Levels at which every Brownian excursion is exceptional

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1. Introduction. Let B_t be a Brownian motion starting at 0, defined on a complete filtered probability space (Ω, F, F_t, P) which satisfies the usual conditions. For each $x \in \mathbb{R}$, let $\Lambda^x(B)$ be the set of starting times of excursions of B above the level x, and let $\Lambda(B)$ be the set of starting times of excursions of B above some level: that is

$$\Lambda^x(B.(\omega)) = \Lambda^x(B) = \{t : B_t = x,\ B_s > x \text{ for } t < s < t + \varepsilon \text{ for some } \varepsilon > 0\},$$

$$\Lambda = \Lambda(B) = \bigcup_x \Lambda^x(B) .$$

Let $t \in \Lambda(B(\omega))$, and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous strictly increasing function with $f(0) = 0$. We will say that f is a lower function (respectively, upper function) for B at t if there is a $\delta(\omega) > 0$ such that

$$B_{t+u}(\omega) - B_t(\omega) \geq f(u) \qquad \text{for all } u \in [0,\delta]$$

(respectively,

$$B_{t+u}(\omega) - B_t(\omega) \leq f(u) \qquad \text{for all } u \in [0,\delta]) .$$

If $A \subset \Lambda$, and f is a lower (upper) function for B at t for all $t \in A$, we will say that f is a uniform lower (upper) function for B on A.

For fixed x, it is well known which functions f are upper or lower functions on Λ^x. Let X be a 3-dimensional Bessel process (we will say a Bes (3) process). It follows from the decomposition of Brownian excursions given in [9, p. 74] that, with probability 1, each excursion of B from x begins in the same way that X leaves 0. As there are only countably many excursions of B from x, a function f is, a.s., a uniform lower, or upper, function for B on Λ_x if and only if f is a lower, or upper, function for X at 0. An integral test for this last property is known (see [11, p. 144, 147]). Let $f(t) = t^{1/2}\phi(t)$. Then f is an upper function for X at 0 if $\phi(t) \uparrow \infty$ as $t \downarrow 0$, and $\int_{0+} \phi^3(t)e^{-(1/2)\phi^2(t)}t^{-1}dt < \infty$, and f is a lower function for X at 0 if $\phi(t) \downarrow 0$ as $t \downarrow 0$, and $\int_{0+} \phi(t)t^{-1}dt < \infty$. In particular, for each $\varepsilon > 0$, $\sqrt{2}(1+\varepsilon)t^{1/2}(\log\log 1/t)^{1/2}$ is an upper function, and $t^{1/2}(\log 1/t)^{-(1+\varepsilon)}$ is a lower function.

By Fubini's theorem, if f is a lower function for X at 0, then $\{x : f$ is a uniform lower function on $\Lambda^x\}$ is of full measure. However, there may be times $t \in \Lambda(\omega)$ for which f fails to be a lower function, and therefore levels x such that f fails to be a lower function for one or more excursions above x.

We may consider 4 types of 'bad behaviour':

(i) times at which functions f which are lower functions for X at 0 fail to be lower functions,

(ii) times at which functions f which are upper functions for X at 0 fail to be upper functions,

(iii) times at which functions f which are not lower functions for X at 0 are lower functions,

(iv) times at which functions f which are not upper functions for X at 0 are upper functions.

(i) In Section 2 we show that, given any continuous strictly increasing function f there are, a.s., times $t \in \Lambda(\omega)$ such that f fails to be a lower function at t. Remarkably, even more is true: there are levels x at which f fails to be a lower function for every excursion above that level x. In fact, this is a real-variable result, which is a consequence of Baire's category theorem: the only properties of Brownian motion that are used are that it is continuous and nowhere monotonic.

It is also of interest to consider the size of the sets on which some function fails to be a lower function. For $1/2 \le p < \infty$ let

$$\Lambda_p = \{t \in \Lambda : \text{there exists } \delta_n \downarrow 0 \text{ with } B_{t+\delta_n} - B_t < \delta_n^p \text{ for all } n\} .$$

In Section 3 we show that $\dim \Lambda_p = 1/(4p)$. (Here dim denotes the Hausdorff dimension).

(ii) The Lévy modulus of continuity for Brownian motion provides a uniform upper function for B on Λ, but this is not quite the best possible. Let $\phi(t) = t^{1/2}(\log 1/t)^{1/2}$ (the Lévy modulus is $\sqrt{2}\,\phi$). Then (Theorem 4.3), $(1+\varepsilon)\phi$ is a uniform upper function on Λ, and $(1-\varepsilon)\phi$ fails to be a uniform upper function. We also find the dimension of the set on which $\alpha\phi$, for $0 < \alpha < 1$, fails to be an upper function: we only state the result here (Theorem 4.4), as the proof is very similar to the proofs in Section 3.

(iii) and (iv). We shall not consider these here, as fairly precise results about this type of behaviour have recently been obtained elsewhere (see [2], [3], [15], [17]). The situation concerning (iii) is as follows: w.p. 1 there exist a dense set of times t such that $\liminf_{h\downarrow 0} (B(t+h) - B(t))h^{-1/2} = 1$ (see [3]) but there is no t for which the above lim inf is greater than one ([2]); in fact there is no t for which $B(t+h) - B(t) \geq \sqrt{h}$ for all $h \in [0,\Delta]$ for some $\Delta > 0$ ([3]). Regarding (iv), it is shown in [17] that

$$\inf_{t\in\Lambda} \limsup_{h \to 0+} \frac{(B_{t+h}-B_t)}{\sqrt{h}} = c$$

where $c(> 1)$ is the smallest positive zero of the unique (up to a multiplicative constant) solution of

$$\frac{1}{2}\left(\frac{d^2}{dx^2} - x\frac{d}{dx}\right)\psi(x) = -\psi(x)\ ,\ \psi \in L^2([0,\infty),\ e^{-x^2/2}\,dx)$$

$$\psi(0) = 0\ .$$

2. Lower Functions

The key to the results of this section is a real variable theorem, (Proposition 2.1) which is an easy consequence of Baire's category theorem and may well be known.

If $g : \mathbb{R}_+ \to \mathbb{R}$ is a cadlag function, set $\Delta g(t) = g(t) - g(t-)$.

Proposition 2.1. Let $g_1, g_2, \ldots$ be a sequence of cadlag functions with the property that, for each i, $\{\Delta g_i > 0\}$ is dense in $[0, 1]$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, increasing with $\phi(0) = 0$. Then there is a set $A \subset (0,1)$ with the following properties:

(i) <u>For all</u> $t \in A$ <u>and</u> $i \geq 1$ <u>there is a sequence</u> $t_{i,n} \downarrow t$ <u>with</u> $g_i(t_{i,n}) - g_i(t) > \phi(t_{i,n}-t)$ <u>for all</u> n.

(ii) A <u>is the countable intersection of open dense sets in</u> $(0,1)$.

(iii) A <u>is of second category in</u> $\mathbb{R}$ <u>(and in particular uncountable)</u> <u>and dense in</u> $[0,1]$.

<u>Proof</u>. Let

$$C_{i,n} = \{t \in [0,1]: \text{ for some } h \in (0,n^{-1}) \text{ and some } \varepsilon > 0,$$
$$g_i(t'+h) - g_i(t') > \phi(h) \text{ for all } t' \in (t-\varepsilon, t+\varepsilon)\}.$$

It is clear from the definition that $C_{i,n}$ is open. Let (a,b) be any interval in $[0,1]$: then there exists $s \in (a,b)$ with $\Delta g_i(s) > 0$. Choose $h < n^{-1}$ such that $0 < \phi(h) < \Delta g_i(s)$: then, as g_i has left limits, for some $\varepsilon > 0$ we have $g_i(u+h) - g_i(u) > \phi(h)$ for $s - \varepsilon < u < s$. Thus $C_{i,n} \cap (a,b)$ is non-empty, and so $C_{i,n}$ is dense in $[0,1]$.

Now let $A = \bigcap_{i,n} C_{i,n}$: the set A is the intersection of a countable number of open dense sets, and therefore, by Baire's theorem (see [4, p. 249]), is dense in $[0,1]$, and of the second category. If $t \in A$, the existence of $t_{i,n} \downarrow t$ with the desired properties is immediate from the definition of A. □

We say that a function $g : \mathbb{R}_+ \to \mathbb{R}$ is <u>nowhere monotone</u> if g is not monotone in any interval: that is, given $a < b$ there exist

$a < s_1 < s_2 < s_3 < b$ such that $g(s_2) > g(s_1) \vee g(s_3)$.

Theorem 2.2. Let $b : \mathbb{R}_+ \to \mathbb{R}$ be continuous and nowhere monotone. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be continuous, strictly increasing, and with $f(0) = 0$. There exists an uncountable dense set S , of the second category in $\mathbb{R}$, such that, for all $x \in S$, and $t \in \Lambda^x(b)$, there exist $t_n \downarrow t$ with

$$b(t_n) - b(t) < f(t_n - t) .$$

Proof. For $0 \leq r < s$ let

$$I^{r,s} = (\inf_{r \leq u \leq s} b(u) , b(s)) ,$$

$$g^{r,s}(x) = \sup\{u \in [r,s) : b(u) = x\} \quad \text{for} \quad x \in I^{r,s} .$$

As $b(g^{r,s}(x)) = x < b(s)$ for $x \in I^{r,s}$, the function $g^{r,s}$ is right continuous and increasing on $I^{r,s}$. Further, the set $\{\Delta g^{r,s} > 0\}$ is dense in $I^{r,s}$: if $g^{r,s}$ were continuous on an interval $[u,v]$, this would imply that b was monotone on $[g^{r,s}(u), g^{r,s}(v)]$.

Let ϕ be the inverse function to f ,

$$A^{r,s} = \{x \in I^{r,s} : \text{there exist } x_n \downarrow x \text{ with } g^{r,s}(x_n) - g^{r,s}(x) > \phi(x_n - x) \text{ for each } n\} ,$$

and $C^{r,s} = A^{r,s} \cup (c\ell(I^{r,s}))^c$.

By (ii) of Proposition 2.1 $C^{r,s}$ contains a countable intersection of open sets each dense in $\mathbb{R}$. Therefore the same is true of $S = \bigcap_{\substack{0 \leq r < s \\ r,s \in \mathbb{Q}}} C^{r,s}$.

By Baire's theorem S is an uncountable dense set of the second category in

$\mathbb{R}$. Let $x \in S$ and $t \in \Lambda^x(b)$. If $t' = \inf\{u > t : b(u) = x\}$, then $t' > t$ and there are rationals r, s with $0 \leq r \leq t < s < t'$. Since $x \in C^{r,s} \cap c\ell(I^{r,s})$, there is a sequence $x_n \downarrow x$ such that $g^{r,s}(x_n) - g^{r,s}(x) > \phi(x_n - x)$ for each n . Let $t_n = g^{r,s}(x_n)$: we have $t_n - t > \phi(x_n - x) = \phi(b(t_n) - b(t))$. Thus $b(t_n) - b(t) < f(t_n - t)$ for each n , and S has the required properties. □

As, with probability 1, $B(\omega)$ is nowhere monotone, we deduce immediately

Theorem 2.3. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a continuous strictly increasing function with* $f(0) = 0$. *Then for* a.a. ω *there is an uncountable dense set* $S(\omega) \subseteq \mathbb{R}$ *such that for all* $x \in S(\omega)$, *and* $t \in \Lambda^x(B(\omega))$, *there exists a sequence* $t_n \downarrow t$ with

$$B_{t_n}(\omega) - B_t(\omega) < f(t_n - t) \quad \text{for all } n .$$

We may also look at the starting times of all excursions of B from the level x . Let $\tilde{\Lambda}^x = \{t: B_t = x , B_s \neq x \text{ for } t < s < t + \varepsilon \text{ for some } \varepsilon > 0\}$. Then, by making a few obvious changes in the previous arguments, one obtains

Corollary 2.4. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a continuous strictly increasing function with* $f(0) = 0$. *Then for* a.a. ω *there is an uncountable dense set* $S(\omega) \subseteq \mathbb{R}$ *such that for all* $x \in S(\omega)$, *and* $t \in \tilde{\Lambda}^x(B(\omega))$, *there exists a sequence* $t_n \downarrow t$ *with*

$$|B_{t_n}(\omega) - B_t(\omega)| < f(t_n - t) \quad \text{for all } n .$$

Remark 2.5. It is clear that these last two results also hold for any process with sample paths which are continuous and nowhere monotone. There are continuous Gaussian processes for which the local and global modulus of continuity are identical - see Kahane [10]: nevertheless they still exhibit this kind of sample function irregularity.

Let $f(t) = \exp(-1/t^2)$, and $S(\omega)$ be the set obtained in Corollary 2.4. For $x \in S(\omega)$, we see that every excursion from x begins in an unusually "slow" fashion, and this might suggest that there are asymptotically more excursions of small duration from x than at a typical level.

In fact, this is not the case. If $N_\varepsilon(t,x)$ is the number of excursions from x exceeding ε in length completed by B before time t , then in [13] it is shown that

$$(2.1) \qquad \lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R},\ t \le T} \left| \left(\tfrac{1}{2}\pi\varepsilon\right)^{1/2} N_\varepsilon(t,x) - L_t^x \right| = 0 \quad \text{for all } T \ge 0 \text{ , a.s.,}$$

where L_t^x is the local time of B at x . This extends a well known result of Lévy [12]. Other characterisations of Brownian local time that hold uniformly in x are given in [1] and [14].

In the light of these positive results it is of interest to note that Corollary 2.4 leads to a characterisation of local time that holds a.s. for any fixed level x , and therefore holds a.s. on a set of full Lebesgue measure, but which fails miserably on the uncountable dense set S .

Example 2.6. Recall $f(t) = \exp\{-1/t^2\}$. Let

$$C^x = \{\omega : \mathbb{R}_+ \to \mathbb{R} : \omega \text{ is continuous, } \omega(0) = x \text{ and there exist}$$
$$t_n \downarrow 0 \text{ with } |\omega(t_n) - x| < f(t_n) \text{ for each } n\} ,$$

and let $N'_\varepsilon(t,x)$ be the number of excursions from x in $(C^x)^c$ that are of length greater than ε, and are completed by B by time t. As f is a lower function for the Bes(3) process, $N'_\varepsilon(t,x) = N_\varepsilon(t,x)$ for all $\varepsilon > 0$, $t \geq 0$ a.s., for each fixed x. Hence, by (2.1)

$$\lim_{\varepsilon \downarrow 0} (\tfrac{1}{2}\pi\varepsilon)^{1/2} N'_\varepsilon(t,x) = L^x_t \quad \text{for all } t \geq 0 \text{, a.s. for all } x \in \mathbb{R} .$$

However, by Corollary 2.4, for each $x \in S(\omega)$,

$$\lim_{\varepsilon \downarrow 0} (\tfrac{1}{2}\pi\varepsilon)^{1/2} N'_\varepsilon(t,x) = 0 .$$

Clearly, we could replace f in the definition of C^x above by any lower function for the Bes(3) process at 0.

3. Hausdorff Dimension and Lower Functions

Recall that $\Lambda_p = \{t \in \Lambda :$ there exist $t_n \downarrow t$ with $B_{t_n} - B_t < (t_n - t)^p$ for all $n\}$. In this section we find the Hausdorff dimension of the set Λ_p. Our main result (Theorem 3.6) could be proved using the "first method" of Orey and Taylor [16]. Indeed, our proof of the upper bound for the Hausdorff dimension follows their argument very closely. Their proof for the lower bound is more involved. We present a different argument here.

Lemma 3.1. *Let* $\tau(t)$ *be a stable subordinator of index* α $(0 < \alpha < 1)$. For $\beta \leq \alpha^{-1}$, *let*

$$R_\beta(\omega) = \{t : \limsup_{h \downarrow 0} h^{-\beta}(\tau(t) - \tau(t-h)) = \infty\} .$$

Then $\dim R_\beta \leq \alpha\beta$ a.s.

Proof. Let

$$C_n = \{[k2^{-n}, (k+2)2^{-n}] , k = 0,\ldots,2^{n+1}\}$$

$$S = \{[s,t]: \tau(t) - \tau(s) > (t-s)^\beta\}$$

Now $P(\tau(1) > x) \leq cx^{-\alpha}$, and therefore

$$\text{(3.1)} \qquad P([s,t] \in S) = P(\tau(1) > (t-s)^{\beta-1/\alpha}) \qquad \text{(by scaling)}$$

$$\leq c(t-s)^{1-\alpha\beta} .$$

If $t \in R_\beta(\omega) \cap [0, 1]$, there exist $u_n \uparrow t$, and $k_n \uparrow \infty$, such that $\tau(t) - \tau(u_n) \geq k_n(t-u_n)^\beta$. We may take $k_n \geq 4^\beta$ for all n . Let m_n be such that $2^{-m_n-1} \leq t - u_n < 2^{-m_n}$: then there is an interval $[r_n, s_n]$ in C_{m_n} such that $[u_n, t] \subseteq [r_n, s_n]$. It follows that

$$\begin{aligned}\tau(s_n) - \tau(r_n) &\geq k_n(t-u_n)^\beta \\ &\geq k_n 4^{-\beta}(s_n-r_n)^\beta \\ &\geq (s_n-r_n)^\beta .\end{aligned}$$

Therefore each point in $R_\beta \cap [0, 1]$ is covered infinitely often by intervals in $\bigcup_m C_m \cap S$. Let N_m be the number of intervals in $C_m \cap S$, and $\gamma > \beta\alpha$: then by (3.1),

$$E \sum_m N_m 2^{-m\gamma} \le c \sum_m 2^{m+1} (2^{-m+1})^{1-\alpha\beta} 2^{-m\gamma}$$

$$= c\, 2^{2-\alpha\beta} \sum_m 2^{-m(\gamma-\alpha\beta)} < \infty .$$

Then $\dim \mathcal{R}_\beta \cap [0, 1] \le \gamma$ and the result is now immediate. □

The above result is essentially due to Orey and Taylor [16]. Indeed it follows from the above and (6.3) of [16] that $\dim \mathcal{R}_\beta = \alpha\beta$ a.s.

Proposition 3.2. $\dim \Lambda_p \le 1/4p$ for all $p \ge 1/2$.

Proof. Let $\hat{B}_t^{(r)} = B_r - B_{r-t}$ for $0 \le t \le r$, and $r \ge 0$, let $\hat{M}_t^{(r)} = \sup_{s \le t} \hat{B}_s^{(r)}$, and $S_r = \{t \le r : \hat{B}_t^{(r)} > \hat{B}_s^{(r)}$ for all $s < t\}$.

Then, if $q < p$,

$$\Lambda_p \subseteq \bigcup_{r \in Q_+} \{t : r-t \in S_r , \text{ and } \liminf_{h \downarrow 0} h^{-q}(\hat{M}_{r-t}^{(r)} - \hat{M}_{r-t-h}^{(r)}) = 0\} .$$

It is therefore sufficient to show that, if $S = \{t \ge 0 : B_t > B_s$ for all $s \le t\}$, and $M_t = \sup_{s \le t} B_s$, and

$$A = \{t \in S: \liminf_{h \downarrow 0} h^{-q}(M_t - M_{t-h}) = 0\} ,$$

then $\dim A \le 1/4q$. The image of A under M is the set

$$M(A) = \{y \ge 0 : \limsup_{x \downarrow 0} x^{-1/q}(\tau(y) - \tau(y-x)) = \infty , \ y \text{ is a continuity point of } \tau\} ,$$

where $\tau_x = \inf\{t \ge 0 : M_t = x\}$. Now τ is a stable subordinator of index $\frac{1}{2}$, and so, by Lemma 3.1, $\dim M(A) \le \frac{1}{2q}$ a.s. As $\tau(M_t) = t$ for all $t \in S$, we have $A = \tau(M(A))$. Hawkes and Pruitt, [8] show that, if Y is a stable subordinator of index $\beta < 1$, then, for all Borel sets B simultaneously, $\dim\{Y_t, t \in B\} = \beta \dim B$. Applying this theorem to $M(A)$ and τ , we have $\dim A = \frac{1}{2} \dim M(A) \le \frac{1}{4q}$. Hence $\dim \Lambda_p \le \frac{1}{4q}$, and letting $q \uparrow p$ we deduce the result. □

We now wish to obtain a lower bound on $\dim \Lambda_p(B)$. Let X be a Bes(3) process, $L_x = \sup\{t \ge 0 : X_t = x\}$, and

$$\Gamma_p = \{x : \text{there exists } t_n \downarrow 0 \text{ such that } X_{L_x+t_n} < x + t_n^p \text{ for all } n\} .$$

We begin by obtaining a lower bound on $\dim \Gamma_p$. The idea of the proof is to fix a set A in $[0, 1]$ of small dimension, and to attempt to apply the condensation argument of [16] on the set A . If $A \cap \Gamma_p \ne \emptyset$ for suitable sets A , then $\dim A + \dim \Gamma_p > 1$.

Let $0 < \alpha < 1$, $\delta_n = 2^{-n}$, and let A have the following properties:

(3.2) (a) There exists a sequence of finite sets A_n such that

$$A = c\ell\left(\bigcup_{n=1}^{\infty} A_n\right)$$

(b) $|x-y| \ge \delta_n$ for all $x, y \in A_n$, $x \ne y$.

(c) For all $n \ge 1$, $a \in A_n$ and $\varepsilon > 0$, there exists $M = M(n,a,\varepsilon)$ such that $\#(A_m \cap (a, a+\varepsilon)) \ge \varepsilon\, \delta_m^{-\alpha}$ for all $m \ge M$.

(d) $A_n \subset (0,1)$ for all n .

For $n \ge 1$ let

$$B_n = \{x \in A_n : \text{there exists } 0 < t < L_{x+\delta_n} - L_x < (\log \tfrac{1}{\delta_n})^{-1} \text{ with } X_{L_x+t} - x < t^p\}.$$

Lemma 3.3. *Let* $\alpha > 1 - 1/2p$, *let* $n \geq 1$, $a \in A_n$ *and* $\varepsilon > 0$. *Then* $P(B_m \cap (a, a+\varepsilon) \neq \emptyset$ infinitely many $m \geq n) = 1$.

Proof. Let $m \geq n$, $y \in A_m$, and $V_t = X_{L_y+t} - y$. If $y \in B_m$ then V_t falls below t^p before $L_{\delta_m}(V) = \sup\{t : V_t = \delta_m\}$. Now V is itself a Bes (3) process and if $\gamma > 1$ then for $u \geq 1$, $P(u^\gamma \geq L_1(V) > u) \geq c_\gamma u^{-1/2}$ (see [18] and [19]). Therefore

$$\begin{aligned} P(y \in B_m) &\geq P(V_{L_{\delta_m}(V)} < L_{\delta_m}(V)^p < (\log \tfrac{1}{\delta_m})^{-p}) \\ &= P(\delta_m^{1/p} < L_{\delta_m}(V) < (\log \tfrac{1}{\delta_m})^{-1}) \\ &= P(\delta_m^{-2+1/p} < L_1(V) < \delta_m^{-2} (\log \tfrac{1}{\delta_m})^{-1}) \\ &\geq c_\gamma \, \delta_m^{1-1/2p} , \end{aligned}$$

where $\gamma \in (1, 2(2-p^{-1})^{-1})$. Now let $A_m \cap (a, a+\varepsilon) = \{y_1, \ldots, y_{r_m}\}$: using the last-exit decomposition of X we have that the events $\{y_i \in B_m\}$ are independent. Therefore, if m is chosen large enough so that, by property (c) of A , $r_m \geq \varepsilon \, \delta_m^{-\alpha}$, we have

$$\begin{aligned} P(B_m \cap (a, a+\varepsilon) = \emptyset) &= \prod_{i=1}^{r_m} P(y_i \notin B_m) \\ &\leq (1 - c\, \delta_m^{1-1/2p})^{\varepsilon \, \delta_m^{-\alpha}} . \end{aligned}$$

Since $\delta_m^{1-1/2p-\alpha} \to \infty$ as $m \to \infty$, $P(B_m \cap (a, a+\varepsilon) = \emptyset) \to 0$, and therefore $B_m \cap (a, a+\varepsilon) \neq \emptyset$ for infinitely many m , a.s.

Proposition 3.4. *If* A *satisfies* (a), (b), (c), (d) *and* $\alpha > 1 - 1/2p$, *then* $A \cap \Gamma_p \neq \emptyset$ a.s.

Proof. Let F_n be the P-null set on which, for some $a \in A_n$, the conclusion of Lemma 3.3 fails to hold, and let $F = \cup_n F_n$. Set

$$C_n = \{x \in (0,1) : \text{for some } \varepsilon > 0 \text{ , and } t \in (0,\delta_n) \text{ ,}$$
$$X_{L_y+t} - y < t^p \text{ for all } x - \varepsilon < y < x + \varepsilon\} \text{ .}$$

C_n is clearly open in $[0,1]$: we now show that, for $\omega \in F^c$, $C_n(\omega)$ is dense in A . Let $(a,b) \subset [0,1]$, with $A \cap (a,b) \neq \emptyset$. Then for some n there exists $x \in A_n \cap (a,b)$, and so, by Lemma 3.3, for infinitely many $m \geq n$ there exists $y \in B_m \cap (a,b)$. For large enough m and for some $\varepsilon > 0$, $(y, y+\varepsilon) \subset C_n$, while by property (c) of A_m , $A \cap (y, y+\varepsilon) \neq \emptyset$.

We may now apply Baire's theorem on the closed set A to deduce that $\cap_n C_n \cap A \neq \emptyset$, and since $\cap_n C_n \supseteq \Gamma_p$, this completes the proof.

Now let (Ω',F',P') be another probability space carrying a stable subordinator Y_t of index $\beta > \alpha$, with Y_0 uniformly distributed on $(1/3, 2/3)$. For $n \geq 1$ let

$$T_{n0} = 0 \text{ ,}$$
$$T_{nj} = \inf\{t > T_{n,j-1} : Y_t > Y_{T_{n,j-1}} + \delta_n\} \text{ ,}$$
$$A_n = \{Y_{T_{n,j}} , j \geq 0\} \cap (0,1)$$
$$A = c\ell\left(\bigcup_{n=1}^{\infty} A_n\right) \text{ .}$$

Thus A is the closed range of the subordinator Y in $(0,1)$. We must verify that A satisfies (3.2a-d) : the only one of these which is not obvious is (c). So let $\varepsilon > 0$ and n be fixed, $j \geq 1$, $J_m = A_m \cap (Y_{T_{n,j}}, Y_{T_{n,j}} + \varepsilon)$, and $N_m = \#(J_m)$. Then since $A \cap (Y_{T_{n,j}}, Y_{T_{n,j}} + \varepsilon) \subseteq \bigcup_{y \in J_m} [y, y+\delta_m]$, we have, writing m^γ for the γ-Hausdorff measure function,

$$m^\gamma(A \cap (Y_{T_{nj}}, Y_{T_{nj}} + \varepsilon)) \leq \liminf_{m \to \infty} N_m \delta_m^\gamma .$$

Now if $\gamma < \beta$, the m^γ measure of the range of Y is ∞ a.s. : hence, as the range is the countable union of sets of the form $A \cap (Y_{T_{nj}}, Y_{T_{nj}} + \varepsilon)$, we deduce that $m^\gamma(A \cap (Y_{T_{nj}}, Y_{T_{nj}} + \varepsilon)) = \infty$. Thus $N_m \delta_m^\gamma \to \infty$ as $m \to \infty$, and so $N_m > \delta_m^{-\gamma}$ for all sufficiently large m : choosing $\alpha < \gamma < \beta$ we see that (c) is satisfied.

Proposition 3.5. $\dim \Gamma_p \geq 1/2p$ for $p > 1/2$, a.s.

Proof. A Borel set B is polar for a stable subordinator of index β if $\dim B < 1 - \beta$. Since $\Gamma_p(\omega) \cap A(\omega') \neq \emptyset$ $P \times P'$ a.s. whenever $\beta > \alpha > 1 - 1/2p$, it follows that $\dim \Gamma_p \geq 1 - \beta$ P-a.s. As α and β may be made as close as we wish to $1 - 1/2p$, we conclude that $\dim \Gamma_p \geq 1/2p$ P-a.s.

Theorem 3.6. $\dim \Lambda_p(B) = 1/4p$ for $p \geq 1/2$.

Proof. The result for $p = 1/2$ follows immediately from Proposition 3.2 and the result for $p > 1/2$. Now $\{L_x, x \in \Gamma_p\} \subseteq \Lambda_p(X)$, and therefore, by the theorem of Hawkes and Pruitt, and the fact that L is a stable subordinator of index $1/2$, $\dim \Lambda_p(X) \geq \dim\{L_x, x\in\Gamma_p\} = \frac{1}{2}\dim \Gamma_p \geq 1/4p$.

Finally, as Brownian motion and the 3-dimensional Bessel process are absolutely continuous on $[n^{-1}, n]$, for any $n \geq 1$, we deduce that $\dim \Lambda_p(B) \geq 1/4p$ a.s., and hence, using Proposition 3.2, that $\dim \Lambda_p(B) = 1/4p$ a.s.

4. Upper Functions

Let $\phi(s) = s^{1/2}(\log 1/s)^{1/2}$. If $\Delta > 0$, define

$$\hat{\Lambda}_\Delta = \hat{\Lambda}_\Delta(\omega) = \{t \in \Lambda \mid B_{t+h} > B_t \text{ for all } u \in (0,\Delta)\} .$$

Theorem 4.1. (a) *For a.a.* ω *and each* $\Delta > 0$,

$$\limsup_{h \downarrow 0} \sup_{t\in\hat{\Lambda}_\Delta,\ t\leq\Delta^{-1}} (B_{t+h} - B_t)\phi(h)^{-1} \leq 1 .$$

In particular,

$$\sup_{t\in\Lambda, t\leq 1} \limsup_{h \downarrow 0} (B_{t+h} - B_t)\phi(h)^{-1} \leq 1 \quad \text{a.s.}$$

(b) *For a.a.* ω *there is an uncountable dense set of times* t *in* Λ *satisfying*

$$\limsup_{h \downarrow 0} (B_{t+h} - B_t)\phi(h)^{-1} = 1 . \tag{4.1}$$

Remarks. 1. In light of Lévy's modulus of continuity for B and the fact that Λ is dense, it is clear that (a) cannot be strengthened to

$$\limsup_{h \downarrow 0} \sup_{t\in\Lambda,\ t\leq 1} (B_{t+h} - B_t)\phi(h)^{-1} = 1 .$$

Indeed, the above lim sup equals $\sqrt{2}$.

2. In [15] it is shown that for certain optional sets A

$$(4.2)\qquad A(\omega)\cap\Lambda(\omega)\begin{cases} = \emptyset & \text{if } \dim A(\omega) < 1/2 \\ \neq \emptyset & \text{if } \dim A(\omega) > 1/2 \end{cases}$$

(dim denotes Hausdorff dimension as usual). If

$$A_\alpha(\omega) = \{t \mid \limsup_{h \downarrow 0} (B_{t+h} - B_t)(\phi(h))^{-1} \geq \alpha\} ,$$

then (see [16]) $\dim(A_\alpha) = 1 - \alpha^2/2$ a.s. and so (4.2) would imply that

$$A_\alpha(\omega)\cap\Lambda(\omega)\begin{cases} = \emptyset & \text{if } \alpha > 1 \\ \neq \emptyset & \text{if } \alpha < 1 \end{cases}$$

This would almost prove Theorem 4.1. However, by the section theorem, A_α is not optional. Further, there is an obvious dependence between the random sets A_α and Λ , and as a result a different proof is required.

Let $M_t = \sup_{s\leq t} B_s$ and set

$$U = U(\omega) = \{t \mid M_t = B_t\} .$$

A time reversal argument shows there is a close connection between Λ and U . We prove Theorem 4.1 by first establishing a similar result for U (Theorem 4.3 below). To handle the aforementioned dependency problem we need a lemma.

<u>Lemma 4.2</u>. <u>If</u> $\varepsilon, \alpha > 0$ <u>and</u> $0 < s < t < s + e^{-1}$, <u>then</u>

$$P(|B_t - B_s| > \alpha\, \phi(t-s) ,\ [t, t+\varepsilon] \cap U \neq \emptyset)$$
$$\leq 4\, s^{-1/2}(t-s)^{(1+\alpha^2)/2} + (t-s)^{\alpha^2/2} \exp\{-\alpha^2(2\varepsilon)^{-1}\, \phi^2(t-s)\} .$$

$$2s^{-1/2}(t-s)^{(1+\alpha^2)/2} + \sigma \exp\{-x_0^{\,2}/2\sigma^2\}$$

(4.4) $$\le 2s^{-1/2}(t-s)^{(1+\alpha^2)/2} + (2(t-s)/t)^{1/2} \exp\{\frac{-\alpha^2(2t-s)\ \phi^2(t-s)}{s(t-s)}\}$$

$$\le 4s^{-1/2}(t-s)^{(1+\alpha^2)/2} .$$

(4.3) and (4.4) give the required result. □

For the next theorem, recall that by [5, Cor. to Thm. 2] (and the equivalence in law between M_t and the local time of B at zero), we have

(4.5) $$\limsup_{h \to 0+} \sup_{t \in U \cap [0,\varepsilon]} (M_{t+h} - M_t)\ \phi(h)^{-1} = 1 \quad \text{for all } \varepsilon > 0 \text{ a.s.}$$

Indeed, (b) of the following theorem is proved by applying the condensation argument of [16] to (4.5).

Theorem 4.3. (a) $$\limsup_{h \to 0\pm} \sup_{t \in [0,T] \cap U} |B_{t+h} - B_t|\ \phi(|h|)^{-1} = 1 \quad \text{for all } T > 0$$

(b) For a.a. ω there is an uncountable dense (in U) set of times t in U satisfying

(4.6)$_\pm$ $$\limsup_{h \to 0\pm} |M_{t+h} - M_t|\ \phi(|h|)^{-1} = 1 ,$$

and therefore for all such times t we have

(4.7) $$\limsup_{h \to 0+} (B_{t+h} - B_t)\ \phi(h)^{-1} = 1 ,$$

and

(4.8) $$\limsup_{h \to 0+} (B_t - B_{t-h})\ \phi(h)^{-1} = 1 .$$

Proof. Consider first the lim sup from the left in (a). We may assume

$T = 1$. We follow Lévy's classical derivation of the modulus of continuity of B. Fix $\varepsilon \in (0,1)$ and let $\delta \in (0, \varepsilon(2+\varepsilon)^{-1})$. Define

$$A_n = \{\omega : \text{there exists } j_1, j_2 \in \mathbb{N} \cap [0,2^n] \text{ such that } j_2-j_1 \in [2^{n\delta-1}, 2^{n\delta}],$$
$$[j_2 2^{-n}, (j_2+1)2^{-n}] \cap U \neq \emptyset \text{ and } |B(j_2 2^{-n}) - B(j_1 2^{-n})|$$
$$> (1+\varepsilon)\, \phi((j_2-j_1)2^{-n})\} .$$

Lemma 4.2 implies that

$$P(A_n) \leq 2^{n\delta} \exp\{-\frac{(1+\varepsilon)^2}{2} \log (2^{n(1-\delta)})\}$$
$$+ \sum_{j_1=1}^{2^n} 2^{n\delta} [2^{2+n/2} j_1^{-1/2} \; 2^{n(\delta-1)(1+(1+\varepsilon)^2)/2} + \exp\{-(1+\varepsilon)^2 \; 2^{n-1} 2^{n\delta-1} 2^{-n}\}]$$

$$\leq 2^{n(2\delta-1)} + [2^{n((2+\varepsilon)\delta-\varepsilon)+1} + 2^{n(\delta+1)} \exp\{-(1+\varepsilon^2)\; 2^{n\delta-2}\}] .$$

As the above bound is summable by the choice of δ, we have $P(A_n \text{ i.o.}) = 0$. Fix ω outside a null set so that there is an $N(\omega) \in \mathbb{N}$ such that whenever $n \geq N$ the following conditions hold:

(i) $\omega \notin A_n$

(ii) $s,t \in [0, 1], \; |t - s| \leq 2^{-n} \Rightarrow |B_t - B_s| \leq (1+\varepsilon) \sqrt{2}\, \phi(2^{-n})$

$$< (\varepsilon/2)\, \phi(2^{-(n+1)(1-\delta)})$$

(iii) $2^{-1+\delta} - 2^{1-n\delta} > 1/2$.

Fix $0 \leq s \leq t \leq 1$ such that $t \in U$ and $t-s \leq 2^{-N(1-\delta)}$. Choose non-negative integers $n \geq N$ and $j_1 < j_2 \leq 2^n$ such that

$$2^{-(n+1)(1-\delta)} < t-s \leq 2^{-n(1-\delta)} ,$$

and

$$(j_1-1)2^{-n} < s \le j_1\, 2^{-n} < j_2\, 2^{-n} \le t < (j_2+1)2^{-n} .$$

Then $[j_2\, 2^{-n}, (j_2+1)\, 2^{-n}] \cap U \neq \emptyset$ and

$$2^{n\delta} \ge j_2 - j_1 \ge 2^n(t-s) - 2 \ge 2^{n\delta}(2^{-1+\delta} - 2^{1-n\delta})$$

$$\ge 2^{n\delta-1} \quad \text{(by (iii))} .$$

As $\omega \notin A_n$ we therefore have

$$|B_t - B_s| \le |B_t - B_{j_2 2^{-n}}| + |B_{j_2 2^{-n}} - B_{j_1 2^{-n}}| + |B_{j_1 2^{-n}} - B_s| \le (1+2\varepsilon)\ \phi(t-s)$$

(by (i) and (ii)) . Letting $\varepsilon \downarrow 0$, we get

$$(4.9)_- \qquad \limsup_{h\to 0-}\ \sup_{t\in[0,1]\cap U} (B_t - B_{t+h})\ \phi(|h|)^{-1} \le 1 \text{ a.s.}$$

The analogous result $(4.9)_+$ for the lim sup from the right is similar and simpler, as the required version of Lemma 4.2 is now trivial by independence.

To finish the proof of (a), it suffices to prove (4.6), because (4.7) and (4.8) are then easy consequences of this and (4.9). If $N \in \mathbb{N}$, let

$$D_N^{\pm} = \{t \in U : |M_{t+h} - M_t| > (1-N^{-1})\phi(|h|) \text{ for some } h \text{ between } 0 \text{ and } \pm N^{-1}\} .$$

D_N^+ and D_N^- are open sets in U by sample path continuity. We claim D_N^+ and D_N^- are both a.s. dense in U for each N . By a time reversal argument it suffices to show this for D_N^+ . Indeed, if $\tau = \inf\{t | M_t = 1\}$, then the process $(M_{(\tau-t)+} - B_{(\tau-t)+} , 1 - M_{(\tau-t)+})$ is equal in law to

$(M_{t\wedge\tau} - B_{t\wedge\tau}, M_{t\wedge\tau})$ (see [20]). If $r \geq 0$ let $T_r = \inf\{t \geq r | t \in U\}$ and

$$M_t^{(r)} = M_{T_r+t} - M_{T_r} = \sup_{0 \leq s \leq t} B_{s+T_r} - B_{T_r} \; ; \; U^{(r)} = \{t | M_t^{(r)} = B_{t+T_r} - B_{T_r}\}$$

$$= \{t | t+T_r \in U\} .$$

Fix ω outside a null set so that (4.5) holds with $M^{(r)}$ in place of M for all non-negative rationals r. If $t \in U$, choose non-negative rationals $r_n \uparrow t$ $(r_n \leq t)$. Then by the choice of ω there are $t_n \in (0, t + \frac{1}{n} - T_{r_n}) \cap U^{(r_n)}$ and a sequence $\{h_n\}$ decreasing to zero such that

$$M^{(r_n)}_{t_n+h_n} - M^{(r_n)}_{t_n} = M_{T_{r_n}+t_n+h_n} - M_{T_{r_n}+t_n} > (1-n^{-1})\phi(h_n) .$$

For large n, $T_{r_n} + t_n \in D_N^+$ (this holds if $h_n < \frac{1}{N}$ and $n \geq N$) and as $T_{r_n} + t_n$ converges to t, the claim is proved.

As U is locally compact, it is a Baire space ([4, p. 249]) and so $\bigcap_{N=1}^{\infty} D_N^+ \cap D_N^-$ is a dense set in U which must also be uncountable as U is perfect. If t is in this set then clearly

$$\limsup_{h \to 0\pm} |M_{t+h} - M_t| \; \phi(|h|)^{-1} \geq 1 .$$

By (4.9) we must have equality in the above and the proof is complete. □

<u>Proof of Theorem 4.1</u>. (a) Let $B_t^{(r)} = B_r - B_{(r-t)^+}$ and

$$U^{(r)} = \{t \le r \mid B_t^{(r)} = \sup_{s\le t} B_s^{(r)}\} \qquad (r>0) .$$

Fix ω (outside a null set) such that Theorem 4.3(a) holds when (B,U,T) is replaced by $(B^{(r)}, U^{(r)}, r)$ for any positive rational r. Choose $\Delta = M^{-1}$ for some $M \in \mathbb{N}$. If $t \in \hat{\Lambda}_{M^{-1}} \cap [0,M]$, there is an

$i \le 2M^2 + 1$ such that $r_i - t \in U^{(r_i)} \cap [(2M)^{-1}, r_i]$ where $r_i = (i\Delta)/2$. Therefore if $h \in (0, (2M)^{-1})$

$$\sup_{t\in\hat{\Lambda}_{M^{-1}}\cap[0,M]} (B_{t+h}-B_t)\phi(h)^{-1} \le \max_{i\le 2M^2+1} \sup_{u\in U} (B_u^{(r_i)} - B_{u-h}^{(r_i)})\phi(h)^{-1}$$

The lim sup as $h \to 0+$ of the expression on the right is one.

(b) Let $M_t^{(1)} = \sup_{s\le t} B_s^{(1)}$ where $B^{(1)}$ is defined as above.

By Theorem 4.3 (b) for a.a. ω there is an uncountable set of times t in $U^{(1)}$ satisfying

$$(4.10) \qquad \limsup_{h \to 0-} (M_t^{(1)} - M_{t+h}^{(1)})\phi(|h|)^{-1} = 1 .$$

Clearly for any such $t < 1$ we have $1 - t \in \Lambda$, in fact $B_s > B_{1-t}$ for all $s \in (1-t, 1]$. Moreover, using (4.10) we have

$$\limsup_{h \to 0+} (B_{1-t+h} - B_{1-t})\phi(h)^{-1} = \limsup_{h \to 0+} \frac{B_t^{(1)} - B_{t-h}^{(1)}}{\phi(h)} \ge 1 .$$

It follows that for a.a. ω there is an uncountable collection of times t satisfying (4.1) in each interval with rational end points. This completes the proof. □

By taking into account the two-sided nature of (4.6) in the above proof of (b), one could prove the existence of an uncountable dense set of times, t , in Λ satisfying (4.1) and

$$\liminf_{h \to 0+} (B_t - B_{t-h})\phi(h)^{-1} = 1 .$$

Recall that

$$A_\alpha = \{t \mid \limsup_{h \to 0+} (B_{t+h} - B_t)\, \phi(h)^{-1} \geq \alpha\} .$$

In view of Theorem 3.6 it is natural to ask for the Hausdorff dimension of $\Lambda \cap A_\alpha$. Recall ([16]) that $\dim(A_\alpha) = 1 - \alpha^2/2$.

<u>Theorem 4.4</u>. $\dim(\Lambda \cap A_\alpha) = (1-\alpha^2)/2$ a.s. □

We omit the proof as it is similar to that of Theorem 3.6. For the upper bound on $\dim(\Lambda \cap A_\alpha)$, one can first show that

$$\dim(U \cap A_\alpha^-) \leq (1-\alpha^2)/2 , \tag{4.11}$$

where

$$A_\alpha^- = \{t \mid \limsup_{h \to 0+} (B_t - B_{t-h})\phi(h)^{-1} \geq \alpha\}$$

The required result now follows easily by time reversal as in the proof of Theorem 4.1. The proof of (4.11) is similar to the proof of Lemma 3.1 (and even closer to the argument in [16, p. 80]). The role of (3.1) is now played by Lemma 4.2.

The lower bound on $\dim \Lambda \cap A_\alpha$ can be obtained by using the techniques of Proposition 3.5.

5. <u>Concluding Remarks</u>

If A_1 and A_2 are subsets of $\mathbb{R}$, one frequently finds that

(5.1) $$\dim(A_1 \cap A_2) = \dim A_1 + \dim A_2 - 1 .$$

Here a negative dimension indicates the set is empty and so

(5.2) $$A_1 \cap A_2 \begin{cases} \neq \emptyset & \text{if } \dim A_1 + \dim A_2 > 1 \\ = \emptyset & \text{if } \dim A_1 + \dim A_2 < 1 \end{cases}$$

In particular these relations hold if A_1 and A_2 are the ranges of independent stable subordinators (see [6]). In general, (5.1) and (5.2) seem to hold if A_1 and A_2 are in some sense "independent". This idea is pursued in [7]. However, the results of sections 2 and 3 indicate how one can construct a sequence of independent random sets $\{A_i\}$, each of Hausdorff dimension zero, whose intersection is an uncountable dense set. By "independent" we mean each A_i is defined in terms of X_i, where $\{X_i\}$ is a collection of independent processes. For example, if $\{\tau^i\}$ is a collection of independent stable subordinators of index $\alpha \in (0,1)$, and

$$A_i = \{t \mid \limsup_{h \downarrow 0} (\tau_i(t) - \tau_i(t-h)) \log(h^{-1}) = \infty\} ,$$

then by Lemma 3.1 $\dim(A_i(\omega)) = 0$ for all i a.s. Nonetheless, Proposition 2.1 shows that $\bigcap_i A_i(\omega)$ is a.s. an uncountable dense set in $[0, \infty)$.

As a second example, consider a sequence of independent Brownian motions $\{B_i\}$ and define

$$C_i = \{t \mid \limsup_{h \downarrow 0} |B_i(t+h) - B_i(t)| \phi(h)^{-1} = \sqrt{2}\} ,$$

where $\phi(h) = (h \log 1/h)^{1/2}$. Then ([16]) $\dim C_i(\omega) = 0$ a.s. but

$\bigcap_i C_i(\omega)$ is uncountable and dense. To see this latter result, note that $C_i(\omega)$ is the countable intersection of open dense sets and apply Baire's theorem. This kind of behaviour does not occur for the sets of slow points

$$S^i(c) = \{t \mid \limsup_{h \downarrow 0} | B_i(t+h) - B_i(t)|h^{-1/2} \leq c\} .$$

Indeed, it is shown in [2] that if $c_i > 1$ (so that $\dim S(c_i) > 0$ a.s.), then

$$\dim(\bigcap_{i=1}^n S^i(c_i)) = \sum_{i=1}^n \dim(S^i(c_i)) - (n-1) \quad \text{a.s.}$$

In particular $\bigcap_{i=1}^n S^i(c) = \emptyset$ a.s. if $n > (1 - \dim S(c))^{-1}$,

and the natural extensions of (5.1) and (5.2) to n sets are valid for $\{S^1(c_1),\ldots,S^n(c_n)\}$.

Finally, using arguments similar to those in section 2, it is not hard to show that w.p. 1 there is an uncountable dense set, S , of the second category, in $\mathbb{R}$ such that for all x in S , all $i \in \mathbb{N}$ and all t in $\Lambda^x(B_i)$ we have

$$\limsup_{h \downarrow 0} (B_i(t+h) - B_i(t))\phi(h)^{-1} = 1$$

and

$$\liminf_{h \downarrow 0} (B_i(t+h) - B_i(t)) e^{1/h^2} = 0 .$$

In particular, there are times t in Λ which exhibit both the types of bad behaviour, (i) and (ii), discussed in the introduction.

References

1. R.V. Chacon, Y. Le Jan, E. Perkins, S.J. Taylor. Generalized arc length for Brownian motion and Lévy processes. Z.f.W. 57, 197-211 (1981).

2. B. Davis. On Brownian slow points. Z.f.W. 64, 359-367 (1983).

3. B. Davis, E. Perkins. Brownian slow points: the critical cases (preprint).

4. J. Dugundji. Topology. Boston, Allyn and Bacon, Inc., 1966.

5. J. Hawkes. A lower Lipschitz condition for the stable subordinator. Z.f.W. 17, 23-32 (1971).

6. J. Hawkes. On the Hausdorff dimension of the intersection of the range of a stable process with a Borel set. Z.f.W. 19, 90-102 (1971).

7. J. Hawkes. Hausdorff measure, entropy, and the independence of small sets. Proc. London Math. Soc. (3) 28, 700-724 (1974).

8. J. Hawkes, W.E. Pruitt. Uniform Dimension Results for Processes with Independent Increments. Z.f.W. 28, 277-288 (1974).

9. K. Itô, H.P. McKean. Diffusion Processes and Their Sample Paths. Berlin-Heidelberg-New York, Springer, 1974.

10. J.-P. Kahane. Slow points of Gaussian processes. Conference on Harmonic Analysis in Honor of Antoni Zygmund, I, 67-83, Wadsworth, 1981.

11. F. B. Knight. Essentials of Brownian Motion and Diffusion. Amer. Math. Soc. Surveys 18, 1981.

12. P. Lévy. Processus Stochastiques et Mouvement Brownien. Paris, Gauthier-Villars, 1948.

13. E. Perkins. A global intrinsic characterization of Brownian local time. Ann. Probability 9, 800-817 (1981).

14. E. Perkins. The exact Hausdorff measure of the level sets of Brownian motion. Z.f.W. 58, 373-388 (1981).

15. E. Perkins. On the Hausdorff dimension of the Brownian slow points. Z.f.W. 64, 369-399 (1983).

16. S. Orey, S.J. Taylor. How often on a Brownian path does the law of the iterated logarithm fail? Proc. London Math. Soc. (3) 28, 174-192 (1974).

17. P. Greenwood, E. Perkins. A conditioned limit theorem for random walk and Brownian local time on square root boundaries, Ann. Probability 11, 227-261 (1983).

18. R.K. Getoor. The Brownian escape process. Ann. Probability 7, 864-867 (1974).

19. D. Williams. Path decompositions and continuity of local time for one-dimensional diffusions I. Proc. London Math. Soc. 28, 738-768 (1974).

20. M. Emery, E. Perkins. On the filtration of $B + L$. Z.f.W. 59, 383-390 (1982).

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