$L(B_t,t)$ is not a semimartingale

by

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Let B be a one-dimensional Brownian motion, with $B_0 = 0$, and let $L(a,t)$, $a \in \mathbb{R}$, $t \geq 0$ be a continuous version of its local time. We shall show that the process Y , defined by $Y_t = L(B_t,t)$, is not a semimartingale. The essence of the proof is the remark that whereas the paths of a continuous semimartingale satisfy a Holder condition of order $\frac{1}{2} - \varepsilon$ almost everywhere, for any $\varepsilon > 0$, the paths of Y just fail to satisfy a Holder condition of order $\frac{1}{4}$.

For a process or function X set

$$D^{\alpha}(X) = \{t \geq 0 : \limsup_{\varepsilon\to 0} \varepsilon^{-1/\alpha}|X_{t+\varepsilon} - X_t| > 0\} \quad .$$

LEMMA <u>Let</u> $\alpha > 1$, <u>and</u> $f : \mathbb{R}_+ \to \mathbb{R}$ <u>be a function such that</u> $D^{\alpha}(f) = \phi$. <u>Let</u> $\tau(t)$ <u>be an increasing function, and</u> $g(t) = f(\tau(t))$. <u>Then</u> $|D^{\alpha}(g)| = 0$.

<u>Proof</u> By Lebesgue's density theorem, $\tau'(t)$ exists and is finite almost everywhere. For such a t

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon^{-1/\alpha} &|g(t+\varepsilon) - g(t)| \\ &= \lim_{\delta\to 0} (\tau'(t))^{1/\alpha} \delta^{-1/\alpha} |f(\tau(t) + \delta) - f(\tau(t))| \\ &= 0 \quad ,\end{aligned}$$

so that $t \notin D^{\alpha}(g)$.

PROPOSITION *Let* X *be a continuous semimartingale. Then for* $\alpha > 2$, $|D^\alpha(X)| = 0$. *a.s.*

Proof Let $X = M + A^+ - A^-$ be the decomposition of X into the sum of a martingale and the difference of two increasing processes. It is plain that $D^\alpha(X) \subset D^\alpha(M) \cup D^\alpha(A^+) \cup D^\alpha(A^-)$. By the lemma, setting $f(t) = t$ and $\tau(t) = A_t^+$ or A_t^-, we have $|D^\alpha(A^+)| = |D^\alpha(A^-)| = 0$.

Now let τ_t be the right-continuous inverse of $\langle M \rangle$, and $U_t = M_{\tau_t}$. Then U is a Brownian motion, and $M_t = U_{\langle M \rangle_t}$. By Lévy's Hölder condition on the variation of Brownian paths, for $\alpha > 2$, $D^\alpha(U) = \phi$ a.s., and thus, by the lemma, $|D^\alpha(M)| = 0$ a.s.

THEOREM (i) *For each* $t > 0$, $B_t \in D^2(L(\cdot,t))$ a.s.

(ii) $D^4(Y)$ *is of full Lebesgue measure* a.s.

(iii) Y *is not a semimartingale*.

Proof From the results of Ray [1] on Brownian local time, $0 \in D^2(L(\cdot,t))$ a.s. Let t be fixed, and $\tilde{B}_s = B_t - B_{t-s}$ for $0 \le s \le t$. Then $\tilde{B}$ is a Brownian motion, and if $\tilde{L}$ denotes its local time, $\tilde{L}(a,t) = L(B_t - a, t)$, so that $B_t \in D^2(L(\cdot,t))$ whenever $0 \in D^2(\tilde{L}(\cdot,t))$, establishing (i).

We may restate (i) as follows: there exist $\mathcal{B}_t$-measurable random variables A_n and C with $|A_n - B_t| < 1/n$, and $C > 0$ a.s., such that

$$|L(A_n,t) - L(B_t,t)| \ge |A_n - B_t|^{\frac{1}{2}}.C \quad \text{for all } n .$$

If (a_n) is a sequence converging to 0, and $T_n = \inf\{t \ge 0: B_t = a_n\}$, then $P(T_n < a_n^2) = k > 0$, for some constant k. Thus $P(T_n < a_n^2$ for infinitely many $n) = 1$ by the Borel-Cantelli lemmas, and the Blumenthal 01 law.

Now let $S_n = \inf\{u > t: B_u = A_n\}$. By the preceding argument, and the Markov property of B at t,

$$S_n - t < (A_n - B_t)^2 \text{ for infinitely many } n \text{ , a.s.}$$

Thus

$$\begin{aligned}
\limsup_{n\to\infty} (S_n - t)^{-\frac{1}{4}} |Y_{S_n} - Y_t| &= \limsup_{n\to\infty} (S_n - t)^{-\frac{1}{4}} |L(A_n, t) - L(B_t, t)| \\
&\geq \limsup_{n\to\infty} (S_n - t)^{-\frac{1}{4}} |A_n - B_t|^{\frac{1}{2}} C \\
&\geq C \qquad \text{a.s.} \\
&> 0 \qquad \text{a.s.}
\end{aligned}$$

Therefore $t \in D^2(Y)$ a.s., and (ii) follows by a Fubini argument. (iii) is an immediate consequence of (ii) and the proposition.

Reference

1. D.B. Ray : Sojourn times of a diffusion process. Illinois J. Math. 7; 615-630. (1963).

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