WIENER-HOPF FACTORIZATION FOR MATRICES

by

M.T. Barlow, L.C.G. Rogers, and David Williams

1. *The main results.* Let E be a finite set. Let $\mathcal{Q}(E)$ denote the set of (real) $E \times E$ matrices Q such that, for $i, j \in E$,

$$Q(i,j) \geq 0 \quad (i \neq j), \quad \sum_{k \in E} Q(i,k) \leq 0 .$$

Let Q now denote some fixed element of $\mathcal{Q}(E)$. Let v be a function from E to $\mathbb{R} \setminus \{0\}$, and let V be the diagonal $E \times E$ matrix $\operatorname{diag}\{v(i) : i \in E\}$. Let $E^+ = \{i \in E : v(i) > 0\}$, and $E^- = \{i \in E : v(i) < 0\}$. Let I denote the identity $E \times E$ matrix, I^+ the identity $E^+ \times E^+$ matrix, and I^- the identity $E^- \times E^-$ matrix.

Let c be a strictly positive real number.

THEOREM I. *There exists a unique pair* (Π_c^+, Π_c^-), *where* Π_c^+ *is an* $E^- \times E^+$ *matrix and* Π_c^- *is an* $E^+ \times E^-$ *matrix, such that, if*

$$S = \begin{pmatrix} I^+ & \Pi_c^- \\ \Pi_c^+ & I^- \end{pmatrix}, \tag{1}$$

then S *is invertible and*

$$S^{-1}[V^{-1}(Q - cI)]S = \begin{pmatrix} \tilde{Q}_c^+ & 0 \\ 0 & -\tilde{Q}_c^- \end{pmatrix}, \tag{2}$$

where $\tilde{Q}_c^+ \in \mathcal{Q}(E^+)$ *and* $\tilde{Q}_c^- \in \mathcal{Q}(E^-)$. *Moreover,* Π_c^+ *and* Π_c^- *are strictly substochastic: thus, for* $i \in E^-$, $j \in E^+$,

$$\Pi_c^+(i,j) \geq 0, \quad \sum_{k \in E^+} \Pi_c^+(i,k) < 1 .$$

Theorem I will be said to yield the 'Wiener-Hopf factorization' of the matrix $V^{-1}(Q - cI)$.

Now let X be a Markov chain on $E \cup \{\partial\}$ (∂ is the cemetery state) with Q-matrix Q. Thus the transition matrix function of X is $P(t) = \exp(tQ)$. For $t \geq 0$, define:

$$\phi(t) = \int_0^t v(X_s)ds , \quad \tau^+(t) = \inf\{s : \phi(s) > t\} .$$

As usual, we shall (for example) write τ_t^+ for $\tau^+(t)$ when more convenient. Note that $X(\tau_t^+) \in E^+ \cup \{\partial\}$.

THEOREM II. For $i \in E^-$ and $j \in E^+$,

$$(3) \qquad \underset{\sim}{E}^i[\exp(-c\tau_0^+) ; \ X(\tau_0^+) = j] = \Pi_c^+(i,j) .$$

For $i \in E^+$, $j \in E^+$, and $t \geq 0$,

$$(4) \qquad \underset{\sim}{E}^i[\exp(-c\tau_t^+) ; \ X(\tau_t^+) = j] = [\exp(t\tilde{Q}_c^+)](i,j) .$$

The corresponding 'minus' results follow, on replacing ϕ by $(-\phi)$.

The problem of finding the joint distribution of τ_t^+ and $X(\tau_t^+)$ is of course solved by Theorems I and II. The way in which Π_c^+ and Π_c^- may be calculated will be clear from the proofs.

Comment. The reader may feel that the martingale techniques used in this paper are more sophisticated than those required for this Markov chain problem. The following two statements are therefore apposite. First, we do not know how to prove the purely algebraic Theorem I without appealing to probability theory (and ultimately to martingale theory). Second, the martingale technique generalises to other (more interesting) cases, though the problem of obtaining explicit answers proves to be very difficult. Work on the 'continuous' state-space case will be published later.

2. Basic martingales. It is important to regard the strictly positive number c as fixed throughout the remainder of the paper, except in section 7.

Let f be a function on $(E \cup \{\partial\}) \times \mathbb{R} \times [0,\infty)$ such that $f(\partial,\cdot,\cdot) = 0$. A natural extension of Dynkin's formula shows that (for every initial distribution)

$$f(X_t,\phi_t,t) - \int_0^t \mathcal{A}f(X_s,\phi_s,s)\,ds$$

is a local martingale, where

$$\mathcal{A}f(x,\phi,t) = Qf + V\frac{\partial f}{\partial \phi} + \frac{\partial f}{\partial t} .$$

Here, of course,

$$Qf(x,\phi,t) = \sum_{y \in E} Q(x,y)f(y,\phi,t) .$$

In particular, if g is any vector on E, and

(5) $$f(x,\phi,t) = \{\exp[-ctI - \phi V^{-1}(Q-cI)]g\}(x) \quad \text{on } E,$$

then $f(X_t,\phi_t,t)$ is a local martingale (in fact, a martingale, because it is bounded on every finite interval).

3. Definition of $\mathcal{N}$. Before recalling part of the theory of the Jordan form, we recall the proof of the well-known fact that $V^{-1}(Q-cI)$ cannot have an eigenvalue on the imaginary axis. For suppose that μ lies on the imaginary axis and that

(6) $$(Q-cI)g = \mu Vg$$

for some non-zero vector g. Choose i in E with $|g(i)| \geq |g(j)|$ for all j in E. The i-th coordinate of (6) reads :

$$[Q(i,i) - c - \mu v(i)]g(i) = -\sum_{j \neq i} Q(i,j)g(j) .$$

But the left-hand side has modulus at least equal to $(|Q(i,i)| + c)|g(i)|$, while the right-hand side has modulus at most equal to $|Q(i,i)||g(i)|$. The contradiction establishes the 'well-known fact'.

One of the main steps in Jordan-form theory shows that the space of complex vectors on E has a basis $\mathcal{G}$ such that every g in $\mathcal{G}$ solves an equation

(7) $$[V^{-1}(Q-cI)-\mu I]^k g = 0 ,$$

where μ is an eigenvalue of $V^{-1}(Q-cI)$ and k is a positive integer. Fix $\mathcal{G}$, and let $\mathcal{N}$ [respectively, $\mathcal{P}$] *denote the set of those vectors* g *in* $\mathcal{G}$ *for which the associated* μ-*value has (strictly) negative [respectively, positive] real part*.

4. *The structure of* $\mathcal{N}$. Let $g \in \mathcal{N}$, so that g satisfies (7) for some k and some μ with negative real part. Then the function f at (5) may be written

$$f(\cdot,\phi,t) = \exp(-ct-\mu\phi)\exp\{-\phi[V^{-1}(Q-cI)-\mu I]\}g ,$$

and the second exponential may be expanded in a power series in which all terms after the (k-1)-th annihilate g. Hence, since μ has negative real part,

(8) *for* $g \in \mathcal{N}$ *and* f *as at* (5), $f(X_s,\phi_s,s)$ *is bounded on* $[0,\tau_t^+]$ *for every* t.

In particular, on applying the optional stopping theorem at time τ_0^+, we find that (for all $i \in E$)

$$\mathbb{E}^i[\exp(-c\tau_0^+)g \circ X(\tau_0^+)] = g(i) .$$

Now *define* Π_c^+ via the probabilistic formula (3). We have just shown that

(9) *if* $g \in \mathcal{N}$, *then* $g = \begin{pmatrix} I^+ \\ \Pi_c^+ \end{pmatrix} g^+$ *where* g^+ *denotes the restriction of* g *to* E^+.

Hence $\mathcal{N}$ has at most $|E^+|$ elements, and, by a similar argument, $\mathcal{P}$ has at most $|E^-|$ elements. The only explanation is that

(10) $\mathcal{N}$ *has precisely* $|E^+|$ *elements and the elements* g^+ *where* $g \in \mathcal{N}$ *form a basis for the space of vectors on* E^+.

5. *Proof of the uniqueness of Wiener-Hopf factorization*. Suppose that for some $E^- \times E^+$ matrix K_c^+, some $E^+ \times E^-$ matrix K_c^-, some $\hat{Q}_c^+$ in $\mathcal{Q}(E^+)$,

and some $\hat{Q}_c^-$ in $\mathcal{Q}(E^-)$, we have that $\begin{pmatrix} I^+ & K_c^- \\ K_c^+ & I^- \end{pmatrix}$ is invertible, and

$$\begin{pmatrix} I^+ & K_c^- \\ K_c^+ & I^- \end{pmatrix}^{-1} V^{-1}(Q - cI) \begin{pmatrix} I^+ & K_c^- \\ K_c^+ & I^- \end{pmatrix} = \begin{pmatrix} \hat{Q}_c^+ & 0 \\ 0 & -\hat{Q}_c^- \end{pmatrix}.$$

Then the eigenvalues of $V^{-1}(Q - cI)$ with negative real part must coincide with the eigenvalues of $\hat{Q}_c^+$. Moreover, if

$$(11) \qquad (\hat{Q}_c^+ - \mu I^+)^k u^+ = 0$$

for some positive integer k, some μ(with negative real part) and some vector u^+ on E^+, then

$$\{V^{-1}(Q - cI) - \mu I\}^k \begin{pmatrix} I^+ \\ K_c^+ \end{pmatrix} u^+ = 0,$$

so that, from the argument leading to (9),

$$(12) \qquad \begin{pmatrix} I^+ \\ K_c^+ \end{pmatrix} u^+ = \begin{pmatrix} I^+ \\ \Pi_c^+ \end{pmatrix} u^+.$$

By the theory of the Jordan canonical form for $\hat{Q}_c^+$, equation (11) holds for a set of vectors u^+ spanning the vectors on E^+, and so therefore does equation (12). Hence

$$K_c^+ = \Pi_c^+$$

and the required uniqueness follows from this fact and its 'minus' analogue.

6. <u>Existence of the Wiener-Hopf factorization.</u> For the moment, regard $t \geq 0$ as fixed. Let $g \in \mathcal{N}$, define f as at (5), and recall that $f(X_s, \phi_s, s)$ is a martingale. Using (8) as justification, apply the optional stopping theorem at time τ_t^+ to obtain

$$\underset{\sim}{E}^{\cdot}[\exp(-c\tau_t^+)h \circ X(\tau_t^+)] = g = \exp[tV^{-1}(Q - cI)]h,$$

where $h = \exp[-tV^{-1}(Q - cI)]g$. Note that h will automatically satisfy the same version of (7) as does g, so that h, like g, has property (9). Hence, since $X(\tau_t^+) \in E^+$, we have

$$(13)\qquad \begin{pmatrix} I^+ \\ \Pi_c^+ \end{pmatrix} \underset{\sim}{E}^{\cdot}[\exp(-c\tau_t^+)h^+\circ X(\tau_t^+)] = \exp[tV^{-1}(Q-cI)] \begin{pmatrix} I^+ \\ \Pi_c^+ \end{pmatrix} h^+ .$$

We have obtained (13) for a class of vectors h^+ which clearly spans the space of all vectors on E^+, so that <u>(13) holds for all vectors h^+ on</u> E^+.

It is almost immediate from the strong Markov property of X that

$$\tilde{P}_c^+(t;i,j) = \underset{\sim}{E}^i[\exp(-c\tau_t^+);\ X(\tau_t^+) = j] \qquad (i,j \in E^+)$$

defines a subMarkovian transition function on E^+, so that

$$\tilde{P}_c^+(t) = \exp(t\tilde{Q}_c^+)$$

for some $\tilde{Q}_c^+$ in $\mathcal{Q}(E^+)$. (One can alternatively deduce this from (13).) On differentiating (13) with respect to t and setting $t = 0$, we obtain

$$(14)\qquad \begin{pmatrix} I^+ \\ \Pi_c^+ \end{pmatrix} \tilde{Q}_c^+ = V^{-1}(Q-cI) \begin{pmatrix} I^+ \\ \Pi_c^+ \end{pmatrix} .$$

Theorems I and II now follow from (14) and its 'minus' analogue.

7. <u>The case $c = 0$</u>. As has already been noted, Theorems I and II solve the problem of calculating the joint distribution of τ_t^+ and $X(\tau_t^+)$, but the involvement of the positive parameter c complicates the formulae, and is irrelevant if only the law of $X(\tau_t^+)$ is sought. This corresponds to the case $c = 0$, in which case we have the following result.

THEOREM III. <u>There exists a unique pair</u> (Π^+, Π^-), <u>where</u> Π^+ <u>is an</u> $E^- \times E^+$ <u>matrix, and</u> Π^- <u>is an</u> $E^+ \times E^-$ <u>matrix, such that, for some</u> $Q^+ \in \mathcal{Q}(E^+)$ <u>and</u> $Q^- \in \mathcal{Q}(E^-)$,

$$(15)\qquad V^{-1}QS = S\begin{pmatrix} Q^+ & 0 \\ 0 & -Q^- \end{pmatrix}$$

where

$$(16)\qquad S = \begin{pmatrix} I^+ & \Pi^- \\ \Pi^+ & I^- \end{pmatrix}.$$

Moreover, Π^+ and Π^- are substochastic, and for $i \in E^-$, $j \in E^+$, $k \in E^+$,

$$P^i[X(\tau_0^+) = j] = \Pi^+(i,j) \tag{17}$$

and

$$P^k[X(\tau_t^+) = j] = [\exp(tQ^+)](k,j) \tag{18}$$

for each $t \geq 0$.

Remarks. The matrix S need no longer be invertible, and it is also possible that both Q^+ and Q^- may be conservative.

Proof. The existence of such a decomposition follows easily from Theorem I: by considering a sequence $(c_n : n \geq 0)$ of reals decreasing to 0, we may suppose, by taking a subsequence if necessary, that $\Pi^+_{c_n}$, $\Pi^-_{c_n}$, $\tilde{Q}^+_{c_n}$, and $\tilde{Q}^-_{c_n}$ each converge entry by entry to Π^+, Π^-, Q^+, and Q^- respectively. Thus (15) follows immediately from (2), (17) follows from (3), and (18) follows from (4).

As to the uniqueness of the decomposition (15), the argument of section 5 serves again with minor modification. Suppose that there exists $\hat{Q}^+ \in \mathcal{Q}(E^+)$ and an $E^- \times E^+$ matrix K^+ such that

$$V^{-1}Q \begin{pmatrix} I^+ \\ K^+ \end{pmatrix} = \begin{pmatrix} I^+ \\ K^+ \end{pmatrix} \hat{Q}^+ .$$

Consider a basis $\mathcal{B}$ for the space of complex vectors on E^+ such that for each $u^+ \in \mathcal{B}$ there exists an integer $k \geq 1$ and an eigenvalue μ of $\hat{Q}^+$ such that

$$(\hat{Q}^+ - \mu I^+)^{k-1} u^+ \neq 0 , \quad (\hat{Q}^+ - \mu I^+)^k u^+ = 0 . \tag{19}$$

(Such a basis is guaranteed by the theory of the Jordan form.)

As the argument of section 3 shows, the non-zero eigenvalues of $\hat{Q}^+$ have strictly negative real part, so if $u^+ \in \mathcal{B}$ is associated through (19) with a non-zero eigenvalue, the argument of section 5 leads to

$$\begin{pmatrix} I^+ \\ K^+ \end{pmatrix} u^+ = \begin{pmatrix} I^+ \\ \Pi^+ \end{pmatrix} u^+ . \tag{20}$$

To complete the proof, we must notice that if $a^+ \in \mathcal{B}$ is associated through (19) with a zero eigenvalue, then $\hat{Q}^+ a^+ = 0$. [Indeed, were this not so, then $\hat{P}^+(t)\, a^+ \equiv \exp(t\hat{Q}^+)\, a^+$ would not remain bounded as $t \to \infty$.] Accordingly, if a is defined by

$$a \equiv \begin{pmatrix} I^+ \\ K^+ \end{pmatrix} a^+ ,$$

then $a \in \ker(Q)$ and so $a(X_t)$ is a martingale, whence (20) is satisfied by a, and we conclude as before that $K^+ = \Pi^+$.

8. Examples. We examine two cases in which explicit expressions can be found for Q^+.

Proposition I.

Let R^+ (respectively, R^-) be an $E^+ \times E^-$ matrix (respectively, an $E^- \times E^+$ matrix) of non-negative entries, with row sums ≤ 1. Let $0 < a \leq b$, *let* $\alpha \equiv a + c$, $\beta \equiv b + c$, *and suppose that*

$$Q = \begin{pmatrix} -aI^+ & aR^+ \\ bR^- & -bI^- \end{pmatrix} \in \mathcal{Q}(E) .$$

If $v(i) = 1$ *if* $i \in E^+$, $v(i) = -1$ *if* $i \in E^-$, *then the Wiener-Hopf factorisation of* $V^{-1}(Q - cI)$ *has the explicit form* :

(21) $$\Pi_c^+ = bR^-(\beta I^+ - \tilde{Q}_c^+)^{-1} ,$$

(22) $$\tilde{Q}_c^+ \equiv \alpha \left(- I^+ + \sum_{n=1}^{\infty} p_n \frac{\alpha+\beta}{2\alpha} [\frac{4ab}{(\alpha+\beta)^2} R^+R^-]^n \right) ,$$

where $p_n \equiv 2 . 4^{-n} . \dfrac{(2n-2)!}{n!(n-1)!}$. *The corresponding formulae for the negative parts are*

(23) $$\Pi_c^- = aR^+(\alpha I^- - \tilde{Q}_c^-)^{-1} ,$$

(24) $$\tilde{Q}_c^- \equiv -\beta I^- + \alpha \sum_{n=1}^{\infty} p_n \frac{\alpha+\beta}{2\alpha} [\frac{4\,ab}{(\alpha+\beta)^2} R^-R^+]^n .$$

Remarks. Suppose $\{T_i^+ ; i = 1,2,\ldots\}$ are independent $\exp(\alpha)$ r.v.'s, and $\{T_i^- ; i = 1,2,\ldots\}$ are independent $\exp(\beta)$ r.v.'s, independent of the T_i^+. Let $X_i = T_i^+ - T_i^-$, let $S_0 = 0$ and let $S_n = S_{n-1} + X_n$ $(n \geq 1)$. The classical Wiener-Hopf theory concerns itself with the joint law of τ and S_τ, where $\tau = \inf\{n ; S_n > 0\}$. This particular case is one of the very few where an explicit answer can be found; if $p_n^+ \equiv P(\tau = n)$, then for $0 \leq s \leq 1$

$$\psi^+(s) \equiv \sum_{n=1}^{\infty} p_n^+ s^n = \frac{(\alpha+\beta)}{2\alpha}\left[1 - \left(1 - \frac{4\alpha\beta s}{(\alpha+\beta)^2}\right)^{\frac{1}{2}}\right]$$

$$= \frac{(\alpha+\beta)}{2\alpha} \sum_{n=1}^{\infty} p_n \left[\frac{4\alpha\beta s}{(\alpha+\beta)^2}\right]^n,$$

(see, for example, Feller [1] p608).

The original process X visits E^+ and E^- alternately, staying an $\exp(\alpha)$ time in E^+ and an $\exp(\beta)$ time in E^-. If the process starts at $i \in E^+$, then the probability that it is in $j \in E^+$ the next time it visits E^+ is $(R^+R^-)_{ij}$. This makes (22) obvious.

Proof. Firstly, we must check that $\tilde{Q}_c^+$ as defined by (22) is well-defined, and is a Q-matrix. If $|E^+| = n$, equipping $\mathbb{R}^n$ with the norm $\|x\| = \max\{|x_i| ; i = 1, \ldots, n\}$ makes $4ab(\alpha+\beta)^{-2} R^+R^-$ into a linear operator of norm < 1, so the series is absolutely convergent, and $\tilde{Q}_c^+$ is well defined. To see that $\tilde{Q}_c^+ \in (E^+)$, note that the off diagonal entries are non-negative, and the row sums are negative.

Since $(\tilde{Q}_c^+)^2 - (\beta - \alpha)Q_c^+ = \alpha\beta I^+ - abR^+R^-$, it is a simple exercise to verify that the stated forms for $\Pi_c^+, \tilde{Q}_c^+$ given in (21) and (22) and for $\Pi_c^-, \tilde{Q}_c^-$ given in (23) and (24) satisfy the factorisation identity (2) so, by Theorem I, they are the unique solutions.

Proposition II.

Suppose that $|E^+| = |E^-|$, and that for some $Q \in \mathcal{Q}(E)$ we have

$$Q - cI = \begin{pmatrix} A & I^+ \\ I^- & D \end{pmatrix} .$$

Suppose also that $AD = DA$. Let $B = \frac{1}{2}(A + D)$.

Then, if $v(i) = 1$ for $i \in E^+$, $v(i) = -1$ for $i \in E^-$, we have

$$\Pi_c^+ = -B \sum_{n=1}^{\infty} p_n B^{-2n} \tag{25}$$

$$\tilde{Q}_c^+ = A + \Pi_c^+ . \tag{26}$$

Proof. Notice that $B' \equiv B + I^+ \in \mathcal{Q}(E^+)$, so that $(I^+ - B')^{-1}$ exists and is a resolvent matrix - in particular, it is substochastic. Thus $-B^{-1}$ is a substochastic matrix, and the expression (25) for Π_c^+ takes the form

$$\Pi_c^+ = \sum_{n=1}^{\infty} p_n (-B^{-1})^{2n-1}$$

which clearly exists (by monotone convergence) and equally clearly is substochastic. It is simple to verify that

$$(\Pi_c^+)^2 + 2B\Pi_c^+ + I = 0 ,$$

and from this, the verification of the identify (2) is an easy exercise.

Reference

[1] FELLER, W., An introduction to probability theory and its applications, Vol.II, John Wiley & Sons, New York

Department of Pure Mathematics
University of Liverpool
P.O. Box 147
Liverpool, L69 3BX

(Barlow)

Department of Pure Mathematics
University College of Swansea
Singleton Park
Swansea, SA2 8PP

(Rogers, Williams)