On Brownian Local Time

by M.T. Barlow

Let B be a Brownian motion starting at 0, and L_t^a denote its local time - as usual we take a version of L which is jointly continuous in (a,t). Recently, Perkins has proved that, for fixed t, the process $a \to L_t^a$ is a semimartingale relative to the excursion fields. It is natural to ask about L_T^a, where T is a stopping time: in this note we give an example to show that L_T^a may be very far from being a semimartingale.

Given a stopping time T (which will be defined later) let

$$M = \inf \{ a : L_T^a > 0 \} = \inf_{s \le T} B_s ,$$

$$Y_a = L_T^{M+a} ,$$

$\underline{\underline{Y}}_a$, $a > 0$, be the (usual augmentation of the) natural filtration of Y

We will choose T so that, for some fixed $x > 0$, if $R = \inf\{a: Y_a = x\}$, then the process $(t,\omega) \to Y_{R+t}(\omega)$ is $B([0,\infty)) \otimes \sigma(R)$ measurable with positive probability. Since Y is never of finite variation, it follows that Y is not a semimartingale $/\underline{\underline{Y}}_t$.

Let $\psi : C[0,\infty) \to [0,1]$ be injective and measurable. Set $S = \inf \{ t : |B_t| = 1 \}$, and let ε, x be positive reals. On $\{B_S = 1\}$ let $T = S$, and on $\{B_S = -1\}$ define

$$U = \inf \{ a : L_S^a \geq x \} ,$$

$$V = \psi (L_S^{U+\cdot})$$

$$W = \max \{ a < 1 : a + n\varepsilon = U - \varepsilon V \text{ for some } n \geq 0 \}$$

Thus $-(1 + \varepsilon) \leq W < -1$, and $U - W = \varepsilon(V + n)$ for some $n(\omega) \geq 0$. Now on $\{B_S = -1\}$ let $T = \inf \{ t > S : B_t = U \text{ or } W \}$. Then, if

$$F = \{B_S = -1\} \cap \{B_T = W\} \cap \{ L_T^a < x , \text{ for } a \geq U \} ,$$

it is evident that ε, x may be chosen so that $P(F) > 0$. However, on F $V = [R / \varepsilon]$ ($[x]$ denotes the fractional part of x), and thus if $X_{R+\cdot} = \psi^{-1} ([R/\varepsilon])$, $1_F Y_{R+\cdot} = 1_F X_{R+\cdot}$. Thus T has the required properties, and it is clear, from, for example, the characterization of semimartingales as stochastic integrators, that Y is not a semimartingale.

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