.16)$$

provided $r_1 \leq r \leq R$, $u \leq T'_B$, and $Y_{S_{i-1}} \in B(x_0, \frac{6}{9}R)$. As $d(Y_0, Y_{S_m}) \leq mr \leq \rho < R/9$, we have $S_m \leq \tau(x, \rho)$ and $Y_{S_j} \in B(x_0, \frac{6}{9}R)$ for $0 \leq j \leq m$. Using Lemma 3.6, and writing $p = \frac{1}{2}$, $a = c_4/r^2$, we deduce that

$$\log P^x(\tau(x, \rho) < t) \leq \log P^x(S_m < t) \leq 2(amt/p)^{1/2} - m \log p^{-1}.$$

Simplifying this expression we obtain

$$\log P^x(\tau(x, \rho) < t) \leq -c_5 m \left(1 - \left(\frac{c_6 t m}{\rho^2}\right)^{1/2}\right). \quad (3.17)$$

Let $\lambda = \rho^2/(2c_6 t)$; if we can choose $m \in \mathbb{N}$ with $\frac{1}{2}\lambda \leq m < \lambda$, and so that the estimate (3.16) is valid, then (3.17) implies (3.15).

If $\lambda \leq 1$ then, adjusting the constant c_2 appropriately, (3.15) is immediate. If $\lambda > 1$ then let $m = \lfloor \frac{1}{2}\lambda \rfloor + 1$. Since then $m \geq 1$ we have $s \leq t \leq T'_B$, and $r \leq \rho \leq R$, while the condition $\rho \leq c_6 t/r_1$ ensures that $r \geq r_1$.

Finally let ρ satisfy (3.14) but with $\rho \geq R/9$. Then (adjusting c_1 if necessary) we can apply the argument above to $\rho_0 = R/9$, and adjusting the constant c_3 we obtain (3.15). \square

Theorem 3.8. *Let $x_0 \in G$, and $B(x_0, R)$ be very good. Let $x \in B(x_0, \frac{1}{2}R)$, $y \in G$, and assume that*

$$N_B^{2d+1} \vee d(x, y) \leq t \leq \frac{c_1 R^2}{\log R}. \quad (3.18)$$

Then

$$q_t(x, y) \leq c_2 t^{-d/2} e^{-c_3 d(x, y)^2/t}. \quad (3.19)$$

Proof. Let $D = d(x, y)$. Using (1.5) we have, since $D \leq t$,

$$q_t(x, y) \leq c_5 e^{-2c_4 D^2/t}.$$

If $t \log t \leq 2c_4 d^{-1} D^2$ then $\exp(-c_4 D^2/t) \leq t^{-d/2}$ and we deduce that

$$q_t(x, y) \leq c_5 t^{-d/2} e^{-c_4 D^2/t}.$$

Suppose therefore that $t \log t \geq 2c_4 d^{-1} D^2$. Note that this implies that $y \in B(x, \frac{5}{9}R)$, provided $R \geq c$ and c_1 in (3.18) is chosen small enough. Let $A_x = \{z : d(x, z) \leq d(y, z)\}$, $A_y = G - A_x$, $s = t/2$, $\rho = D/2$. Note that $B(x, \rho) \subseteq A_x$. Then

$$\mu(x)P^x(Y_t = y) = \mu(x)P^x(Y_t = y, Y_s \in A_y) + \mu(x)P^x(Y_t = y, Y_s \in A_x). \quad (3.20)$$

To bound the first term in (3.20) we write

$$\begin{aligned} P^x(Y_t = y, Y_s \in A_y) &= P^x(\tau(x, \rho) < s, Y_s \in A_y, Y_t = y) \\ &\leq E^x(1_{\{\tau(x, \rho) < s\}} P^{Y_\tau}(Y_{t-\tau} = y)) \\ &\leq P^x(\tau(x, \rho) < s) \sup_{z \in \partial B(x, \rho), s \leq t} q_{2t-s}(z, y) \mu(y). \end{aligned} \quad (3.21)$$

Since $2T_B < \frac{1}{2}N_B^{2d+1} \leq s < T'_B$, by Corollary 3.2 the second term in (3.21) is bounded by $ct^{-d/2}$. To control the first term we use Proposition 3.7. We have $\rho < D < R$ and $s < T'_B$, while since $t \geq N_B^{2d+1}$,

$$c_{3.7.1} N_B^d (\log N_B)^{1/2} \rho \leq c N_B^d (\log N_B)^{1/2} (t \log t)^{1/2} \leq \frac{1}{2}t = s,$$

the three conditions in (3.14) are satisfied, and by (3.15),

$$P^x(Y_t = y, Y_s \in A_y) \leq ct^{-d/2} e^{-c'D^2/s}.$$

By symmetry the second term in (3.20) equals

$$\mu(y)P^y(Y_t = x, Y_s \in A_x), \quad (3.22)$$

and so can be bounded in the same way as the first term. Combining these estimates completes the proof. \square

4. A weighted Poincaré inequality

While the weak Poincaré inequality of section 2 is enough for upper bounds on the transition density, to obtain lower bounds we will need a weighted Poincaré inequality, which we will derive using the methods of [J], [SCS].

We continue with the notation and assumptions of the previous section. Fix $x_0 \in G$, $R \in \mathbb{N}$, and let $B = B(x_0, R)$ be a very good ball with $R_0 = N_B \leq R^{1/(1+d)}$. For each $x, y \in G$ we write $\gamma(x, y)$ for a shortest path $x = z_0, \dots, z_{d(x,y)} = y$ between x and y .

We begin with a Whitney decomposition of B , which we need to adapt to our situation. We have two differences from [J], which both arise on small length scales. The first, minor, difficulty is that in our discrete setting we cannot use balls of size smaller than 1. The second difficulty is that we do not have any volume doubling estimate for balls smaller than R_0 .

Let (X, d) be the metric space obtained as the ‘cable system’ of \mathcal{G} . This is the metric space obtained by replacing each edge e by a copy of $(0, 1)$, linked in the obvious way at the vertices $x \in G$. We define a measure $\tilde{\mu}$ on X by taking $\tilde{\mu}$ to be Lebesgue measure on each cable. See, for example, [BB3] for further details of this construction. We write $\tilde{B}(x, r)$ for balls in X . Since $R \in \mathbb{N}$ the boundary of \tilde{B} is contained in G . For $x \in \tilde{B} = \tilde{B}(x_0, R)$ we write $\rho(x) = d(x, \tilde{B}^c)$; note that if $x \in G$ then $\rho(x) = d(x, G - B)$. We will frequently use the inequality

$$|\rho(x) - \rho(y)| \leq d(x, y). \quad (4.1)$$

Let $\lambda \geq 10^3 \vee (21C_W)$, $10 \leq K \leq \lambda/10$ be fixed constants. We can assume that $R_0 > \lambda$.

Lemma 4.1. *There exists a sequence of disjoint balls $\tilde{B}_i = \tilde{B}(x_i, r_i)$, $i \geq 1$ such that $r_1 \geq r_2 \geq \dots$ and*

- (a) $\tilde{B} = \cup_{i=1}^{\infty} \tilde{B}_i$,
- (b) $\rho(x_i) = \lambda r_i$ for each i .
- (c) If $y \in \tilde{B}(x_i, Kr_i)$ then

$$(\lambda - K)r_i \leq \rho(y) \leq (\lambda + K)r_i. \quad (4.2)$$

Proof. This is standard. We start by choosing a ball \tilde{B}_1 of maximal size satisfying (b), and continue, so that \tilde{B}_n is chosen to be the largest ball contained in $\tilde{B} - \cup_{i=1}^{n-1} \tilde{B}_i$ satisfying (b). To prove (a), suppose $y \notin \cup \tilde{B}_i$. Since $\tilde{\mu}(\tilde{B}) < \infty$ and $\tilde{\mu}(\tilde{B}_i) \geq r_i$ we must have $r_i \rightarrow 0$. Let $t_i = d(y, x_i) \geq 2r_i$, $t = \inf t_i$. Then by (4.1)

$$\rho(y) \leq \rho(x_i) + t_i = \lambda r_i + t_i \leq (1 + \frac{1}{2}\lambda)t_i,$$

so that $t \geq c\rho(y) = t' > 0$, contradicting the definition of \tilde{B}_i if $r_i < t$.

(c) follows immediately from (4.1) and (b). \square

We now adapt this construction to our discrete setup. Let N be defined by $r_N \geq R_0 + 1 > r_{N+1}$. For each $i \leq N$ the centre x_i of $\tilde{B}_i = \tilde{B}(x_i, r_i)$ will lie on a cable $[y_i, y'_i]$ where $y_i, y'_i \in G$. We label these so that y_i is the point in G closest to x_i , and set $s_i = r_i - d(x_i, y_i)$. Then we have $B(y_i, s_i) \subseteq \tilde{B}(x_i, r_i) \cap G$, and $r_i \geq s_i \geq r_i - \frac{1}{2} > R_0$.

We set $\lambda_1 = \lambda - 2K$, $\lambda_2 = \lambda + 2K$.

Lemma 4.2. *The sequence of disjoint balls $B_i = B(y_i, s_i)$, $1 \leq i \leq N$, defined above satisfy the following:*

(a) For each $i \leq N$

$$\lambda s_i - \frac{1}{2} \leq \rho(y_i) \leq \frac{1}{2}(1 + \lambda) + \lambda s_i. \quad (4.3)$$

(b) If $x \in B - \cup_{i=1}^N B(y_i, 3s_i)$ then $\rho(x) < \lambda_2 R_0$. Further

$$B(x_0, R - \lambda_2 R_0) \subseteq \cup_{i=1}^N B(y_i, 3s_i - 1) \subseteq \cup_{i=1}^N B(y_i, \lambda_1 s_i) \subseteq B(x_0, R). \quad (4.4)$$

(c) If $x \in B(y_i, Ks_i)$ then

$$\lambda_1 s_i \leq (\lambda - K)s_i - \frac{1}{2} \leq \rho(x) \leq (\lambda + K)s_i + \frac{1}{2}(1 + \lambda) \leq \lambda_2 s_i. \quad (4.5)$$

(d) There exists a constant c_1 such that

$$|\{i \leq N : x \in B(y_i, Ks_i)\}| \leq c_1. \quad (4.6)$$

Proof. Since $\rho(x_i) = \lambda r_i$ and $|\rho(y_i) - \rho(x_i)| \leq \frac{1}{2}$ (a) is immediate.

(b) Let $x \in B$. Then $x \in \tilde{B}(x_i, 2r_i)$ for some i , and so $\rho(x) \leq (2 + \lambda)r_i$. If $i > N$ then $r_i < 1 + R_0$ and so $\rho(x) \leq (2 + \lambda)(1 + R_0) < \lambda_2 R_0$, which implies that $d(x_0, x) > R - \lambda_2 R_0$. So, if $\rho(x) \geq \lambda_2 R_0$ then $x \in \tilde{B}(x_i, 2r_i)$ for some $i \leq N$. We then have $d(x, y_i) \leq d(x, x_i) + d(x_i, y_i) < 2s_i + \frac{3}{2}$. Since each $s_i > 3$ this implies that $x \in \cup B(y_i, 3s_i - 1)$.

The final inclusion in (4.4) is immediate from (a), and the first follows from the inequality $d(x_0, x) + \rho(x) \geq R$.

(c) is immediate from (a) and (4.1).

(d) If $x \in B(y_i, Ks_i)$ then by (c) $\rho(x) \geq \lambda_1 s_i$ and $B_i \subseteq B(x, (1 + K)c_1 \rho(x))$, where $c_2 = (1 + K)/\lambda_1$. Also, $\mu(B_i) \geq C_V s_i^d \geq C_V \lambda_2^{-1} \rho(x)^d$. So writing $I = \{i : x \in \tilde{B}(y_i, Ks_i)\}$ we have

$$C_0 c_2^d \rho(x)^d \geq \mu(B(y, c_2 \rho(x))) \geq \sum_{i \in I} \mu(B_i) \geq |I| C_V \lambda_2^{-d} \rho(x)^d, \quad (4.7)$$

which proves (4.6). \square

Let

$$B'_i = B(y_i, 3s_i), \quad 1 \leq i \leq N.$$

Let $\eta = 2\lambda_2$, and set

$$B''_i = B(y_i, 10s_i), \quad \text{if } s_i \geq \eta R_0.$$

If $s_i < \eta R_0$ we will call B_i a *boundary ball*, and define B''_i to be the connected component of $B(y_i, 2\lambda s_i) \cap B$ which contains y_i . (While balls are connected, the intersection of two balls need not be.) We relabel the balls B_i so that $x_0 \in B_1$, and B_i is a boundary ball if and only if $M + 1 \leq i \leq N$.

Lemma 4.3. (a) $B = (\cup_{i=1}^M B'_i) \cup (\cup_{i=M+1}^N B''_i)$.

(b) There exists a constant c_1 such that for any $x \in B$

$$|\{i \leq N : x \in B''_i\}| \leq c_1. \quad (4.8)$$

Proof. (a) Suppose $x \in B$ but $x \notin \cup_i B'_i$. Then $\rho(x) < \lambda_2 R_0 = t$. Now choose $x' \in \gamma(x, x_0)$ with $1 + t \geq \rho(x') > t$, and $x'' \in \gamma(x_0, x')$ with $d(x', x'') = 1$. Then $x' \in B(y_i, 3s_i - 1)$ for some i , so and $\lambda s_i < \rho(x'') < t < \lambda_2 R_0$. Hence

$$R_0 < s_i \leq \frac{\lambda_2 R_0}{\lambda_1} < \eta R_0,$$

so that B_i is a boundary ball. Now $d(x, y_i) \leq d(x, x') + 3s_i - 1 \leq t + 3s_i < 2\lambda s_i$, which proves that $x \in B(y_i, 2\lambda s_i)$. The same argument proves that each $y \in \gamma(x, x')$ is also in $B(y_i, 2\lambda s_i)$. Hence x is connected to x' (and so y_i) by a path in $B(y_i, 2\lambda s_i) \cap B$, and so $x \in B''_i$.

(b) Since $K \geq 10$ we have a bound on $|\{i : x \in B(y_i, 10s_i)\}|$. So it is enough to control $|I'|$ where $I' = \{i : x \in B(y_i, 2\lambda s_i), s_i < \eta R_0\}$. The argument is almost exactly the same as in Lemma 4.2(d): if $i \in I'$ then $s_i < \eta R_0$ and $d(x, y_i) \leq 2\lambda_2 s_i < 2\lambda_2 \eta R_0$. So $B_i \subseteq B(x, cR_0)$ and we use volume bounds as in (4.7). \square

For each i define

$$\begin{aligned}\mathcal{F}(i) &= \{j : \gamma(x_0, y_i) \cap B(y_j, Ks_j) \neq \emptyset\}, \\ \mathcal{F}(i, r) &= \{j \in \mathcal{F}(i) : r \leq s_j \leq 2r\}.\end{aligned}$$

Lemma 4.4. (a) If $j \in \mathcal{F}(i)$ then $s_j \geq \frac{1}{2}\lambda_1\lambda_2^{-1}s_i \geq \frac{1}{4}s_i$.

(b) If $j \in \mathcal{F}(i)$ then $d(y_i, y_j) \leq (K + \lambda_2)s_j$.

(c) $|\mathcal{F}(i, r)| \leq c_1 C_0 / C_V$.

Proof. Let $j \in \mathcal{F}(i)$, so that there exists $x \in B(y_j, Ks_j) \cap \gamma(x_0, y_i)$. Since $x \in \gamma(x_0, y_i)$, $d(x, y_i) = d(x_0, y_i) - d(x_0, x) < R - d(x_0, x) \leq \rho(x)$. Thus using (4.5)

$$\lambda_1 s_i \leq \rho(y_i) \leq d(x, y_i) + \rho(x) \leq 2\rho(x) \leq 2\lambda_2 s_j,$$

which proves (a).

For (b) note that $d(y_i, y_j) \leq d(y_i, x) + Ks_j \leq \rho(x) + Ks_j$.

From the estimates above, if $j \in \mathcal{F}(i, r)$ then $y_j \in B(y_i, 2(K + \lambda_2)r)$, so that $B(y_j, s_j) \subseteq B(y_i, 3\lambda r)$. Hence

$$c_2 C_0 r^d \geq \sum_{j \in \mathcal{F}(i, r)} \mu(B_j) \geq |\mathcal{F}(i, r)| C_V r^d,$$

proving (c). \square

Corollary 4.5. $|\mathcal{F}(i)| \leq c_1 \log(R/s_i)$.

Proof. This follows easily from Lemma 4.4(c). \square

Now let

$$\begin{aligned}\mathcal{F}^*(j) &= \{i : j \in \mathcal{F}(i)\} \\ \mathcal{F}^*(j, r) &= \{i : j \in \mathcal{F}(i), r \leq s_i \leq 2r\}.\end{aligned}$$

Lemma 4.6. There exists $\alpha = \alpha(d, C_0, C_V) > 0$ such that for each $1 \leq j \leq N$ we have, for $r > R_0$,

$$\sum_{i \in \mathcal{F}^*(j, r)} \mu(B'_i) \leq c\mu(B'_j) \left(\frac{r}{s_j}\right)^\alpha.$$

Proof. This argument runs along the same lines as the proof of Lemma 5.9 in [J]. Note first that we can assume that $r \leq 4s_j$, since if $i \in \mathcal{F}^*(j, r)$ then by Lemma 4.4(a) we have $s_i \leq 4s_j$.

Write $\partial B = \{y : d(x_0, y) = R\}$. Fix j ; using (4.5) we have $\rho(y_j) \leq \lambda_2 s_j$, so we can choose $z \in \partial B$ with $d(y_j, z) \leq \lambda_2 s_j$. Set $t = (4 + K + 4\lambda_2)s_j$ and for $u > 0$ let

$$\Lambda(u) = B(z, t + 2u) \cap \{x \in B : \rho(x) \leq u\}.$$

Suppose that $i \in \mathcal{F}^*(j, r)$. By Lemma 4.4(b) $d(y_i, y_j) \leq (K + \lambda_2)s_j$, so that $d(y_i, z) \leq (K + 2\lambda_2)s_j$, and

$$B'_i \subseteq B(z, (K + 2\lambda_2 + 4)s_j).$$

By (4.5) $\rho(x) \leq 2\lambda_2 r$ on B'_i , so $B'_i \subseteq \Lambda(2\lambda_2 r)$. As the balls B'_i are disjoint,

$$\sum_{i \in \mathcal{F}^*(j, r)} \mu(B'_i) \leq \mu(\Lambda(2\lambda_2 r)). \quad (4.9)$$

Now fix $u > 0$, and choose a maximal set of points $\{z_1, \dots, z_m\} \subseteq \partial B$ such that $B(z_k, u)$ are disjoint and $d(z, z_k) \leq t + u$ for each k . We next show that

$$\Lambda(u/4) \subseteq \bigcup_{k=1}^m B(z_k, 3u). \quad (4.10)$$

Let $x \in \Lambda(u/4)$, so that $d(x, z) \leq t + u/2$ and there exists $z' \in \partial B$ with $d(x, z') \leq u/4$. Hence $d(z', z) \leq t + \frac{3}{4}u < t + u$. As $\{z_1, \dots, z_m\}$ is maximal we must have $d(z', z_k) < 2u$ for some k . Thus $d(x, z_k) < 2u + \frac{1}{4}u < 3u$, proving (4.10).

For each z_j we have $d(x_0, z_j) = R$ by construction. Choose w_j on $\gamma(x_0, z_j)$ such that

$$\frac{1}{2}u < d(z_j, w_j) \leq \frac{2}{3}u;$$

this will be possible provided $6 < u < R$. We have $d(w_k, w_l) > d(z_k, z_l) - \frac{4}{3}u \geq \frac{2}{3}u$, so the balls $B(w_k, \frac{1}{4}u)$ are disjoint. The choice of w_k implies that $\rho(w_k) = d(w_k, z_k) > \frac{1}{2}u$, and therefore

$$B(w_k, \frac{1}{4}u) \cap \Lambda(u/4) = \emptyset. \quad (4.11)$$

We also have

$$B(w_k, \frac{1}{4}u) \subseteq \Lambda(u). \quad (4.12)$$

To check this, if $x \in B(w_k, \frac{1}{4}u)$ then $\rho(x) \leq d(x, z_k) \leq \frac{3}{4}u$, while $d(x, z) \leq d(x, z_k) + d(z_k, z) \leq \frac{3}{4}u + t + u < t + 2u$. By (4.10), we deduce that

$$\mu(\Lambda(u/4)) \leq \sum_{k=1}^m \mu(B(z_k, 3u)) \leq mC_0(3u)^d, \quad (4.13)$$

while by (4.11) and (4.12)

$$\mu(\Lambda(u)) \geq \mu(\Lambda(u/4)) + \sum_{k=1}^m \mu(B(w_k, \frac{1}{4}u)). \quad (4.14)$$

So, provided $\frac{1}{4}u \geq R_0$,

$$\mu(\Lambda(u)) \geq \mu(\Lambda(u/4)) + mC_V(\frac{1}{4}u)^d \geq (1 + c_1)\mu(\Lambda(u/4)). \quad (4.15)$$

Note that the constant c_1 here depends only on C_0 , C_V and d .

Now let $R_0 \leq r \leq 4s_j$. Choose $n \in \mathbb{Z}_+$ as large as possible so that $4^n r \leq 4s_j$. Then

$$\mu(\Lambda(2\lambda_2 r)) \leq (1 + c_1)^{-n} \mu(\Lambda(2\lambda_2 4^n r)) \leq (1 + c_1)^{-n} \mu(\Lambda(2t)). \quad (4.16)$$

Set $\alpha = \log(1 + c_1)/\log 4$, then $(1 + c_1)^n \geq (s_j/r)^\alpha$. We have $\mu(\Lambda(2t)) \leq \mu(B(z, 5t)) \leq C_0(5t)^d$, and also $\mu(B'_j) \geq C_V s_j^d \geq ct^d$. Combining this with (4.16) and (4.9) completes the proof of the lemma. \square

Proposition 4.7. *For each $1 \leq j \leq N$ we have*

$$\sum_{i \in \mathcal{F}^*(j)} |\mathcal{F}(i)| \frac{\mu(B'_i)}{\mu(B'_j)} \leq c_1 \log(R/s_j).$$

Proof. We can write

$$\mathcal{F}^*(j) = \cup_{n=-1}^{\infty} \mathcal{F}^*(j, 2^{-n}s_j).$$

Hence using Corollary 4.5 and Lemma 4.6

$$\begin{aligned} \sum_{i \in \mathcal{F}^*(j)} |\mathcal{F}(i)| \mu(B'_i) &\leq \sum_{n=-1}^{\infty} \sum_{i \in \mathcal{F}^*(j, 2^{-n}s_j)} |\mathcal{F}(i)| \mu(B'_i) \\ &\leq c \sum_{n=-1}^{\infty} \log(2^n R/s_j) 2^{-n\alpha} \mu(B_j) \leq c \mu(B_j) \log(R/s_j). \end{aligned} \quad \square$$

Set

$$\varphi(y) = \left(\frac{R \wedge \rho(y)}{R} \right)^2, \quad y \in G. \quad (4.17)$$

For any set A let $\widehat{\mu}(A) = \int_A \varphi d\mu$, and $\overline{f}_A = \widehat{\mu}(A)^{-1} \int_A f \varphi d\mu$. For an edge $e = \{x, y\}$ define $\widetilde{\varphi}(e) = \varphi(x) \wedge \varphi(y)$; note that if $e \in E(B)$ then $\widetilde{\varphi}(e) \geq R^{-2}$.

Theorem 4.8. Let $B = B(x_0, R)$ be very good, and $R_0 = N_B \leq R^{1/(d+2)}$. Then

$$\int_B (f(x) - \bar{f}_B)^2 \varphi(x) d\mu \leq c_1 R^2 \int_{E(B)} |\nabla f|^2 \tilde{\varphi} d\nu. \quad (4.18)$$

Proof. We follow the proof in [SCS], but need some extra care close to the boundary of B .

For $1 \leq i \leq N$ set

$$B_i^* = \begin{cases} B(y_i, 10C_W s_i), & 1 \leq i \leq M, \\ B_i'', & M+1 \leq i \leq N. \end{cases}$$

As $10C_W s_i \leq \frac{1}{2}\lambda s_i - 1$ we have $B_i^* \subseteq B$ for each $i \leq M$, while $B_i^* \subseteq B$ by definition if $M+1 \leq i \leq N$. Note that for any ball $B(y_i, cs_i)$ with $c \leq \frac{1}{2}\lambda$ we have from (4.17)

$$\varphi(x) \leq \frac{3\lambda_2}{2\lambda_1} \varphi(y) \text{ for all } x, y \in B(y_i, cs_i). \quad (4.19)$$

Let P_j be the best constant in the weighted Poincaré inequality

$$\int_{B_j''} (f(x) - \bar{f}_{B_j})^2 \varphi(x) d\mu \leq P_j \int_{E(B_j^*)} |\nabla f|^2 \tilde{\varphi} d\nu. \quad (4.20)$$

Then for $j \leq M$ as each B_j'' is good we have using (4.19) $P_j \leq cC_P s_j^2$. If $M < j \leq N$ then using Corollary 1.5(a)

$$P_j \leq 2\mu(B_j'')^2 \sup_{x, y \in B_j''} \frac{\varphi(x)}{\varphi(y)} \leq c(C_0 \lambda R_0^d)^2 (\lambda R_0)^2 \leq c' R_0^{2d+2}. \quad (4.21)$$

Fix for the moment a ball B_n . We define a sequence of balls $D_i = B(w_i, t_i)$, $1 \leq i \leq L_n$, with $D_1 = B_1$ and $D_{L_n} = B_n$, as follows. Suppose D_1, \dots, D_{k-1} have been defined. Let z_k be the point in $\gamma(x_0, y_n) \cap \cup_{i=1}^{k-1} D_i'$ which is furthest from x_0 . (If $D_i = B_j$ then $D_i' = B_i'$, $D_i'' = B_j''$.) If $z_k = y_n$ then we let $D_k = B_n$, $L_n = k$ and stop. Suppose that $z_k \neq y_n$. If $z_k \in B(y_i, 3s_i - 1)$ for some i then we take $D_k = B_i$, and continue. (We choose the largest such i if this i is not unique.) Note that D_k must be distinct from D_1, \dots, D_{k-1} .

Finally, suppose $z_k \neq y_n$ and $z_k \notin \cup B(y_i, 3s_i - 1)$. Then $\rho(z_k) < \lambda_2 R_0$, and $\rho(y_n) < d(y_n, z_k) + \rho(z_k) < 2\lambda_2 R_0$. Hence $s_n < 2\lambda_2 \lambda_1^{-1} R_0 < \eta R_0$, so that B_n is a boundary ball. We also have $\rho(w_{k-1}) \leq 3t_{k-1} + \rho(z_k)$, so that D_{k-1} is also a boundary ball. In this case we take $D_k = B_n$, $L_n = k$ and stop. Note that each D_k is a ball in $\{B_j : j \in \mathcal{F}(n)\}$, so that $L_n \leq |\mathcal{F}(n)|$.

We now show that

$$D_{k-1}' \cup D_k' \subseteq D_{k-1}'' \cap D_k'', \quad 2 \leq k \leq L_n. \quad (4.22)$$

First, if $z_k \neq y_n$ and $z_k \notin \cup B(y_i, 3s_i - 1)$ then both D_{k-1} and D_k are boundary balls, and $d(w_{k-1}, w_k) \leq 3t_{k-1} + \lambda_2 R_0$, from which (4.22) follows easily. Now let $z_k \in B(w_k, 3t_k - 1)$. Then $d(w_{k-1}, z_k) < 3t_{k-1}$, and so $d(w_{k-1}, w_k) < 3t_{k-1} + 3t_k$. Since $z_k \in B(w_j, 4t_j)$ for $j = k-1, k$, by (4.5)

$$\lambda_2(t_k \vee t_{k-1}) \geq \rho(z_k) \geq \lambda_1(t_k \wedge t_{k-1}).$$

Since $\lambda_2/\lambda_1 < 10/9$ this implies (4.22).

Let $\bar{f}_1 = \bar{f}(B'_1)$. Then

$$\begin{aligned} \int_{B''_n} (f(x) - \bar{f}_1)^2 \varphi(x) d\mu &= \int_{B''_n} \left(f(x) - \bar{f}_1 + \sum_{k=1}^{L_n-1} (\bar{f}(D''_k) - \bar{f}(D''_{k+1})) \right)^2 \varphi(x) d\mu \\ &\leq L_n \int_{B''_n} (f(x) - \bar{f}_1)^2 \varphi(x) d\mu + L_n \sum_{i=1}^{L_n-1} (\bar{f}(D''_k) - \bar{f}(D''_{k+1}))^2 \hat{\mu}(B''_n). \end{aligned} \quad (4.23)$$

To control the terms in (4.23) note that

$$\begin{aligned} \hat{\mu}(B''_n) (\bar{f}(D''_k) - \bar{f}(D''_{k+1}))^2 &= \frac{\hat{\mu}(B''_n)}{\hat{\mu}(D''_k \cap D''_{k+1})} \int_{D''_k \cap D''_{k+1}} (\bar{f}(D''_k) - \bar{f}(D''_{k+1}))^2 \varphi d\mu, \\ &\leq 2 \frac{\hat{\mu}(B''_n)}{\hat{\mu}(D''_k \cap D''_{k+1})} \left(\int_{D''_k} (f - \bar{f}(D''_k))^2 \varphi d\mu + \int_{D''_{k+1}} (f - \bar{f}(D''_{k+1}))^2 \varphi d\mu \right) \\ &\leq 2 \frac{\hat{\mu}(B''_n)}{\hat{\mu}(D''_k)} \int_{D''_k} (f - \bar{f}(D''_k))^2 \varphi d\mu + 2 \frac{\hat{\mu}(B''_n)}{\hat{\mu}(D''_{k+1})} \int_{D''_{k+1}} (f - \bar{f}(D''_{k+1}))^2 \varphi d\mu. \end{aligned}$$

Using (4.19), and Lemma 4.4(a),

$$\frac{\hat{\mu}(B''_n)}{\hat{\mu}(D''_k)} \leq c \frac{\varphi(y_n)^2 \mu(B''_n)}{\varphi(w_k)^2 \mu(D'_k)} \leq c' \frac{\mu(B''_n)}{\mu(D'_k)} \leq c'' \frac{\mu(B'_n)}{\mu(D'_k)}.$$

Combining these estimates we obtain from (4.23)

$$\begin{aligned} \int_{B''_i} (f(x) - \bar{f}_1)^2 \varphi(x) d\mu &\leq c |\mathcal{F}(i)| \sum_{j \in \mathcal{F}(i)} \frac{\mu(B'_i)}{\mu(B'_j)} \int_{B''_j} (f - \bar{f}(B'_j))^2 \varphi d\mu \\ &\leq c |\mathcal{F}(i)| \sum_{j \in \mathcal{F}(i)} \frac{\mu(B'_i)}{\mu(B'_j)} P_j \int_{E(B_j^*)} |\nabla f|^2 \varphi d\mu. \end{aligned} \quad (4.24)$$

Summing the inequalities (4.24) gives

$$\begin{aligned} \int_B (f(x) - \bar{f}_1)^2 \varphi(x) d\mu &\leq c \sum_{i=1}^N |\mathcal{F}(i)| \mu(B'_i) \sum_{j \in \mathcal{F}(i)} \mu(B'_j)^{-1} P_j \int_{E(B_j^*)} |\nabla f|^2 \tilde{\varphi} d\nu \\ &= c \sum_{j=1}^N \left(\sum_{i \in \mathcal{F}^*(j)} |\mathcal{F}(i)| \frac{\mu(B'_i)}{\mu(B'_j)} \right) P_j \int_{E(B_j^*)} |\nabla f|^2 \tilde{\varphi} d\nu \\ &\leq c \sum_{j=1}^N \log(R/s_j) P_j \int_{E(B_j^*)} |\nabla f|^2 \tilde{\varphi} d\nu. \end{aligned}$$

If $j \leq M$ then $P_j \leq cs_j^2 C_P$ and so since $t^p \log(R/t) \leq cR^p$ for $t > 0$ we have $P_j \log(R/s_j) \leq cC_P R^2$. If $M+1 \leq j \leq N$ then $R_0 \leq s_j \leq \eta R_0$, $P_j \leq cR_0^{2d+2}$ and so since $R_0^{d+2} \leq R$

$$\log(R/s_j)P_j \leq cR_0^{2d+2} \log(R/R_0) \leq cR^2.$$

So since any edge in B is contained in at most c of the B_i^* ,

$$\int_B (f(x) - \bar{f}_1)^2 \varphi(x) d\mu \leq cC_P R^2 \sum_{j=1}^N \int_{E(B_j^*)} |\nabla f|^2 \tilde{\varphi} d\nu \leq c'C_P R^2 \int_{E(B)} |\nabla f|^2 \tilde{\varphi} d\nu,$$

which completes the proof of the theorem. \square

We can also take $\varphi = 1$ in (4.17) and use the same argument to obtain a (strong) Poincaré inequality from the family of weak ones. The condition on N_B is slightly weaker than in Theorem 4.8, since we have $P_j \leq cR_0^{2d}$ in (4.21).

Lemma 4.9. *Let $B = B(x_0, R)$ be very good, and $R_0 = N_B \leq R^{1/(d+2)}$. Then*

$$\min_a \int_B (f(x) - a)^2 d\mu \leq c_1 R^2 \int_{E(B)} |\nabla f|^2 d\nu.$$

Remark 4.10. The weight function φ in (4.17) is similar to that in [SCS]. [FS] and [SZ] use weight functions which are supported on the whole space. In particular [SZ] uses

$$\psi_R(x) = e^{-d(x_0, x)/R}, \quad x \in \mathbb{Z}^d,$$

and proves a weighted Poincaré inequality of the form

$$\min_a \int_{\mathbb{Z}^d} (f - a)^2 \psi_R d\mu \leq c_1(d) R^2 \int_{\mathbb{E}_d} |\nabla f|^2 \tilde{\psi}_R d\nu. \quad (4.25)$$

It is interesting to note that this fails for percolation clusters when $d \geq 3$.

To see this fix a point $x_0 \in \mathcal{C}_\infty$ and $R \geq 1$ large enough so that $B(x_0, R)$ is good. If we look far enough from x_0 we can, \mathbb{P}_p -a.s., find a cube Q of side R with $Q \subseteq \mathcal{C}_\infty$ and such that Q is only connected to the rest of \mathcal{C}_∞ by one edge $\{x_1, x_2\}$. We take $x_1 \in Q^c$, $x_2 \in Q$ and write $s = d(x_0, x_2)$; we can assume $s \gg R$. We have $e^{-(s+dR)/R} \leq \psi \leq e^{-s/R}$ on Q .

Let $f = 1_Q$. Then as $\int \psi_R \asymp R^d$,

$$\bar{f}_R = \frac{\int_Q \psi_R}{\int \psi_R} \leq ce^{-s/R} \leq \frac{1}{4},$$

and

$$\min_a \int_{\mathbb{Z}^d} (f - a)^2 \psi_R d\mu = \int (f - \bar{f}_R)^2 \psi_R d\mu \geq \frac{1}{2} \int_Q \psi_R d\mu \geq \frac{1}{2} e^{-d} R^d e^{-s/R}.$$

On the other hand

$$R^2 \int |\nabla f|^2 \tilde{\psi} d\nu = R^2 \psi(x_2) = R^2 e^{-s/R}.$$

Thus (4.25) cannot hold with a constant c_1 independent of R .

5. Lower bounds

In this section we use the weighted Poincaré inequality and the technique of [FS] to prove lower bounds for $q_t(x, y)$. We continue with the notation and assumptions of Section 3.

Proposition 5.1. *Let $x_0 \in G$, and $B_1 = B(x_0, R \log R)$ be a very good ball with $N_B \leq R^{1/(d+2)}$. Then if $x_1 \in B(x_0, \frac{1}{2}R \log R)$*

$$q_t(x_1, x_2) \geq c_1 t^{-d/2}, \quad \text{for } x_2 \in B(x_1, R), \quad \text{and } \frac{1}{8}R^2 \leq t \leq R^2. \quad (5.1)$$

Proof. Let x_3 be such that $x_1, x_2 \in B(x_3, R/2)$. Write $R_1 = R \log R$, and let $\rho = R/6$, and $T = c_2 R^2$. Let $x_4 \in B(x_3, R/2)$. We apply Proposition 3.7 to B_1 with $t = T$; since $T'_{B_1} = cR_1^2 \log R_1 \geq c'R^2 \log R$ the third condition in (3.14) holds, while the other two are evident. So, if $t \leq T$ then

$$\begin{aligned} \sum_{x \in B(x_3, 2R/3)^c} q_t(x_4, x) \mu(x) &= P^{x_4}(Y_t \notin B(x_1, 2R/3)) \\ &\leq P^{x_4}(\tau(x_4, R/6) < t) \\ &\leq P^{x_4}(\tau(x_4, R/6) < T) \leq ce^{c'R^2/T} \leq \frac{1}{2}, \end{aligned} \quad (5.2)$$

provided c_2 is chosen small enough. We can assume that $c_2 \leq 1/8$.

Let $B = B(x_3, R)$, and for $x \in B$ set $\rho(x) = d(x, B^c)$, and

$$\varphi(y) = \left(\frac{R \wedge \rho(y)}{R} \right)^2, \quad y \in G, \quad \text{and } V_0 = \sum_{x \in B} \varphi(x) \mu(x).$$

Then we have

$$c_3 R^d \leq V_0 \leq \mu(B) \leq c_4 R^d. \quad (5.3)$$

Write $u(s, x) = u_s(x) = q_s(x_4, x)$, $s \geq 0$, $x \in G$. Set

$$w(s, y) = w_t(x) = \log V_0 u(s, y), \quad H(t) = H(x_4, t) = V_0^{-1} \int_B \varphi w_t d\mu.$$

Then

$$V_0 H'(t) = \int_B \varphi \frac{\partial w_t}{\partial t} d\mu = \int_B \varphi \frac{1}{u_t} \frac{\partial u_t}{\partial t} d\mu = \sum_{x \in B} \frac{\varphi(x)}{u_t(x)} \mathcal{L} u_t(x) \mu(x).$$

Hence, writing $f = 1_B \varphi / u_t$ we have

$$V_0 H'(t) = -\frac{1}{2} \sum_{x \in G} \sum_{y \in G} a_{xy} (f(x) - f(y)) (u_t(x) - u_t(y)).$$

Now we use the elementary inequality (see [SZ])

$$-\left(\frac{d}{b} - \frac{c}{a}\right)(b - a) \geq \frac{1}{2}(c \wedge d)(\log b - \log a)^2 - \frac{(d - c)^2}{2(c \wedge d)},$$

which holds for any strictly positive a, b, c, d . If $f(x) > 0$ and $f(y) > 0$ then $x, y \in B$ and

$$\begin{aligned} -(f(x) - f(y))(u_t(x) - u_t(y)) &= -\left(\frac{\varphi(x)}{u_t(x)} - \frac{\varphi(y)}{u_t(y)}\right)(u_t(x) - u_t(y)) \\ &\geq \frac{1}{2}(\varphi(x) \wedge \varphi(y))(\log u_t(x) - \log u_t(y))^2 - \frac{(\varphi(x) - \varphi(y))^2}{2(\varphi(x) \wedge \varphi(y))}. \end{aligned}$$

If both $x \in B^c$ and $y \in B^c$ then $f(x) = f(y) = 0$, while if $x \in B$ and $y \in B^c$ then

$$-(f(x) - f(y))(u_t(x) - u_t(y)) = -\varphi(x)\left(1 - \frac{u_t(y)}{u_t(x)}\right).$$

We therefore have

$$V_0 H'(t) \geq \frac{1}{4} \sum_{x \in B} \sum_{y \in B} a_{xy} (\varphi(x) \wedge \varphi(y)) (w_t(x) - w_t(y))^2 \quad (5.4)$$

$$- \frac{1}{4} \sum_{x \in B} \sum_{y \in B} a_{xy} \frac{(\varphi(x) - \varphi(y))^2}{\varphi(x) \wedge \varphi(y)} \quad (5.5)$$

$$- \frac{1}{2} \sum_{x \in B} \sum_{y \in B^c} a_{xy} \varphi(x) \left(1 - \frac{u_t(y)}{u_t(x)}\right). \quad (5.6)$$

Call the sums in (5.4), (5.5), (5.6), S_1, S_2, S_3 respectively. To bound S_2 note that if $x \sim y$ with $x, y \in B$ and $k = \rho(x) \wedge \rho(y)$ then $k \geq 1$ and

$$\frac{(\varphi(x) - \varphi(y))^2}{\varphi(x) \wedge \varphi(y)} = R^{-2} \frac{(2k + 1)^2}{k^2} \leq \frac{9}{R^2}.$$

So

$$S_2 \geq -\frac{9}{4} R^{-2} \sum_{x \in B} \mu(x) \geq -\frac{9}{4} R^{-2} \mu(B) \geq -c R^{-2} V_0.$$

Also, if $x \in \partial_i(B)$ then $\varphi(x) = R^{-2}$, so that

$$S_3 \geq - \sum_{x \in B} \sum_{y \in B^c} a_{xy} R^{-2} \geq -R^{-2} \sum_{x \in B} \mu(x) \geq -c R^{-2} V_0.$$

So the terms S_2 and S_3 are controlled by bounds of the same size, and we deduce, using Theorem 4.8,

$$\begin{aligned} H'(t) &\geq \frac{1}{4} V_0^{-1} \sum_{x \in B} \sum_{y \in B} a_{xy} (\varphi(x) \wedge \varphi(y)) (w_t(x) - w_t(y))^2 - c_5 R^{-2} \\ &\geq -c_5 R^{-2} + c_6 R^{-2} V_0^{-1} \sum_{x \in B} (w_t(x) - H(t))^2 \varphi(x) \mu(x). \end{aligned} \quad (5.7)$$

Let $I = [\frac{1}{2}T, T]$; we will only use (5.7) for $t \in I$. Note that by Theorem 3.8 we have $V_0 u_t(x) \leq c_7$ for $t \in I$. Since $v \rightarrow (\log v - h)^2/v$ is decreasing on $[e^{2+h}, \infty)$ we have

$$\begin{aligned} \sum_{x \in B} (w_t(x) - H(t))^2 \varphi(x) \mu(x) &= \sum_{x \in B} \frac{(\log V_0 u_t(x) - H(t))^2}{u_t(x)} \varphi(x) u_t(x) \mu(x) \\ &\geq \frac{(\log c_7 - H(t))^2}{c_7} \sum_{x \in B: V_0 u_t(x) > e^{2+H(t)}} \varphi(x) u_t(x) \mu(x). \end{aligned} \quad (5.8)$$

Then since $\varphi(x) \geq \frac{1}{9}$ on $B(x_1, 2R/3)$,

$$\begin{aligned} &\sum_{x \in B: V_0 u_t(x) > e^{2+H(t)}} \varphi(x) u_t(x) \mu(x) \\ &= \sum_{x \in B} \varphi(x) u_t(x) \mu(x) - \sum_{x \in B: V_0 u_t(x) \leq e^{2+H(t)}} \varphi(x) u_t(x) \mu(x) \\ &\geq \frac{1}{9} \sum_{x \in B(x_1, 2R/3)} u_t(x) \mu(x) - \sum_{x \in B} \varphi(x) V_0^{-1} e^{2+H(t)} \mu(x) \\ &\geq \frac{1}{9} \left(1 - \sum_{x \in B(x_1, 2R/3)^c} u_t(x) \mu(x)\right) - e^{2+H(t)} \geq \frac{1}{18} - e^{2+H(t)}, \end{aligned} \quad (5.9)$$

where we used (5.2) in the final line.

Combining the estimates (5.7), (5.8) and (5.9) we deduce that

$$TH'(t) \geq -c_5 + c_8(\log c_7 - H(t))^2 \left(\frac{1}{18} - e^{2+H(t)}\right).$$

Since $(a - h)^2 \geq \frac{1}{2}h^2$ if $h < -a$ this implies that there exists c_9 such that

$$TH'(t) \geq c_{10}H(t)^2 \text{ provided } H(t) < -c_9, t \in I. \quad (5.10)$$

If $\sup_{t \in I} H(t) < -c_9$ then (5.10) implies that $H(T) \geq -c_{11}$, while since $H(t) + c_5 T^{-1}t$ is increasing, if $\sup_{t \in I} H(t) \geq -c_9$ then $H(T) \geq -c_9 - c_2 c_5$. We therefore deduce that $H(T) \geq -c_{12}$, and it follows that

$$H(t) = H(x_4, t) \geq -c_{13}, \quad T \leq t \leq R^2. \quad (5.11)$$

To conclude the argument, we have, for $x_1, x_2 \in B(x_3, R/2)$, $t \in [T, R^2]$,

$$\begin{aligned} V_0 q_{2t}(x_1, x_2) &= V_0^{-1} \sum_y V_0 q_t(x_1, y) V_0 q_t(x_2, y) \mu(dy) \\ &\geq V_0^{-1} \sum_{y \in B} V_0 q_t(x_1, y) V_0 q_t(x_2, y) \varphi(y) \mu(dy). \end{aligned}$$

So, using Jensen's inequality:

$$\begin{aligned} \log(V_0 q_{2t}(x_1, x_2)) &\geq V_0^{-1} \sum_{y \in B} \left(\log(V_0 q_t(x_1, y)) + \log(V_0 q_t(x_2, y)) \right) \varphi(y) \mu(dy) \\ &= H(x_1, t) + H(x_2, t) \geq -2c_{13}. \end{aligned}$$

Using (5.3) this completes the proof of (5.1). \square

Lemma 5.2. *Let $x, y \in G$. Suppose there exists $r \geq 1$ and a path $x = z_0, \dots, z_m = y$ such that for each $i = 0, \dots, m$, $B(z_i, r \log r)$ is very good with $N_{B(z_i, r \log r)}^{d+2} \leq r$. Then*

$$q_{mr}(x, y) \geq c(mr)^{-d/2} e^{-c_1 m/r}. \quad (5.12)$$

Proof. This uses a chaining argument similar to that in Proposition 3.7. Choose points $x = w_0, w_1, \dots, w_k = y$ along the path $\{z_0, \dots, z_m\}$, such that $d(w_{i-1}, w_i) < r/3$ and $3m/r \leq k \leq 4m/r$. Let $s = mr/k$, so that $\frac{1}{4}r^2 \leq s \leq \frac{1}{3}r^2$. Let $B_j = B(w_j, r/3)$; we have $d(x', y') < r$ whenever $x' \in B_{j-1}$ and $y' \in B_j$. So by Proposition 5.1

$$q_s(x', y') \geq c_2 s^{-d/2} \geq c_3 \mu(B_j)^{-1}, \quad x' \in B_{j-1}, y' \in B_j.$$

So $P^{x'}(Y_s \in B_j) \geq c_3$ and therefore

$$P^x(Y_{ks} = y) \geq P^x(Y_{js} \in B_j, 1 \leq j \leq k-1, Y_{ks} = y) \geq \left(\prod_{j=1}^{k-1} c_3 \right) c_3 s^{-d/2}. \quad \square$$

Theorem 5.3. *Let $B = B(x_0, R \log R)$ be a very good ball with $N_B \leq R^{1/(d+2)}$. Let $d(x_0, x_1) \leq \frac{1}{2}R \log R$ and $x, y \in B(x_1, R)$. Then*

$$q_t(x, y) \geq c_1 t^{-d/2} e^{-c_2 d(x, y)^2/t}, \quad (5.13)$$

provided

$$N_B^{2(2+d)} \leq t \leq R^2, \quad (5.14)$$

and

$$N_B^{2+d} d(x, y) \leq t. \quad (5.15)$$

Proof. Let $D = d(x, y)$. If $D^2/t \leq 1$ then, set $r = t^{1/2}$ and apply Proposition 5.1 to $B_1 = B(x, r \log r)$. Since $D \leq r \leq R$, $B_1 \subseteq B$ and $N_{B_1} \leq N_B$. So $N_{B_1}^{d+2} \leq N_B^{d+2} \leq t^{1/2} = r$, the hypotheses of Proposition 5.1 hold and we deduce that $q_t(x, y) \geq ct^{-d/2}$.

For the case $D^2/t > 1$ we use Lemma 5.2. Let $x = z_0, \dots, z_D = y$ be a shortest path between x and y , and $r = t/D$. By (5.15) $r \geq N_B^{d+2}$, while $r \leq t^{1/2} \leq R$, so that $B_i = B(z_i, r \log r) \subseteq B$, and hence $N_{B_i}^{d+2} \leq N_B^{d+2} \leq r$. So the hypotheses of Lemma 5.2 are satisfied, and we obtain (5.13). \square

Remark. The restriction in (5.15) is weaker than the hypotheses in the upper bound of Theorem 3.8, where we were able to use global upper bounds on q_t to restrict to the case when t was close to $D^{1/2}$. The lower bound argument for $cD \leq t \leq cDN_B^{2+d}$ requires the existence of a chain of small balls (of size roughly $r = t/D$) on which the lower bounds of Proposition 5.1 are valid. If $r = O(1)$, so that $t \simeq D$, then one can just use a path in the graph and (1.4), to deduce that $q_t(x, y) \geq e^{-ct} \simeq e^{-c'D^2/t}$. But if $D = t^{1-\varepsilon}$ then one needs $r \simeq t^\varepsilon \gg 1$, and elementary bounds are not enough.

For a very good ball B the volume condition or the Poincaré inequality, will fail for some sub-balls of size $r < N_B$. However these may hold for enough small sub-balls so that one can still use Lemma 5.2, and in Section 2 it is proved, in the percolation context, that it is possible to find such a chain. Fix $C_E \geq 1$, $C_F > 1$.

Definition 5.4. A ball $B = B(x_0, R_1)$ is *exceedingly good* if:

- (1) B is very good with $N_B^{10(d+2)} \leq R_1$.
- (2) For each $x_1, x_2 \in B(x_0, R_1)$ with $d(x_1, x_2) \geq R_1^{1/4}$ and $C_E \leq r \leq N_B^{2+d}$ there exists a path $y_1 = z_0, \dots, z_k = y_2$ with the following properties.
 - (a) $B_i = B(z_i, r \log r)$ is very good with $N_{B_i}^{2+d} \leq r$,
 - (b) $k \leq C_F d(x, y)$,
 - (c) $d(x_j, y_j) \leq R_1^{1/4}$, $j = 1, 2$.

Remark 5.5. If B is very good and $N_B^{2+d} \leq r \leq R/\log R$ then taking $m = d(x, y)$ and z_0, \dots, z_m to be a shortest path between x_1 and x_2 we get a path satisfying 2(a)–2(c) above, with $y_j = x_j$.

Theorem 5.6. Let $B = B(x_0, R \log R)$ be exceedingly good, and $x_1, x_2 \in B(x_0, R)$. Then there exist constants c_i (depending on C_E, C_F) such that

$$q_t(x_1, x_2) \geq c_1 t^{-d/2} e^{-c_2 d(x_1, x_2)^2/t}, \quad R^{1/2} \vee d(x, y) \leq t \leq R^2. \quad (5.16)$$

Proof. Let $R_1 = R \log R$, $D = d(x_1, x_2)$. By Theorem 5.3 it is enough to consider the case when t satisfies the condition in (5.16) but fails (5.14) or (5.15). Since $t \geq R^{1/2} \geq R_1^{1/3} \geq N_B^{3(d+2)}$, (5.14) must hold. So we just need to consider the case when $D \leq t \leq N_B^{2+d} D$. Note this implies that

$$D \geq \frac{D^2}{t} \geq \frac{t}{N_B^{2(d+2)}} \geq \frac{R^{1/2}}{R_1^{1/5}} \geq R_1^{1/4}. \quad (5.17)$$

In particular $D^2/t \gg \log t$ so that, for this range of t and D , terms of the order of $t^{-d/2}$ can be absorbed into the constant c_2 in (5.16).

Let $r = t/(2C_F D)$, so that $(2C_F)^{-1} \leq r \leq (2C_F)^{-1} N_B^{2+d}$. We have to consider two cases. If $1 \leq r \leq C_E$ then we use (1.4) directly. Set $s = t/D$, so that $c_3 \leq s \leq c_4$, and we obtain $q_{Ds} \geq e^{-cD} \geq e^{-cD^2/t}$.

If $C_E \leq r$ then we use the fact that B is exceedingly good so that there exists a path z_0, \dots, z_k satisfying conditions 2(a)–(c) of Definition 5.4. We have $D \leq k \leq C_F D$, so that $(2C_F)^{-1} t \leq kr \leq \frac{1}{2}t$, and $k/r \leq 2C_F^2 D^2/t$. Applying Lemma 5.2

$$q_{kr}(z_0, z_k) \geq c(kr)^{-d/2} e^{-c_5 k/r} \geq c t^{-d/2} e^{-cD^2/t}. \quad (5.18)$$

By (5.17) $D^2/t \geq R_1^{1/4} \geq d(x_1, y_1) \vee d(x_2, y_2)$. Let $D_i = d(x_i, z_i)$; by (5.17) $D_i \leq R_1^{1/4} \leq t/8$. Using (1.4)

$$q_{m_j}(x_j, z_j) \geq e^{-c_6 m_j} \geq e^{-c_6 D^2/t}. \quad (5.19)$$

Let $u = D_1 + D_2 + kr$; then $\frac{1}{2}t \leq u \leq \frac{3}{4}t$. By (1.4)

$$q_t(x_0, x_1) \geq q_u(x_0, x_1) q_{t-u}(x_1, x_1) \geq q_{D_0}(x_0, z_0) q_{kr}(z_0, z_k) q_{D_1}(z_k, x_1) q_{t-u}(x_1, x_1).$$

Using the bounds (5.18) and (5.19), and Proposition 5.1 to control the final term, we obtain (5.16). \square

Theorem 5.7. Let $d \geq 2$, C_0 , C_V , C_P , and C_W be constants. Let $\mathcal{G} = (G, E)$ be an infinite graph satisfying (1.1), and let $x \in G$.

(a) Suppose that there exists $R_0 = R_0(x)$ such that $B(x, R)$ is very good with $N_{B(x, R)}^{3(d+2)} \leq R$ for each $R \geq R_0$. There exist constants $c_i = c_i(d, C_0, C_V, C_P, C_W)$ such that if t satisfies

$$t \geq R_0^{2/3} \quad (5.20)$$

then

$$q_t(x, y) \leq c_1 t^{-d/2} e^{-c_2 d(x, y)^2/t}, \quad d(x, y) \leq t, \quad (5.21)$$

and

$$q_t(x, y) \geq c_3 t^{-d/2} e^{-c_4 d(x, y)^2/t}, \quad d(x, y)^{3/2} \leq t. \quad (5.22)$$

(b) If in addition there exists $R_E = R_E(x)$ such that $B(x, R)$ is exceedingly good for each $R \geq R_E$ then the lower bound (5.22) holds (with constants depending in addition on C_E , C_F) whenever $t \geq R_E$ and $t \geq d(x, y)$.

Proof. Fix $y \in G$ and let $D = d(x, y)$. We need to check that we can find R so that we can apply Theorems 3.8, 5.3 and 5.6. We begin with (a). Let $R = t^{2/3}$, so that $R \geq D$; by (5.20) we have $R \geq R_0$. Let $B = B(x, R)$; then $N_B^{2(2+d)} \leq R$.

To apply Theorem 3.8 and obtain (5.21) we need (3.18) to hold, but this is clear since $t \geq D^{3/2} \geq D$ and

$$N_B^{2d+1} \leq N_B^{2(d+2)} \leq R = t^{2/3} \leq t = R^{3/2} \leq \frac{cR^2}{\log R}.$$

To obtain (5.22) we use Theorem 5.3. Since $R \geq D$ we have $y \in B(x, R)$. Let $R_1 = R \log R$. Since $R_1 \geq R_0$, $B_1 = B(x, R_1)$ is very good with $N_{B_1}^{3(2+d)} \leq R_1$. So

$$N_{B_1}^{2(2+d)} \leq R \log R \leq R^{3/2} = t \leq \frac{1}{4} R^2,$$

and (5.14) is verified. Also,

$$cDN_{B_1}^{2+d} \leq c't^{2/3}R_1^{1/3} \leq t^{2/3}R^{1/2} \leq t,$$

so (5.15) holds, and we obtain (5.22).

(b) We now take $R = t$, so that $R \geq R_E$. Then $D \leq t = R$, and we apply Theorem 5.6 to $B(x, R_1)$ with $R_1 = R \log R$. The condition (5.16) is easily verified, and the bound in (5.22) is immediate. \square

We now prove an elliptic Harnack inequality for graphs satisfying the conditions of Theorem 5.3. The approach is the same as in [FS, Section 5], but we need an additional argument in Theorem 5.11.

Lemma 5.8. *Let $B = B(x_0, R \log R)$ be a very good ball with $N_B^{2(d+2)} \leq R$. Let $d(x_0, x_1) \leq \frac{1}{2}R \log R$, $B_0 = B(x_1, R)$ and let $q_t^0(x, y)$ be the density of the process Y killed at the exit time τ of Y from B_0 . Then*

$$q_t^0(x, y) \geq c_1 t^{-d/2}, \quad x, y \in B(x_1, 3R/4), \quad c_2 R^2 \leq t \leq R^2. \quad (5.23)$$

Proof. As this argument is quite standard – see for example [FS, Lemma 5.1], we only give a sketch. For $x, y \in B_0$ we have

$$\begin{aligned} q_t^0(x, y) &\geq q_t(x, y) - E^x 1_{(\tau < t)} q_{t-\tau}(Y_\tau, y) \\ &\geq q_t(x, y) - \sup_{0 \leq s \leq t} \sup_{z \in \partial B_0} q_s(z, y). \end{aligned} \quad (5.24)$$

Let $\delta \in (0, \frac{1}{8})$, $d(x, y) \leq \delta R$ and $t = \delta^2 R^2$. Using the estimates in Theorem 5.3 and Theorem 3.8 in (5.24) we obtain, provided t and $s = \theta t$ satisfy the conditions (3.18), (5.14), (5.15),

$$q_t^0(x, y) \geq t^{-d/2} \left[c_1 e^{-c_2} - \sup_{0 \leq \theta \leq 1} c_3 \theta^{-d/2} e^{-c_4/(\theta \delta^2)} \right].$$

(If s is too small to satisfy (3.18) we use Lemma 1.1.) Hence if δ is chosen small enough we obtain

$$q_t^0(x, y) \geq c_5 t^{-d/2}. \quad (5.25)$$

A chaining argument now gives (5.23). \square

Definition. Write $\overline{B}(x, R) = B(x, R) \cup \partial_e(B(x, R))$. A function $h : \overline{B}(x, R) \rightarrow \mathbb{R}$ is *harmonic* on $B(x, R)$ if

$$\mathcal{L}h(x) = 0, \quad x \in B(x, R).$$

We write $\text{Osc}(h, A) = \sup_A h - \inf_A h$.

The following oscillation bound follows from (5.23) just as in [FS, Lemma 5.2].

Lemma 5.9. *Let $B = B(x_0, R \log R)$ be a very good ball with $N_B^{2(d+2)} \leq R$. Let $d(x_0, x_1) \leq \frac{1}{2}R \log R$, $B_0 = B(x_1, R)$, $B_1 = B(x_1, \frac{1}{2}R)$ and h be harmonic in B_0 . There exists $c_1 \in (0, 1)$ such that*

$$\text{Osc}(h, B_1) \leq (1 - c_1) \text{Osc}(h, B_0).$$

Proof. By a linear transformation we can assume $\min_{B_0} h = 0$, $\max_{B_0} h = 1$ and $\int_{B_1} h d\mu \geq \frac{1}{2}\mu(B_1)$. Then if $x \in B_1$ by Lemma 5.8

$$h(x) \geq \int_{B_1} q_{R^2}(x, y) h(y) \mu(dy) \geq \frac{1}{2} c_2 R^{-d} \mu(B_1) \geq c_3.$$

So $\text{Osc}(h, B_1) \leq (1 - c_3)$. \square

We also need an intermediate range Harnack inequality.

Lemma 5.10. Let $B = B(x_0, R \log R)$ be a very good ball with $N_B^{3(d+2)} \leq R$. Let $d(x_0, x_1) \leq \frac{1}{2}R \log R$, $B_0 = B(x_1, R)$, and h be non-negative and harmonic in B_0 . Then if $d(x_1, y) \leq R/2$ and $r = R^{1/2}$,

$$\sup_{B(y, r)} h \leq c_1 \inf_{B(y, r)} h.$$

Proof. Let $B_1 = B(x_1, 3R/4)$. We can assume $\inf_{B_1} h = 1$. The local Harnack inequality for graphs implies that if $x \sim y$, $x, y \in B_1$, then $h(x) \leq C_0 h(y)$. Hence $h(x) \leq e^{c_2 R}$, $x \in \overline{B_0}$. We extend h to G by taking $h \equiv 1$ outside $\overline{B_0}$. Let τ be the first exit time of Y from B_1 . As $h(Y_t)$ is a martingale on $[0, \tau_0]$, for $x \in B(x_1, R/2)$,

$$\begin{aligned} h(x) &= E^x h(Y_{t \wedge \tau}) \\ &= E^x 1_{B_1}(Y_t) h(Y_{t \wedge \tau}) + E^x 1_{B_1^c}(Y_t) h(Y_\tau) \\ &= E^x 1_{B_1}(Y_t) h(Y_t) + E^x 1_{B_1}(Y_t) 1_{(\tau < t)} (h(Y_\tau) - h(Y_t)) + E^x 1_{B_1^c}(Y_t) h(Y_\tau) \\ &\leq \int_{B_1} q_t(x, y) h(y) d\mu(y) + e^{c_2 R} P^x(\tau < t). \end{aligned}$$

Using Proposition 3.7 the final term above is bounded by $c_3 \exp(c_2 R - c_4 R^2/t) \leq c_3$ if $t \leq Rc_2/c_4$.

Now let $d(z, x) \leq R^{1/2}$, and $s = \lambda t$, with $\lambda > 1$. Then if $\lambda \theta \leq c_2/c_4$,

$$\begin{aligned} h(z) &\geq E^z 1_{B_1}(Y_s) h(Y_s) - E^z 1_{(\tau < t)} h(Y_s) \\ &\geq \int_{B_1} q_s(z, y) h(y) d\mu(y) - e^{c_2 R} P^x(\tau < s) \\ &\geq \int_{B_1} q_s(z, y) h(y) d\mu(y) - c_3. \end{aligned}$$

Also, if $u = R^2$, by Lemma 5.8,

$$h(z) \geq \int_{B_1} q_t^0(z, y) h(y) d\mu(y) \geq \int_{B_1} c_5 u^{-d/2} h(y) d\mu(y).$$

Hence

$$2h(z) \geq \int_{B_1} (c_5 R^{-d} + q_s(z, y)) h(y) d\mu(y) - c_3.$$

Using Lemma 1.1, and Theorems 3.8 and 5.3 we can choose λ so that

$$c_5 R^{-d} + q_s(z, y) \geq c_6 q_t(x, y), \quad y \in B_1.$$

It follows that $2h(z) \geq c_6(h(x) - c_3) - c_3$, so that (as $h(z) \geq 1$) we have $c_6 h(x) \leq (2 + c_3(1 + c_6))h(z)$. \square

Theorem 5.11. Let $B = B(x_0, R \log R)$ be very good, with $N_B^{4(d+2)} \leq R$. Then if $d(x_0, x_1) \leq \frac{1}{3}R \log R$, $B_0 = B(x_1, R)$ and $h : \overline{B}_0 \rightarrow \mathbb{R}$ is non-negative and harmonic in B_0 ,

$$\sup_{B(x_1, R/2)} h \leq c_1 \inf_{B(x_1, R/2)} h.$$

Proof. We begin in the same way as in [FS]. Let $\alpha_3 = 1/4$. From Lemma 5.9 it follows that there exists A such that

$$\text{Osc}(h, B(y, r)) \leq (2e^d)^{-1} \text{Osc}(h, B(y, Ar)) \quad \text{if } R^{\alpha_3} \leq r, Ar \leq R/8. \quad (5.26)$$

We normalize h so that $\min_{B(x_1, R/4)} h = 1$. Let $x_2 \in B(x_1, R/4)$ satisfy $h(x_2) = 1$. By Lemma 5.8, if $y \in B(x_1, 3R/4)$, and $B(y, s)$ is good,

$$1 = h(x_2) \geq c_2 \int_{B(y, s)} R^{-d} h(y) d\mu(y) \geq c_3 (s/R)^d \inf_{B(y, s)} h.$$

Thus

$$\inf_{B(y, s)} h \leq c_3^{-1} \frac{R^d}{s^d}. \quad (5.27)$$

Now let $M_r = 2c_3^{-1}e^{rd}$, and $a_r = Re^{-r}$.

Suppose that there exists $y_r \in B(x_1, R/2)$ with $h(y_r) \geq M_r$. Then, by (5.27), $\text{Osc}(h, B(y_r, a_r)) \geq c_3^{-1}e^{rd}$. So, provided

$$a_r \geq R^{\alpha_3} \text{ and } Aa_r \leq R/8, \quad (5.28)$$

it follows that $\text{Osc}(h, B(y_r, Aa_r)) \geq 2e^d c_3^{-1}e^{rd} = M_{r+1}$. Hence there exists y_{r+1} with $d(y_r, y_{r+1}) \leq Aa_r$ with $h(y_{r+1}) \geq M_{r+1}$.

Choose r_0 so that $\sum_{r_0}^{\infty} Aa_r \leq R/8$. Then if $\sup_{B(x_1, R/4)} h \geq M_{r_0}$, the argument above implies that we can construct a sequence y_j , $r_0 \leq j \leq k$ with $y_j \in B(x_1, R/2)$ and $h(y_j) \geq M_j$. Here k is the largest r such that (5.28) holds. We have $M_k = 2c_3^{-1}e^{kd} = cR^d a_k^{-d} \geq c_4 R^{d(1-\alpha_3)}$. The local Harnack inequality in Lemma 5.10 implies that

$$\inf_{B(y_k, R^{1/2})} h \geq c_5 c_4 R^{d(1-\alpha_3)}, \quad (5.29)$$

while (5.27) implies that

$$\inf_{B(y_k, R^{1/2})} h \leq c_3^{-1} R^{d/2}. \quad (5.30)$$

Since $\alpha_3 < \frac{1}{2}$ this gives a contradiction if $R \geq c_6$. So we deduce that

$$\sup_{B(x_1, R/4)} h \leq M_{r_0} \inf_{B(x_1, R/4)} h.$$

The Harnack inequality for $B(x_1, R/2) \subseteq B(x_1, R)$ now follows by an easy chaining argument. \square

6. Random walks and percolation

In this section we tie together the results of Section 2 and those of Section 3–5, and prove Theorems 1–5. We recall the notation for bond percolation: we take $\Omega = \{0, 1\}^{\mathbb{E}_d}$, write $\eta_e, e \in \mathbb{E}_d$ for the coordinate maps and take \mathbb{P}_p to be the probability on Ω which makes the η_e i.i.d. Bernoulli r.v. with mean p . We assume that $p > p_c$ so that there exists Ω_0 with $\mathbb{P}_p(\Omega_0) = 1$ such that for $\omega \in \Omega_0$ there is a unique infinite cluster $\mathcal{C}_\infty(\omega)$. As in the introduction we take Y to be the Markov process with generator \mathcal{L}_ω given by (0.1), and write $q_t^\omega(x, y)$ for the transition density of Y as given by (0.2). Write $(P_\omega^x, x \in \mathcal{C}_\infty)$ for the probability law of Y .

Proposition 6.1. *Let $p > p_c$. There exists $\Omega_1 \subseteq \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$, and $S_x, x \in \mathbb{Z}^d$ such that $S_x(\omega) < \infty$ for each $\omega \in \Omega_1, x \in \mathcal{C}_\infty(\omega)$. There exist constants $c_i = c_i(d, p)$ such that for $x, y \in \mathcal{C}_\infty(\omega), t \geq 1$ with*

$$S_x(\omega) \vee d_\omega(x, y) \leq t, \quad (6.1)$$

the transition density $q_t(x, y)$ of Y satisfies:

$$c_1 t^{-d/2} e^{-c_2 d_\omega(x, y)^2/t} \leq q_t^\omega(x, y) \leq c_3 t^{-d/2} e^{-c_4 d_\omega(x, y)^2/t}. \quad (6.2)$$

Proof. Let the constants $C_0, C_V, C_P, C_W, C_E, C_F$ (depending on d and p) be as in Section 2, and as in Section 2 let $\alpha_2^{-1} = 11(d+2)$. Let $(N_x, x \in \mathbb{Z}^d)$ be as in Lemma 2.24 and $\Omega_1 = \{\omega : N_x(\omega) < \infty \text{ for all } x\}$. Let $\omega \in \Omega_1$, and $x \in \mathcal{C}_\infty(\omega)$. We set $S_x = R_E(x) = N_x$, and check the hypotheses of Theorem 5.7. Let $R \geq R_E, B = B_\omega(x, R) \subseteq Q = Q(x, R)$, and $n = 2R = s(Q)$.

As $\omega \in L(Q) \subseteq D(Q, \alpha_2) \cap H(Q, \alpha_2)$, applying Theorem 2.18(c) we deduce that B is very good with $N_B \leq C_F n^{\alpha_2} \leq R^{1/(10(d+2))}$.

Now let $C_E \leq r \leq N_B^{d+2}$, and $x_1, x_2 \in B$ with $d_\omega(x_1, x_2) \geq R^{1/4}$. As $D(Q, \alpha_2)$ holds, $d_\omega(x_1, x_2) \geq \frac{1}{3}n^{1/4}$. Choose m so that $m/16 = \lfloor r \log r \rfloor$, and apply Theorem 2.18(b) to deduce that there exists a path $x'_1 = z_0, \dots, z_j = x'_2$ satisfying condition (b) of Theorem 2.23. We need to verify (a)–(c) of Definition 5.4(2). (a) holds since $B_i = B_\omega(z_i, r \log r) = B_\omega(z_i, m/16)$ is very good, and $N_{B_i} \leq m^{1/(d+4)} < r^{1/(d+2)}$. For (b) we have $j \leq c_{2.23.3} |x_1 - x_2|_\infty \leq d c_{2.23.3} d_\omega(x_1, x_2)$. (c) is easily verified as $d_\omega(x_i, x'_i) \leq \frac{1}{3}n^{1/4} < R^{1/4}$. Thus B is exceedingly good.

So we can apply Theorem 5.7 to deduce that the bound (6.2) holds for $t \geq R_E$ and y such that $d(x, y) \leq t$. \square

Proof of Theorem 1. Given Proposition 6.1, all that remains is to replace the chemical distance $d_\omega(x, y)$ by $|x - y|_1$. As $|x - y|_1 \leq d_\omega(x, y)$ the upper bound in (0.4) is immediate. For the lower bound, we take S_x as in Proposition 6.1, let $x, y \in \mathcal{C}_\infty$ and $t \geq S_x$ with $|x - y|_1 \leq t$. Choose the smallest cube Q of side n containing x and y and with $n \geq S_x$. Since Q is very good, by Proposition 2.17(d) we have $|x - y|_\infty \leq C_F(n^{\alpha_2} + 1) \vee |x - y|_\infty$. If $|x - y|_\infty \geq 1 + n^{\alpha_2}$ then it follows that $d_\omega(x, y) \leq c|x - y|_\infty$, so the lower bound in

(0.4) follows. If $|x - y|_\infty < 1 + n^{\alpha_2}$ then $d_\omega(x, y) \leq n^{\alpha_2}$ so as $t \geq S_x$ both $d_\omega(x, y)^2/t$ and $|x - y|_1^2/t$ are less than 1, and again the lower bound in (0.4) holds. \square

Proof of Theorem 2. Fix $x, y \in \mathbb{Z}^d$, let $D = |x - y|_1$ and fix $t \geq D$. Write $A = \{x, y \in \mathcal{C}_\infty\}$. Since $c_5 \leq \mathbb{P}_p(x, y \in \mathcal{C}_\infty) \leq 1$ it is enough to prove (0.6) for $\mathbb{E}_p(q_t^\omega(x, y)1_A)$. Let S_x be as in the proof of Theorem 1, and $n = t/2$. Then

$$\mathbb{E}_p(q_t^\omega(x, y)1_A) = \mathbb{E}_p(q_t^\omega(x, y)1_A 1_{\{S_x < n\}}) + \mathbb{E}_p(q_t^\omega(x, y)1_A 1_{\{S_x \geq n\}}). \quad (6.3)$$

By the proof of Theorem 1, if $\omega \in A$ and $S_x(\omega) \leq n$ then $q_t^\omega(x, y)$ satisfies the bounds in (0.4). So the lower bound in (0.6) is immediate since

$$\begin{aligned} \mathbb{E}_p(q_t^\omega(x, y)1_A) &\geq \mathbb{E}_p(q_t^\omega(x, y)1_A 1_{\{S_x < n\}}) \\ &\geq c_6 e^{-c_7 D^2/t} (\mathbb{P}_p(A) - \mathbb{P}_p(A \cap \{S_x \geq n\})) \\ &\geq c_6 e^{-c_7 D^2/t} (c_5 - c_8 e^{-t^{\alpha_\beta}}). \end{aligned} \quad (6.4)$$

So if $t \geq c_9$ then the final term in (6.4) is greater than $\frac{1}{2}$ and we obtain (0.6). If $t < c_9$ then with probability at least $c_5 p^D$ there is a path of length D in \mathcal{C}_∞ joining x and y , and using (1.4) we deduce that on this event $q_t^\omega(x, y) \geq c^D \geq c'$. So, (0.6) holds in this case also.

To prove the upper bound, recall that by Lemma 1.1 we always have (when $t \geq D$) that

$$q_t^\omega(x, y) \leq c e^{-2c_{10} d_\omega(x, y)^2/t} \leq c e^{-2c_{10} D^2/t}.$$

If $e^{-c_{10} D^2/t} \leq t^{-d/2}$ then the upper bound in (0.6) is then immediate. If not, then by Theorem 1 we have

$$\mathbb{E}_p(q_t^\omega(x, y)1_A 1_{\{S_x < n\}}) \leq c_{11} e^{-c_{12} D^2/t},$$

so it remains to control the second term in (6.3). We have, provided $t \geq c_{13}$,

$$\mathbb{E}_p(q_t^\omega(x, y)1_A 1_{\{S_x \geq n\}}) \leq \mathbb{P}_p(S_x \geq n) \leq c e^{-c t^{\alpha_2 \beta}} \leq t^{-d} \leq t^{-d/2} e^{-c_{10} D^2/t}.$$

If $t \leq c_{13}$ then (as $D \leq t$) we obtain the upper bound in (0.6) by taking the constant c_2 large enough. \square

Proof of Theorem 3. This follows easily from Theorem 5.11 and Proposition 6.1. \square

Proof of Theorem 4. (a) This is a well known consequence of the Harnack inequality. Let $h : \mathcal{C}_\infty \rightarrow (0, \infty)$ be a global harmonic function. Replacing h by ch if necessary, we can assume $h \geq 1$ everywhere, and that there exists x_0 with $h(x_0) < 2$. Applying Theorem 3 to $B_\omega(x_0, R) \subseteq B_\omega(x_0, 2R) \subseteq \mathcal{C}_\infty$, when R is large enough we deduce that $\sup_{B_\omega(x_0, R)} h \leq 2c_1$, so that h is bounded.

Let $B_n = B(x_0, 2^n R)$. Then Theorem 3 implies the oscillation inequality $\inf_{B_n} h \geq c_1^{-1} \sup_{B_{n+1}} h$, so that $\text{Osc}(h, B_n) \leq (1 - c_1^{-1}) \text{Osc}(h, B_{n+1})$. Iterating one deduces that

$\text{Osc}(h, B_n) \geq (1 - c_1^{-1})^{-n} \text{Osc}(h, B_0)$, and since h is bounded this implies that $\text{Osc}(h, B_0) = 0$. Thus h is constant on any large ball, and so constant.

(b) This is also standard – [FS, Lemma 5.2]. Lemma 5.8 gives lower bounds on the transition probability $q_t^{0,\omega}(x, y)$ for Y killed outside a ball $B(x_0, R)$, for all sufficiently large R . Let F be an event in the tail field, and set $f(s, x) = P_\omega(F | Y_s = x)$. Then $0 \leq f \leq 1$, and f satisfies

$$f(s, x) = \int_{\mathcal{C}_\infty} q_{t-s}^\omega(x, y) f(t, y) \mu(dy), \quad s < t. \quad (6.5)$$

Fix $x_0 \in \mathcal{C}$, $t_0 \geq 0$ and set

$$A(R) = B(x_0, \tfrac{1}{2}R) \times [t_0, t_0 + R^2].$$

Let $g(s, x)$ satisfy (6.5) with $\min_{A(2R)} g = 0$, $\max_{A(2R)} g = 1$, and

$$\int_{B(x_0, \frac{1}{2}R)} g(t_0 + 4R^2, y) \mu(dy) \geq \tfrac{1}{2}.$$

Then if $(s, x) \in A(R)$,

$$\begin{aligned} g(s, x) &\geq \int_{B(x_0, \frac{1}{2}R)} q_{t_0+4R^2-s}^{0,\omega}(x, y) g(t_0 + 4R^2, y) \mu(dy) \\ &\geq \int_{B(x_0, \frac{1}{2}R)} c_2 R^{-d} g(t_0 + 4R^2, y) \mu(dy) \\ &\geq \tfrac{1}{2} c_2 R^{-d} \mu(B(x_0, \tfrac{1}{2}R)) \geq c_3. \end{aligned}$$

Hence there exists $\delta > 0$ such that

$$\text{Osc}(f, A(R)) \leq (1 - \delta) \text{Osc}(f, A(2R)),$$

and iterating it follows that f is constant. □

Proof of Theorem 5. This is immediate on integrating the bounds (0.4), and using (1.6) to control $q_t(x, y)$ for $|x - y|_1 \geq t$. □

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Version 2.0 8 August 2003