On Countable Dense Random Sets

by

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We shall discuss point processes whose realisations consist typically of a countable dense set of points. In particular, we discuss when such a process may be regarded as Poisson.

The most primitive way to describe a point process on $[0,\infty)$ is as a subset B of $\Omega \times [0,\infty)$, where the section B_ω represents the times of the "points" in realisation ω. In the locally finite case, there are the more familiar descriptions using the counting process

$$N_t(\omega) = \#(B_\omega \cap [0,t]) \qquad \text{(as in [BJ])}$$

or using the random measure

$$\xi(\omega,D) = \#(B_\omega \cap D) \qquad \text{(as in [K])}$$

Our point processes will generally not be locally finite, so we cannot use these familiar descriptions: we revert to describing a process as a subset B.

We first describe an (obvious) construction of a countable dense Poisson process. Let Θ be a countable infinite set. Let (F_t) be a filtration (all filtrations are assumed to satisfy the usual conditions). Suppose $\{S_i^\theta : i \geq 1, \theta \in \Theta\}$ are optional times such that each counting process $N_t^\theta = \sum_i 1(S_i^\theta \leq t)$ is a Poisson process of rate 1 with respect to (F_t), and suppose the process N^θ are independent. Let ξ be the random measure on $\Theta \times [0,\infty)$ whose realisation $\xi(\omega)$ has the set of atoms $\{(\theta, S_i^\theta(\omega)) : i \geq 1, \theta \in \Theta\}$. Then ξ describes a uniform Poisson process on $\Theta \times [0,\infty)$, with respect to (F_t). But we can also think of ξ as a marked point process on the line. That is, each realisation is an a.s. countable dense set $\{S_i^\theta(\omega) : i \geq 1, \theta \in \Theta\}$ of points in $[0,\infty)$, and each point is marked by some θ. The corresponding unmarked process can be described by

(1) $$B = \{(\omega,t) : S_i^\theta(\omega) = t \text{ for some } i,\theta\} = \{(\omega,t) : \xi(\omega,\Theta\times\{t\})=1\} .$$

Think of B as a σ-finite Poisson process. We are concerned with the converse procedure: given a set B, when can we assign marks θ to the points of B to construct a uniform Poisson process ξ satisfying (1)? To allow external randomisation is assigning marks, we make the following definitions:

(2) Definition. (G_t) is an extension of (F_t) if for each t

(i) $G_t \subset F_t$

(ii) G_t and F_∞ are conditionally independent given F_t.

(3) Definition B is a σ-finite Poisson process with respect to (F_t) if

(i) B is (F_t)-optional

(ii) There exists a uniform Poisson process ξ on $\Theta \times [0,\infty)$ with respect to some extension (G_t) of (F_t) such that (1) holds.

Theorem 4 below gives a more intrinsic description of σ-finite Poisson processes. First we recall some notation. An optional time T has <u>conditional intensity</u> $a(\omega,s)$ if T has compensator $A_t = \int_0^t a(s)ds$. We may assume $a(\omega,s)$ is previsible by [D.V. 19]. Replacing (F_t) by an extension does not alter the conditional intensity of an (F_t)-optional time T.

Recall also the notation

$$T_D = T \text{ on } D$$
$$= \infty \text{ elsewhere.}$$

Let λ be Lebesgue measure on $[0,\infty)$.

(4) THEOREM. Let (F_t) be a filtration. Let B be an optional set whose sections B_ω are a.s. countable. The following are equivalent

(a) B is a σ-finite Poisson process

(b) There exists a family (T^n) such that

(5) T^n is optional; the graphs $[T^n]$ are disjoint; $B = \cup[T^n]$ a.s.;

(6) T^n has a conditional intensity, say $a_n(\omega,s)$;

(7) $\sum_n a_n(\omega,s) = \infty$ a.e. $(P \times \lambda)$

(b') Every family (T^n) satisfying (5) also satisfies (6) and (7)

(c) For every previsible set C

$$\{\omega : C_\omega \cap B_\omega = \emptyset\} = \{\omega : \lambda(C_\omega) = 0\} \quad \text{a.s.}$$

Remark Families satisfying (5) certainly exist, by the section theorem and transfinite induction [D. VI. 33] .

The next result comes out of the proof of Theorem 4.

(8) PROPOSITION. Let μ be a probability measure on $[0,\infty)$ which is equivalent to Lebesgue measure .

(a) Let (Y_i) be i.i.d. with law μ , and let (F_t) be the smallest filtraiton making each Y_i optional - that is, the filtration generated by the processes $1_{[Y_i,\infty)}$. Then $B = U[Y_i]$ is a σ-finite Poisson process with respect to (F_t).

(b) Conversely, let B be a σ-finite Poisson process with respect to some (F_t) . Then there exist times (Y_i) such that $B = U[Y_i]$ a.s., (Y_i) are i.i.d. with law μ , and (Y_i) are optional with respect to some extension of (F_t) .

Before the proofs, here is an amusing example.

Example There exists a process X_t and filtrations (F_t) , (G_t) such that X is optional with respect to each of (F_t) and (G_t) , but X is not optional with respect to $F_t \cap G_t$.

To construct the example, let $(Y_i), B, (F_t)$ be as in part (a) of Proposition 8, and let $X = 1_B$. Let Π be the set of finite permutations $\pi = (\pi(1), \pi(2), \ldots)$ of $(1,2,\ldots)$. Since Π is countable we can construct a random element π^* of Π such that $P(\pi^* = \pi) > 0$ for each $\pi \in \Pi$. Take π^* independent of $\underset{\sim}{Y} = (Y_1, Y_2, \ldots)$. Define $\underset{\sim}{V} = (V_1, V_2, \ldots) = (Y_{\pi^*(1)}, Y_{\pi^*(2)}, \ldots)$. Let (F_t) be the smallest filtration making each V_i optional. Since $X_t = \Sigma 1_{(Y_i = t)} = \Sigma 1_{(V_i = t)}$, plainly X is both (F_t)- and (G_t)-optional. But $F_\infty \cap G_\infty$ is trivial! For let $D \in F_\infty \cap G_\infty$. Then there exist measurable functions f,g such that

$$1_D = f(\underset{\sim}{Y}) = g(\underset{\sim}{V}) \quad \text{a.s.}$$

So $f(\underset{\sim}{Y}) = h(\underset{\sim}{Y},\pi^*)$ a.s., where $h(y_1 y_2, \ldots ;\pi) = g(y_{\pi(1)}, y_{\pi(2)}, \ldots)$ But π^* is independent of $\underset{\sim}{Y}$ with support Π , so

$$F(\underset{\sim}{Y}) = h(\underset{\sim}{Y},\pi) \quad \text{a.s., each } \pi \in \Pi .$$

So, putting $G = \{g = 1\}$,

$$D = \{(Y_{\pi(1)}, Y_{\pi(2)}, \ldots) \in G\} \quad \text{a.s., each } \pi \in \Pi .$$

Thus D is exchangeable, and so is trivial by the Hewitt-Savage zero-one law.

We now start the proof of Theorem 4. The lemma below shows that (b) and (b') are equivalent.

(9) LEMMA. Let (T^n) be optional times whose graphs $[T^n]$ are disjoint. Let $(\hat{T}^m)$ be a similar family, and suppose $U[T^n] = U[\hat{T}^m]$. Suppose T^n has conditional intensity a_n .

Then $\hat{T}^m$ has a conditional intensity, $\hat{a}_m$ say, and $\Sigma\hat{a}_m = \Sigma a_n$ a.e. $(P \times \lambda)$.

<u>Proof</u> Put $U_{m,n} = T^n_{(T^n=\hat{T}^m)}$. Then $U_{m,n}$ has a conditional intensity, $a_{m,n}$ say. It is easy to verify

$$a_n = \sum_m a_{m,n} \quad \text{a.e.}$$

$\hat{a}_m \equiv \sum_n a_{m,n}$ is the conditional intensity of $\hat{T}^m$, where the sum is a.e. finite because

$$E\int \sum_{n=1}^{N} a_{m,n}(s)\, ds = \sum_{n=1}^{N} P(U_{m,n} < \infty) \le P(T^m < \infty) \le 1 \ .$$

Hence $\Sigma a_n = \Sigma\Sigma a_{m,n} = \Sigma\hat{a}_m \le \infty$ a.e.

Lemmas 10 and 13 show that conditions (b') and (c) are equivalent.

(10) LEMMA. For B as in theorem 4, the following are equivalent

(i) $\{\omega : C_\omega \cap B_\omega = \emptyset\} \supset \{\omega : \lambda(C_\omega) = 0\}$ a.s., each previsible C .

(ii) Each family (Y^n) satisfying (5) also satisfies (6).

<u>Proof: (ii) implies (i)</u> Let C be previsible. Put $T = \inf\{t : \lambda(C_\omega \cap [0,t]) > 0\}$. Then T is optional, so $C' = C \cap [0,T]$ is previsible. Now $\lambda(C'_\omega) = 0$ a.s. We must prove

(11) $C'_\omega \cap B_\omega = \emptyset$ a.s.

Let (T^n) satisfy (5) and (6). Then

$$P(T^n \in C'_\omega) = E\int 1_{C'}\, d1_{[T^n,\infty)}$$
$$= E\int 1_{C'}(s)\, a_n(s)\, ds$$
$$= 0\ .$$

Since $B = \cup[T^n]$, (11) follows.

<u>(i) implies (ii)</u>. Let T be optional, $[T] \subset B$. Let A_t be the compensator of T . From the proof of the Lebesgue decomposition theorem, we can write $A_t = \hat{A}_t + \int_0^t a(s)ds$, where there exists a progressive set D such that

(12) $\lambda(D_\omega) = 0$ a.s.; the measure $d\hat{A}(\omega)$ is carried on D_ω a.s.

Let $C = \{^p(1_D) > 0\}$; then C is previsible and since

$$\hat{A}_t \geq \int_0^t 1_C(s)d\hat{A}_s \geq \int_0^t {}^p(1_D)(s)d\hat{A}_s = \int_0^t 1_D(s)d\hat{A}_s = \hat{A}_t\ ,$$

and

$$\int_0^t {}^p(1_D)(s)ds = \int_0^t 1_D(s)ds = 0\ ,$$

C satisfies (12). However

$$E\hat{A}_\infty = E\int 1_C(s)d\hat{A}_s$$
$$= E\int 1_C(s)dA_s$$
$$= P(T \in C_\omega) = 0 \quad \text{by (11).}$$

So $\hat{A} \equiv 0$.

(13) LEMMA. For B as in Theorem 4, the following are equivalent.

(i) $\{\omega : C_\omega \cap B_\omega \neq \emptyset\} \supset \{\omega : \lambda(C_\omega) > 0\}$ a.s., each previsible C.

(ii) Each family (T^n) satisfying (5) and (6) also satisfies (7).

Proof. (ii) implies (i) Let C be previsible. Define optional times :

$$T = \inf\{t : \lambda(C_\omega \cap [0,t]) > 0\}$$
$$S = \inf\{t : t \in B_\omega \cap C_\omega\}.$$

It is sufficient to prove

(14) $S \le T$ a.s.

Consider the previsible set $C' = C \cap (T,S]$

Let (T^n) satisfy (5), (6). By definition of S , the sets $\{\omega : T^n \in C'_\omega\}$ are disjoint. So $\sum_n P(T^n \in C'_\omega) \le 1$. But

$$\Sigma\, P(T^n \in C'_\omega) = \Sigma E\int 1_{C'}\, d1_{[T^n,\infty)}$$

$$= \Sigma E\int 1_{C'}(s) a_n(s) ds$$

$$= \Sigma E\int 1_{C'}(s)\ \Sigma a_n(s)\ ds\ .$$

But $\Sigma a_n = \infty$ a.e., and so $\lambda(C'_\omega) = 0$ a.s. But by definition of T, we have $\lambda(C'_\omega) > 0$ on $\{T < S\}$. This proves 14.

<u>(i) implies (ii)</u> Let (T_n) satisfy (5), (6). Fix $N < \infty$. Consider the previsible set $H = \{(\omega,s): \Sigma a_n \le N-1\}$. We must prove $P \times \lambda(H) = 0$. Suppose not : then for some $\varepsilon > 0$ we have

$$P(\Omega_0) \geq \varepsilon \text{ , where } \Omega_0 = \{\omega : \lambda(H_\omega) > \varepsilon\}$$

Define optional times

$$S_i = \inf\{t : \lambda(H_\omega \cap [0,t]) > i\varepsilon/N\} \qquad i = 0,\dots,N .$$

Consider the previsible sets

$$H^i = H \cap (S_{i-1}, S_i] \qquad i = 1,\dots,N$$

$$\bar{H} = H \cap (S_0, S_n] .$$

By construction, $\lambda(H^i_\omega) = \varepsilon/N$ on Ω_0. So by (i), $B_\omega \cap H^i_\omega$ is a.s. non-empty on Ω_0. So

$$E \sum_n 1_{(T_n \in \bar{H}_\omega)} = E \sum_{in}\sum 1_{(T_n \in H^i_\omega)} \gtrsim N\, P(\Omega_0) \gtrsim N\varepsilon$$

But

$$E \sum_n 1_{(T_n \in \bar{H}_\omega)} = E\sum_n \int 1_{\bar{H}}\, dI_{[T_n,\infty)}$$

$$= E \int 1_{\bar{H}}(s) \,.\, \Sigma a_n(s)\, ds$$

$$\le (N-1)\,\varepsilon$$

because $\Sigma a_n \le N-1$ on H, and $\lambda(\bar{H}_\omega) \le \varepsilon$ by construction. This contradiction establishes the result.

It remains to prove that (b) and (a) are equivalent. Recall from [BJ] that optional times $0<S_1<S_2< \dots$ form a Poisson process of rate 1 with respect to (F_t) iff S_n has conditional intensity $1_{(S_{n-1} < s \le S_n)}$. If moreover this condition holds for each family $(S_i^\theta)_{i\ge 1}$, $\theta\in\Theta$, and if the graphs $\{[S_i^\theta] : i\ge 1, \theta\in\Theta\}$ are disjoint, then the families $\{(S_i^\theta)_{i\ge 1} : \theta\in\Theta\}$ are independent.

The proof that (a) implies (b) is easy. The family (S_i^θ) in (1) plainly satisfy the conditions of (b) with respect to the extension (G_t). Because (b) implies (b'), we deduce that any (G_t)-optional family satisfying (5) will also satisfy (6) and (7) with respect to (G_t). Now, as remarked before, there exists a family satisfying (5) with respect to (F_t); and since conditional intensities are unchanged by extension, this family satisfies (6) and (7) with respect to (F_t).

The proof that (b) implies (a) is harder. There are only two ideas. First, we show how to construct S_1 with $[S_1] \subset B$ such that S_1 has exponential law (Lemma 19). Then we can proceed inductively to construct a uniform Poisson process (S_i^θ). Finally, we must show that $\bigcup_{i,\theta} [S_i^\theta]$ exhausts B.

Here is a strightforward technical lemma.

(14) LEMMA. Let (Q_i) be optional times with conditional intensities a_i. Suppose $Q_i \to \infty$ a.s. and $[Q_i]$ are disjoint. Let $T= \min(Q_i)$. Then

$T_{(T=Q_i)}$ has conditional intensity $a_i 1_{(s\le T)}$

T has conditional intensity $\Sigma a_i\, 1_{(s\le T)}$.

Here is an informal description of the external randomisation. Suppose

(15) T is optional, with conditional intensity a,

$p(\omega,s)$ is previsible, $0\le p\le 1$.

Then we can define Q such that:

$$\text{if } T = t \text{ then } Q = t \text{ with probability } p(\omega,t)$$
$$= \infty \text{ otherwise}$$

It is intuitively obvious that Q has conditional intensity p.a. Here is the formal construction and proof.

(16) LEMMA. Let T,a,p be as in (15), on a filtration $(\hat{\hat{F}}_t)$. Let U be uniform on $[0,1]$, independent of $\hat{F}_\infty$. Define

$$Q = T \text{ if } U \le p(T) \equiv p(\omega,T(\omega))$$
$$= \infty \text{ otherwise.}$$

Let G_t be the usual augmentation of $G_t^0 = \sigma(\hat{F}_t, Q_{(Q\le t)})$. Then (G_t) is an extension of $(\hat{F}_t)$, and Q is (G_t)-optional with conditional intensity p.a.

<u>Proof</u> $Q_{(Q\le t)} \in \sigma(\hat{\hat{F}}_t, U)$, and hence $G_t^0 \subset \sigma(\hat{F}_t, U)$, so (G_t) is indeed an extension of $(\hat{F}_t)$. Plainly Q is (G_t)-optional. To prove the final assertion, let $S<\infty$ be a (G_t)-optional time. It is sufficient to prove

(17) $$P(Q \le S) = E\int_0^S a(s)p(s)ds .$$

We assert

(18) $R = S_{(S<T)}$ is (F_t)-optional.

For $\{R<u\} = \bigcup_{\substack{t<u \\ t \text{ rational}}} \{S<t<T\}$, and $\{S<t<T\}$ is in F_t since $G_t \cap \{T>t\} = F_t \cap \{T>t\}$. To prove (17), note that $\{Q\leq S\} = \{T\leq S, Q<\infty\} = \{T\leq R, Q<\infty\} = \{T\leq R, T<\infty, U\leq p(T)\}$. So

$$
\begin{aligned}
P(Q\leq S) &= P(T\leq R, T<\infty, U\leq p(T)) \\
&= E(1_{(T\leq R, T<\infty)}\, P(U\leq p(T) \mid F_\infty)) \\
&= E\, 1_{(T\leq R, T<\infty)}\, p(T) \quad \text{by the independence of } U \\
&= E \int 1_{(s\leq R)} p(s)\, d1_{[T,\infty)} \\
&= E \int 1_{(s\leq R)} p(s)\, a(s)\, ds.
\end{aligned}
$$

(17) now follows, as $[S,R] \subset [T,\infty)$, and $a = 0$ on this set.

(19) LEMMA. Let $(\hat{F}_t)$ be an extension of (F_t). Suppose (T^n) satisfies condition (b) with respect to $(\hat{F}_t)$. Let $S_0<\infty$ be $(\hat{F}_t)$-optional. Then there exists an extension (G_t) of $(\hat{F}_t)$ and a (G_t)-optional time S with conditional intensity $1_{(S_0<s\leq S)}$ such that $[S] \subset \bigcup [T^n]$.

<u>Proof</u> Define

$$\phi(x) = 1 \quad x \geq 1$$
$$= x \quad 0 \leq x \leq 1$$
$$= 0 \quad x \leq 0$$

Define inductively

$$p_1(\omega,s) = \phi\left(\frac{1}{a_1(\omega,s)}\right) 1_{(s>S_0)}$$

$$p_j = \phi\left(\frac{1 - \sum_1^{j-1} a_i p_i}{a_j}\right) 1_{(s>S_0)}$$

Then p_j is predictable, $0 \leq p_j \leq 1$, and

$$(20) \quad \sum_1^N a_j p_j = \left(1 \wedge \sum_1^N a_j\right) \cdot 1_{(s>S_0)}$$

By Lemma 16 we can construct extensions (G_t^j) of (F_t) and (G_t^j)-optional times Q_j such that

$$[Q_j] \subset [T^j] ,$$

Q_j has conditional intensity $p_j a_j$.

Then

$$\sum_j P(Q_j < t) = \sum_j E\int_0^t p_j(s) a_j(s)\, ds$$
$$= E\int_0^t \Sigma a_j(s) p_j(s)\, ds$$
$$\leq t \qquad \text{by } (20) .$$

By the Borel-Cantelli lemma, $Q_j \to \infty$ a.s.

Set $S = \min (Q_j)$, and let (G_t) be the filtration generated by $(G_t^j, j \geq 1)$. By Lemma 14, S has conditional intensity $\Sigma a_j p_j \; 1_{(s \leq S)}$, and by (20) this equals $1_{(S_0 < s \leq S)}$.

For later use, note that, by Lemma 14, $S_{(S=T^n)}$ has conditional intensity $p_n a_n \; 1_{(s \leq S)}$. In other words, using (20),

(21) $T^n_{(T^n=S)}$ has conditional intensity

$$[(1 \wedge \sum_1^N a_i) - (1 \wedge \sum_1^{N-1} a_i)] \; 1_{(S_0 < s \leq S)} .$$

We can now prove (b) implies (a). Let $(T^{1,n})$ satisfy condition (b). By Lemma 19 we can construct extensions G_t^1, $G_t^2, \ldots$ of F_t and (G_t^i)-optional times S_i^1 such that $[S_i^1] \subset B$ and such that S_i^1 has conditional intensity $1_{(S_{i-1}^1 < s \leq S_i^1)}$. Let F^1 be the filtration generated by $(G^i : i \geq 1)$. Then $(S_i^1)_{i \geq 1}$ is a Poisson process of rate 1 with respect to F^1.

Now let $T^{2,n} = T^{1,n}_{(T^{1,n} \neq S_i^1 \text{ for any } i)}$.

We assert that $(T^{2,n})$ satisfies (b) with respect to (F_t^1), for a certain set B'. We need only check (7). Write $a_{k,n}$ for the conditional intensity of $T^{k,n}$. Write

$$R_{n,i} = T^{1,n}_{(T^{1,n} = S_i^1)}$$

$$R_n = T^{1,n}_{(T^{1,n} = S_i^1 \text{ for some } i)} .$$

Then

$$(22)\quad R_n \text{ has conditional intensity } a_{1,n} - a_{2,n} \geq 0 .$$

But $U[R_n] = \underset{n,i}{U}[R_{n,i}] = \underset{i}{U}[S_i^1]$, so by Lemma 9

$$\sum_n (a_{1,n} - a_{2,n}) = \sum_i 1_{(S_{i-1}^1 < s \leq S_i^1)} = 1 \text{ a.e.}$$

Thus condition (7) extends from $(T^{1,n})$ to $(T^{2,n})$.

Now we may apply Lemma 19 again to construct an extension F^2 and F^2-optional times (S_i^2) with $[S_i^2] \subset \underset{n}{U} [T^{2,n}]$ and such that $(S_i^2)_{i\geq 1}$ is again a Poisson process of rate 1.

Continuing, we obtain a uniform Poisson process $(S_i^k : i,k\geq 1)$ on $\{1,2,\dots\} \times [0,\infty)$. By construction $\underset{i,k}{U}[S_i^k] \subset B$, but we must show there is a.s. equality. Thus we must show that, for each n ,

$$(23)\quad P(T^{k,n} < \infty) = E\int a_{k,n}(s)\, ds \to 0 \quad \text{as } k \to \infty \quad .$$

Define

$$R_n^k = T^{k,n}_{(T^{k,n} = S_i^k \text{ for some } i)}$$

As at (22), R_n^k has conditional intensity $a_{k,n} - {}_n a_{k+1,n}$. But from (21), R_n^k has conditional intensity $(1\wedge \sum_1 a_{k,j}) - (1\wedge \sum_1^{n-1} a_{k,j})$.

So

$$(24)\quad E\int (a_{k,N} - a_{k+1,N})\, ds = E\int (1\wedge \sum_1^N a_{k,j}) - (1\wedge \sum_1^{N-1} a_{k,j})\, ds$$

Now $a_{k,m} \downarrow a_{\infty,n}$, say, as $k \to \infty$. Suppose, inductively, that (23) holds for $n<N$. As $k \to \infty$ the left side of (24) tends to 0, and the right side tends to $E\int(1 \wedge a_{\infty,N})ds$ by the inductive hypothesis. Thus $a_{\infty,N} = 0$ a.e, so (23) holds for N.

Proof of Proposition 8. Put $f(t) = \frac{F'(t)}{1-F(t)}$, where F is the distribution function of μ.

From [BJ], if Y has conditional intensity $f(s)1_{(s\leq Y)}$ then Y has law μ: conversely, if Y has law μ then Y has conditional intensity $f(s)1_{(s\leq Y)}$ with respect to the smallest filtration making Y optional. Thus the random variables (Y_i) in part (a) of Proposition 8 satisfy condition (b) of Theorem 4, so $U[Y_i]$ is indeed a σ-finite Poisson process.

Part (b) is similar to, but simpler than, the proof that (b) implies (a) in Theorem 4. Let B be a σ-finite Poisson process, and let $(T^{1,n})$ satisfy condition (b) of Theorem 4. Lemma 19 showed how to construct an optional time S with conditional intensity $1_{(s\leq S)}$. Essentially the same argument shows we can construct Y_1 with conditional intensity $f(s)1_{(s\leq Y_1)}$, and hence with law μ. Put $T^{2,n} = T^{1,n}_{(T^{1,n}\neq Y_1)}$, and continue. We obtain i.i.d. variables (Y_k), with $U[Y_k] \subset B$: arguing as at (23), we show that there is a.s. equality.

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References

Brémaud, P., Jacod, J. : Processus ponctuels et martingales: résultats récents sur la modelisation et le filtrage. Adv. Appl. Prob. 9, 362-416 (1977)

Dellacherie, C. : Capacités et processus stochastiques. Springer 1972

Kallenberg, O. : Random measures. Academic Press 1976

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