(4.4) REMARK. In the preceding theorem, if $\lambda = 0$ then π is excessive relative to (P_t) and the resulting process P is stationary in time. In this case we say that (P_t) and (Q_t) are in weak duality relative to π. For arbitrary λ we call the P of (4.3) a *quasi-stationary Markov process*.

References

1. B.W. ATKINSON. "Two-sided Markov chains." Preprint (1984).

2. K.L. CHUNG and J.B. WALSH. "To reverse a Markov process." *Acta. Math.* 123, 225-251 (1969).

3. S.E. KUZNETSOV. "Construction of Markov processes with random times of birth and death." *Th. Prob. Appl. 18*, 571-575 (1973).

4. J.B. MITRO. "Dual Markov processes: Construction of a useful auxiliary process." *Z. Wahr. verw. Geb. 47*, 139-156 (1979).

5. D.W. STROOCK. "On the spectrum of Markov semi-groups and the existence of invariant measures." *Functional Analysis in Markov Processes*, Lecture notes in Math. 923, pp. 286-307. Springer-Verlag, New York, 1982.

B.W. ATKINSON
Department of Mathematics
University of Florida
Gainesville, Florida 32611

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THE BEHAVIOUR AND CONSTRUCTION OF LOCAL TIMES FOR LÉVY PROCESSES

by

MARTIN T. BARLOW[1], EDWIN A. PERKINS, S. JAMES TAYLOR[1]

1. Introduction

The local time of a Lévy process X_t at a point a, denoted $\ell_t^a(X)$, is a continuous increasing additive functional, which increases only on $\{t: X_t = a\}$. If X is such that ℓ_t^0 exists, then as the transition probabilities of X are stationary in space, ℓ_t^x will exist for every $x \in \mathbb{R}$, and we may therefore ask about the properties of the map $(x,t,\omega) \to \ell_t^x(\omega)$.

In this paper we give a survey of what is known about this problem, and include some new results of the authors. After establishing our notation in section 2, we review in section 3 known conditions for the existence of a jointly continuous local time, and the properties of ℓ_t^x when a continuous version does not exist. We present a conjecture of J. Hawkes, which gives necessary and sufficient conditions for the existence of a continuous version of $(x,t) \to \ell_t^x$, and formulate some other problems concerning its behaviour.

In section 4 we look at the case when the range of X is nowhere dense: this forces ℓ_t^x to have a very erratic behaviour, and in par-

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ticular we show that, if ℓ_t^x is unbounded, then ℓ_t^x, $x \in \mathbb{R}$, is dense in $[0,\infty)$.

In the final two sections we consider the problem of constructing ℓ_t^x as the limit of a sequence $K_n(x,t)$ of functionals of the path of X_t. If $\lim_{n\to\infty} K_n(0,t) = \ell_t^0$ a.s., then, using Fubini, we have immediately that $K_n(x,t)$ converges to ℓ_t^x on a set of full Lebesgue measure; but to go further than this requires new techniques.

In section 5 we give some examples of constructions which, while converging almost everywhere, fail to converge to ℓ_t^x at some levels: some of these counterexamples are valid for Brownian motion.

Finally, in section 6, we state three positive results on the uniform convergence in x of specific constructions $K_n(x,t)$ to ℓ_t^x. One of these is proved here; the proofs of the others are rather complicated, and will be given in a subsequent paper [4].

2. Preliminaries

We use the framework established by Getoor and Kesten [13] which combines the definition of local time at a fixed level as a continuous additive functional with its definition as an occupation density in state space. In this paper a Lévy process X_t will be a standard Markov process on the line with stationary independent increments whose characteristic function takes the form

$$E\, e^{i\lambda X_t} = E\, e^{i\lambda(X_{t+s}-X_s)} = e^{-t\psi(\lambda)}$$

with

$$\psi(\lambda) = -ia\lambda + \tfrac{1}{2}\sigma^2\lambda^2 - \int_{\mathbb{R}\setminus\{0\}} \left[e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2}\right]\nu(dy). \tag{1}$$

As shown by Kesten [17] and Bretagnolle [7], if either $\sigma^2 > 0$, or

$$\int (|x| \wedge 1)\nu(dx) = \infty \quad \text{and} \quad \int_0^\infty \mathrm{Re}\,\frac{1}{1+\psi(\lambda)}\,d\lambda < \infty, \tag{2}$$

then 0 is regular for both $\{0\}$ and $(\mathbb{R}\setminus\{0\})$, and the local time ℓ_t^x exists for fixed x as a continuous additive functional in t. It will be convenient to have a notation for particular processes satisfying these conditions.

B_t denotes a standard Brownian motion for which $\psi(\lambda) = \frac{1}{2}\lambda^2$.

$S_t^{\gamma,\delta}$ denotes a stable process of index γ with

$$\psi(\lambda) = |\lambda|^\gamma(1 + i\delta \tan\frac{\pi\alpha}{2}) \qquad -1 \le \delta \le 1,\quad 1 < \gamma < 2$$

arising from a Lévy measure of the form

$$\nu(dx) = c|x|^{-1-\gamma}(pI_{(x>0)} + qI_{(x<0)})$$

with $p \ge 0$, $q \ge 0$, $p+q = 1$, $p-q = \delta$.

$S_t^{1,\alpha,\beta}$ is a symmetric process close to Cauchy with

$$\nu(dx) = x^{-2}\, g^{\alpha,\beta}\left(\frac{1}{|x|}\right) \quad \text{and}$$

$$g^{\alpha,\beta}(y) = (\log y)^\alpha(\log\log y)^\beta I_{(y>e)}: \quad \alpha,\beta \in \mathbb{R}.$$

$A_t^{1,\alpha,\beta,p}$ is the corresponding asymmetric process with

$$\nu(dx) = x^{-2}\, g^{\alpha,\beta}\left(\frac{1}{|x|}\right)(pI_{(x>0)} + qI_{(x<0)})$$

and $p \ge 0$, $q \ge 0$, $p+q = 1$, $p \ne \frac{1}{2}$.

It is clear that (2) is satisfied for $S_t^{\gamma,\delta}$, $1 < \gamma < 2$ and Barlow [2] estimates $\psi(\lambda)$ and shows that (2) is satisfied for

$$S^{1,\alpha,\beta} \quad \text{if and only if} \quad \alpha > 1 \ \text{ or } \ \alpha = 1,\ \beta > 1$$

$A^{1,\alpha,\beta,p}$ if and only if $\alpha > -1$ or $\alpha = -1$, $\beta > 1$.

We now summarise the main content of Theorem 4 in [13].

THEOREM 2.1. *Suppose* X_t *is a Lévy process whose exponent* (1) *satisfies either* (2) *or* $\sigma^2 > 0$. *Then, for any* $r > 0$ *there exists a bounded continuous density* u^r *for the potential kernel; that is,*

$$E^x \int_0^\infty e^{-rt} f(X_t)\,dt = \int_{\mathbb{R}} u^r(y-x) f(y)\,dy \tag{3}$$

for each non-negative measurable f. *For each* x *there exists a continuous additive functional* ℓ_t^x *(a local time at* x*) such that*

$$E^x \int_0^\infty e^{-rt} d_t \ell_t^y = u^r(y-x), \tag{4}$$

and for fixed $t \geq 0$, *the map* $(x,\omega) \to \ell_t^x$ *is* $\mathcal{B} \times \mathcal{F}_t$ *measurable, and a.s. for each Borel set* B,

$$\mu_t(B) = |\{s \leq t : X_s \in B\}| = \int_B \ell_t^x\,dx. \tag{5}$$

The probability that ℓ_t^x *has a version continuous in* (x,t) *is zero or one.*

3. Continuity of local time

In general the conditions satisfied by ℓ_t^x in Theorem 2.1 do not determine it uniquely as a function of (x,t). However, if the process is such that a.s. a continuous version exists, it is clear by (5) that this version is unique. We will denote it by L_t^x. It is instructive to see how we can modify L_t^x to obtain a new version which is not continuous, but still satisfies Theorem 2.1.

Suppose $Q_t(\omega)$ is a random subset of $\mathbb{R}$ such that, for each $t > 0$,

$I_{Q_t}(x)$ is $\mathcal{B} \times \mathcal{F}_t$ measurable,

$t' > t \Rightarrow Q_{t'} \supset Q_t$,

$Q_t(\omega)$ is dense in $\{X_s | s \leq t\}$,

$$P\{x \in Q_t(\omega)\} = 0 \text{ for all } x. \tag{6}$$

For the process B_t we give an example of such a set Q_t in section 5 (see Example 5.4). Now define

$$\ell_t^x = L_t^x \text{ for all } t \geq 0 \text{ whenever } x \notin Q = \bigcup_{t \geq 0} Q_t,$$

for $x \in Q$, let $t_0(x) = \inf\{t : x \in Q_t\}$ and let

$$\ell_t^x = L_t^x \text{ for } t \leq t_0; \quad \ell_t^x = L_{t_0}^x \text{ for } t > t_0.$$

It is easy to check that ℓ_t^x satisfies all the conditions of Theorem 2.1 and for each x, ℓ_t^x is monotone and continuous in t. However, if $x_0 \in Q_t(\omega)$ for $t > t_1$ and t_1 is a growth point of $L_t^{x_0}$, then ℓ_t^x will be discontinuous in x at $x = x_0$ for $t > t_1$. By (6) we have still, for each (x,t), $\ell_t^x = L_t^x$ a.s.

In fact, without any continuity assumption, the normalisation (4) ensures that any two versions of ℓ_t^x will agree a.s. for all $t > 0$ and a fixed level x. This agreement therefore extends a.s. to all levels x in a fixed countable set D. We will assume that D is the set of dyadic rationals. We can then study the a.s. properties of any version ℓ_t^x satisfying Theorem 2.1 by looking at its behaviour for $x \in D$, $t > 0$. For example, to show that a jointly continuous version of ℓ_t^x exists, it suffices to show that $\{\ell_t^x, 0 \leq t \leq 1, x \in D\}$ is uniformly continuous.

Necessary and sufficient conditions for the existence of a continuous version of ℓ_t^x are not known. Sufficient conditions have been given by Trotter [31], Boylan [6], Getoor and Kesten [13], and Barlow [2]. Getoor and Kesten also found a condition which ensures that no continuous version of ℓ_t^x exists: this last result was strengthened by Millar and Tran [22], who showed that, under the same conditions, ℓ_t^x is a.s. unbounded.

For the special processes introduced earlier, we have the following table.

Process	Parameter Values	Properties of Local Time
$S^{\alpha,\delta}$	$1<\gamma<2$, $-1\le\delta\le 1$	continuous
$S^{1,\alpha,\beta}$	$\alpha>2$; $\alpha=2$, $\beta>2$	continuous
	$\alpha=2$, $0<\beta\le 2$	unknown
	$\alpha=2$, $\beta\le 0$; $1<\alpha<2$; $\alpha=1$, $\beta>1$	unbounded on D
$A^{1,\alpha,\beta,p}$	$\alpha>0$; $\alpha=0$, $\beta>2$	continuous
	$\alpha=0$, $0<\beta\le 2$	unknown
	$\alpha=0$, $\beta\le 0$; $-1<\alpha<0$; $\alpha=-1$, $\beta>1$	unbounded on D

If the following conjecture is correct, then the local times of $S^{1,2,\beta}$, $A^{1,0,\beta,p}$, $0<\beta\le 2$ are not continuous.

CONJECTURE 3.1 (Hawkes, 1981). *Let*

$$\phi^2(h) = \frac{1}{\pi}\int (1-\cos\lambda h)\,\mathrm{Re}\left(\frac{1}{1+\psi(\lambda)}\right)d\lambda,$$

and let $\bar\phi$ *be the monotone rearrangement of* ϕ. *Set*

$$I(\bar\phi) = \int_{0+} \frac{\bar\phi(u)}{u(\log 1/u)^{\frac{1}{2}}}\,du.$$

Then $I(\bar\phi)<\infty$ *is a necessary and sufficient condition for the existence of a continuous version of* $(x,t)\to\ell_t^x$.

The sufficient condition for continuity given in Barlow [2] is that

$I(\tilde\phi)<\infty$, where $\tilde\phi(h) = \sup_{|u|\le h}\phi(u)$.

A related question is the following

PROBLEM 3.2. *If no jointly continuous version of* ℓ_t^x *exists, is every version of* ℓ_t^x *unbounded for* $t=t_0$, $x\in\mathbb{R}$?

Based on all known examples it seems possible that the process $\ell_{t_0}^x$, for fixed $t_0>0$, exhibits the same sort of dichotomy in behaviour as a stationary Gaussian process.

CONJECTURE 3.3. *A Lévy process satisfying* (2) *either a.s. has a continuous local time, or a.s. every version* ℓ_t^x *of the local time has the property that, for* $t_0>0$, *the values of* $\ell_{t_0}^x$, $x\in\mathbb{R}$, *are dense in* $[0,\infty)$.

We present further evidence in support of this conjecture in the next section.

We remark on another consequence of the improved modulus of continuity in space obtained by Barlow [2]. Hawkes [15] obtained an exact uniform modulus of continuity in t for fixed x for L_t^x, the continuous local time of the stable process $S_t^{\gamma,\delta}$. Perkins [25] has obtained the best modulus in t which is true uniformly in x:

$$\lim_{s\downarrow 0}\ \sup_{0\le t\le 1}\ \sup_{a\in\mathbb{R}} (L_{t+s}^a - L_t^a)\phi_\gamma(s)^{-1} = \theta_0,$$

where $\phi_\gamma(s) = s^{1-1/\gamma}(\log 1/s)^{1/\gamma}$, and θ_0 is a known constant, strictly larger than that of Hawkes [15].

Except for B_t, where one can use the Ray-Knight theorem (see Ray [27] or Knight [30]), the exact modulus in space is not known. This gives

PROBLEM 3.4. *What is the asymptotic behaviour of*

$$W(y) = \sup_{x\in\mathbb{R}} \sup_{|h|\le y} |L_t^{x+h}(S^{\gamma,\delta}) - L_t^x(S^{\gamma,\delta})|,$$

as $y \downarrow 0$?

Barlow [2] obtains

$$W(y) \le c(\sup_{x\in\mathbb{R}} L_t^x)^{\frac{1}{2}} y^{\frac{1}{2}(\gamma-1)} (\log 1/y)^{\frac{1}{2}},$$

which is likely to be the right order of magnitude, since it is for B_t.

4. Processes with a nowhere dense range

We denote the range up to time t by

$$F_t = \{x \in \mathbb{R} : X_s = x \quad \text{for some} \quad s \in [0,t]\}.$$

As remarked in Pruitt, Taylor [26], if X_t is a Lévy process with a local time, then a.s. F_t is a closed subset of $\mathbb{R}$ with positive Lebesgue measure for $t > 0$. The zero–one law of Barlow [1] shows that *either* a.s. F_t is a countable union of disjoint closed intervals; *or* a.s. F_t is a perfect nowhere dense set of positive Lebesgue measure. Both cases can arise. In fact Kesten [18] showed that for $S^{1,\alpha,\beta}$ we have F_t nowhere dense when $\alpha = 1$, $1 < \beta < 2$ and Pruitt, Taylor [26] show that the asymmetric Cauchy process $A_t^{1,0,0,p}$ has this property except for the extreme case $p = 1$ or 0 where jumps in only one direction occur. This leads naturally to the next

PROBLEM 4.1. *Suppose* X_t *is a Lévy process satisfying* (2), *and in* (1), $\nu(-\infty,0) = +\infty = \nu(0,+\infty)$. *If a.s. no continuous version of* ℓ_t^x *exists, does it follow that the range* F_t *is a.s. nowhere dense?*

Note that a nowhere dense F_t implies ℓ_t^x is discontinuous. We cannot omit the extra condition on ν in Problem 4.1 because, if $\nu(-\infty,0)$ is finite, the sample paths of X_t exhibit a local one-sided continuity which forces F_t to be a union of intervals.

The following Proposition shows that, for the asymmetric Cauchy process $A^{1,0,0,p}$, $p < 1$, $\{\ell_t^x, x \in \mathbb{R}\}$ is dense in $[0,\infty)$.

PROPOSITION 4.2. *Suppose* X_t *is a Lévy process with local time* ℓ_t^x *such that*

(7) (i) The range F_t is nowhere dense in $\mathbb{R}$,

(8) (ii) ℓ_t^x is unbounded for $x \in D \cap (-\eta,\eta)$ for all $\eta > 0$.

Then, for any interval (a,b) and $t > 0$, *either*

(iii) $F_t \cap (a,b) = \phi$; *or*

(iv) for all $0 \le u < v < +\infty$ there exists $y \in D \cap (a,b)$ such that $u < \ell_t^y < v$.

PROOF: The idea is to search for the point y as the value of a process Y_s which moves on D to satisfy $u < \ell_s^{Y_s} < v$ for all s greater than the hitting time of (a,b). Let $u < u_0 < v_0 < v$ and put

$$\Gamma = \{(\omega,s) : \ell_s^{X_s}(\omega) = u_0, \ X_s \in D \cap (a,b)\}.$$

Now suppose $S < T$ are any stopping times such that $X_s \in (a,b)$ for $S \le s < T$. Then

$$P\{\omega : (\omega,s) \in \Gamma \quad \text{for some} \quad s \in (S,T)\} = 1. \tag{9}$$

To see this first note that (7) implies that F_S is nowhere dense, so there exists a sequence A_n of $\mathcal{F}_S$-measurable random variables with $A_n \notin F_S$, and $A_n \to X_S$. But then X_S is regular for the set $\{A_n, n \geq 1\}$, since $P^x(y \in F_t) \to 1$ as $y \to x$, and therefore there exists $n = N$ such that $A_N \in F_T$. But now (8) implies that ℓ_T^a, $a \in D$, is unbounded in every interval around A_N, and therefore, for some $Y \in D$, $\ell_T^Y > v_0$ while $Y \notin F_S$. By continuity in t we can find s with $S < s < T$, $X_s = Y$, and $\ell_S^Y = u_0$. This proves (9).

Now put $T_0 = \inf\{s \geq 0: X_s \in (a,b)\}$. Fix $\varepsilon > 0$, We shall define an optimal process Y such that for $s > T_0 + \varepsilon$, $Y_s \in (a,b) \cap D$ and

$$P\{u_0 \leq \ell_s^{Y_s} < v\} \geq 1 - \varepsilon. \tag{10}$$

To construct Y_s we will use the section theorem (see Dellacherie and Meyer [8, p. 137]) to define an increasing sequence of stopping times: Y_s will be constant between terms of the sequence.

We can choose S_0 such that $T_0 \leq S_0 \leq T_0 + \frac{1}{2}\varepsilon$, $X_{S_0} \in D \cap (a,b)$: set $T_0' = \inf\{s > S_0: X_s \notin (a,b)\}$. Now use (9) to apply the section theorem to the set $\Gamma \cap (S_0, T_0' \wedge (T_0 + \varepsilon))$ to give a stopping time S_1 such that

$$P\{S_1 < \infty\} > 1 - \tfrac{1}{2}\varepsilon,\ \ell_{S_1}^{X_{S_1}} = u_0,\ X_{S_1} \in (a,b) \cap D,$$

and $S_1 < T_0' \wedge (T_0 + \varepsilon)$ on $\{S_1 < \infty\}$. Now put $T_1 = S_1' = \inf\{s > S_1: \ell_s^{X_{S_1}} = v_0\}$, and note that a.s. T_1 is a growth point of $\ell_s^{X_{S_1}}$ so that $X_{T_1} = X_{S_1} \in (a,b) \cap D$. Hence $T_1' = \inf\{s > T_1: X_s \notin (a,b)\} > T_1$, and if we put $T_1'' = \inf\{s > S_1: \ell_s^{X_{S_1}} = v\}$ we can again apply the section theorem to $\Gamma \cap (T_1, T_1' \wedge T'')$ to find a stopping time S_2 such that $P\{S_2 = \infty, S_1 < \infty\} < \frac{1}{4}\varepsilon$ and on $\{S_2 < \infty\}$ we have $T_1 < S_2 < T_1' \wedge T_1''$, $\ell_{S_2}^{X_{S_2}} = u_0$, and $X_{S_2} \in (a,b) \cap D$. Continuing inductively, we obtain, except on a set of probability ε, a sequence (S_n), (T_n) of stopping times such that $(T_n - S_n)$ are independent, identically distributed, $T_n < S_{n+1}$, $X_{S_n} \in (a,b) \cap D$ and $\ell_s^{X_{S_n}} \in [u_0, v)$ for $S_n \leq s < S_{n+1}$. If $Y_s = \sum 1_{[S_n, S_{n+1})}(s) X_{S_n}$ (10) is satisfied. But clearly $S_n = S_0 + \sum (S_i - S_{i-1}) \geq S_0 + \sum (T_{i-1} - S_{i-1}) \to \infty$, so Y_s is defined for all $s > T_0 + \varepsilon$ and the construction is valid outside a set of probability ε. Since ε is arbitrary, this completes the proof.

We note that the conclusion of Proposition 4.2 allows us to deduce 'denseness' in two senses

COROLLARY 4.3. *Under the hypothesis of Proposition* 4.2, *for* $t > 0$, $0 \leq u < v \leq +\infty$,

$$\text{a.s. } \{x \in D: u < \ell_t^x < v\} \quad \text{is dense in } F_t.$$

COROLLARY 4.4. *Under the hypothesis of Proposition* 4.2, *for* $t > 0$, *if* I *is an open interval with* $I \cap F_t \neq \phi$,

$$\text{a.s. } \{y: y = \ell_t^x \text{ for some } x \in D \cap I\} \quad \text{is dense in } \mathbb{R}^+.$$

REMARK. The totally asymmetric Cauchy process $A^{1,0,0,1}$ has a range which is a union of intervals, and therefore fails to satisfy (7). However the information in [26] can be used to show that its local time is dense in the sense of Proposition 4.2.

5. Constructions of ℓ_t^x that fail at some level

In the literature there are many distinct ways of obtaining ℓ_t^x as the limit of functionals $K_n(x,t)$ of the sample path X_s, $0 \leq s \leq t$. A systematic approach to these constructions was initiated by Maisonneuve [21], and developed into a unified umbrella method in Fristedt, Taylor [11], to which the reader is referred for a bibliography. Suppose that a construction (K_n) converges a.s. at one level x_0, to give an additive functional $K_t^{x_0} = \lim_{n\to\infty} K_n(x_0,t)$, which is continuous in t, and which is normalised by (4), with $x = y = x_0$. A Fubini argument then shows that $K_n(x,\cdot)$ converges on a set of full measure in $\mathbb{R}_1$ and we can trivially use (K_n) to define a version of the local time ℓ_t^x satisfying Theorem 2.1 by setting

$$\ell_t^x = \limsup_{n\to\infty} K_n(x,t).$$

Remembering from our earlier discussion that we only have a canonical value of ℓ_t^x for all (x,t) whenever there is a continuous version, there are two distinct questions to resolve for any construction $K_n(x,t)$ which converges to ℓ_t^x $t \ge 0$ a.s. for each x.

I. *Is there a fixed null set* N *such that, for* $\omega \notin N$, (K_n) *converges for all* (x,t)? *(If so, the result is automatically a version of* ℓ_t^x *satisfying Theorem* 2.1.)

II. *Suppose* X_t *is such that a continuous version* L_t^x *of the local time exists. Is there a null set* N *such that, for* $\omega \notin N$, (K_n) *converges to* L_t^x *for all* (x,t)?

We now give an example of a construction which fails to converge at some levels, thus showing that the answer to I may be "no." Let $N_t(x,x+\varepsilon)$ be the number of upcrossings made by F from $(-\infty,x)$ to $(x+\varepsilon,\infty)$ before time t. Suppose there exists a sequence $\varepsilon_k \downarrow 0$, and constants a_k such that, for each (x,t) a.s.,

$$(11) \qquad a_k N_t(x,x+\varepsilon_k) \to \ell_t^x \quad \text{as} \quad k \to \infty.$$

THEOREM 5.1. *Suppose* X_t *is a Lévy process with local time* ℓ_t^x, *and satisfying* (7), (8) *and* (11). *Then given* $0 < u < v < +\infty$, $t > 0$ *a.s., there exists a level* $z = z(w)$ *such that*

$$(12) \qquad \limsup_{k\to\infty} a_k N_t(z,z+\varepsilon_k) \ge v > u > \liminf_{k\to\infty} a_k N_t(z,z+\varepsilon_k).$$

PROOF: We use Proposition 4.2 to obtain z as the limit point in a condensation argument. First note that, since D is countable, we can assume that a.s. (11) holds at every point of D. Apply Proposition 4.2 to find $y_0 \in D$ with $\ell_t^{y_0} > v$. But $y \to N_t(y,y+\varepsilon_{k_1})$ is constant in a small closed interval $[y_0', y_0'']$ with $y \in (y_0', y_0'')$. The Proposition now gives a point $y_1 \in (y_0', y_0'') \cap D$ for which $\ell_t^{y_1} < u$, and therefore for some $k_2 > k_1$, $a_{k_2} N_t(y_1, y_1 + \varepsilon_{k_2}) < u$. By induction we obtain a sequence I_r of closed intervals, which we may assume nested, such that

$$r \text{ even}, \ x \in I_r \Rightarrow a_{k_r} N_t(x, x+\varepsilon_{k_r}) < u$$

$$r \text{ odd}, \ x \in I_r \Rightarrow a_{k_r} N_t(x, x+\varepsilon_{k_r}) > v.$$

Clearly $z = \cap I_r$ satisfies (12).

REMARK 1. The asymmetric Cauchy process studied in Pruitt, Taylor [26] satisfies the conditions of the Theorem; the construction of its local time given there involved counting 'passes' of given length across a level, but for a fixed level this is equivalent to counting upcrossings.

REMARK 2. A similar argument, giving non-convergence at some level, will work for any construction $K_n(x,t)$ such that, a.s., $K_n(y,t) = K_n(x,t)$ for y sufficiently close to x. For example, the analogue of Theorem 5.1 is valid for the Getoor-Millar construction (see [14]), which counts jumps across a level, and which we will consider in section 6.

The preceding counterexample deals with processes with a discontinuous local time: it might be thought that if ℓ_t^x is jointly continuous then any construction (K_n) should converge simultaneously at every level x. In fact this is false even for B_t, as is shown in Barlow, Perkins [3]. We now give a generalization of their construction.

We start with a real variable result. Suppose $\psi: [0,1] \to \mathbb{R}$ is a fixed function, and define

(13) $$\Lambda_x(\psi) = \{t \in [0,1): \psi(t) = x \text{ and } \exists\ \delta > 0 \text{ with } \psi(s) \neq x \text{ for } t < s < t+\delta\};$$

denoting the starting points of excursions from x. Let $R_r = \{\psi(s): 0 \le s < r\}$ denote the range of ψ with interior R_r^0 and closure $\bar{R}_r$.

THEOREM 5.2. *Suppose* $\psi:[0,1] \to \mathbb{R}$ *is cadlag, nowhere monotone and satisfies*

(14) $$R_r^0 \text{ is dense in } R_r \text{ for all } r \text{ in } (0,1].$$

Let $f:[0,1] \to [0,\infty)$ *be any continuous strictly increasing function with* $f(0) = 0$. *Then there is a set* S *which is a countable intersection of sets each of which is open and dense in* R_1^0 *such that, for all* x *in* S, *and* t *in* $\Lambda_x(\psi)$, *there is a sequence* $\{t_n\}$ *decreasing to* t *for which*

(15) $$|\psi(t_n) - \psi(t)| < f(t_n - t).$$

PROOF. For $0 < r \le 1$ and $x \in \bar{R}_r$ define

$$g_x^r = \sup\{s < r: \psi(s) = x \text{ or } \psi(s-) = x\}.$$

Then, for fixed r, $g_\cdot^r$ is upper semi-continuous on $\bar{R}_r$, that is,

$$g_x^r \ge \limsup_{\substack{y \to x \\ y \in \bar{R}_r}} g_y^r$$

However, for fixed r, we claim that the discontinuity points of $g_\cdot^r$ are dense in $\bar{R}_r$. For, suppose $(a,b) \cap \bar{R}_r \neq \phi$: using (14) we can assume without loss of generality that $\psi(r\pm) \notin [a,b]$ and $(a,b) \subseteq R_r^0$. If

$$M = \sup\{s < r: \psi(s) \in (a,b)\}$$

then $0 < M < r$, $\psi(M-) \in [a,b]$, and either $\psi(M-) > a$ or $\psi(M-) < b$: we assume $\psi(M-) > a$. The left continuity and nowhere monotonicity of $\psi(s-)$ imply there is a $t_1 < M$ such that

$$a < \inf\{\psi(s-): t_1 \le s \le M\} < \psi(t_1) \wedge \psi(t_1-).$$

Now choose the largest t_0 in $[t_1, M]$ satisfying

$$\psi(t_0) \wedge \psi(t_0-) = \inf\{\psi(s-): t_1 \le s \le M\} = x_0, \text{ say.}$$

Now $x_0 \in (a, \psi(M-)] \subset (a,b]$, For all $u \in (t_1,t_0) \cup (t_0, r)$

(16) $$\psi(u), \psi(u-) \notin (a, x_0).$$

This is clear for $u \in (t_1, t_0)$ by the definition of x_0 and for $u \in (M,r)$ by the definition of M and the fact that $(a,x_0) \subset (a,b)$. Finally, if $t_0 < M$, then $\psi(u) \wedge \psi(u-) > x_0$ for $t_0 < u \le M$ by the choice of t_0. Now (16) implies

$$g_{x_0}^r = t_0 \quad \text{and} \quad \limsup_{y \uparrow x_0,\ y \in \bar{R}_r} g_y^r \le t_1 < t_0.$$

(Note that the existence of $x_n \in \bar{R}_r$ with $x_n \uparrow x_0$ is guaranteed by $(a,b) \subset R_r^0$: this is all we need from (14).) Thus $x_0 \in (a,b]$ is a point of discontinuity of g_x^r, so the discontinuity points are dense.

Now let f^{-1} be the continuous inverse of f, and define for

$r \in (0,1]$ and $n \in \mathbb{N}$,

$$G_n^r = (R_1^0 \setminus \bar{R}_r) \cup \{x \in R_r^0 : \exists\, \varepsilon > 0 \text{ with } (x-\varepsilon, x+\varepsilon) \subset R_r^0 \text{ and for all}$$
$$y \in (x-\varepsilon, x+\varepsilon), \exists\, h = h(y) \in (-n^{-1}, n^{-1}) \text{ with } g_{y+h}^r - g_y^r > f^{-1}(|h|)\}.$$

Clearly, G_n^r is open. We now show it is dense in R_1^0. Suppose $(a,b) \subset R_r^0$: then (a,b) contains a discontinuity of $g_\cdot^r$. Pick $\varepsilon_0 > 0$ and $x, x' \in (a,b)$ such that

$$2\varepsilon_0 < g_{x'}^r - g_x^r,\ |x-x'| < n^{-1} \text{ and } f^{-1}(|x-x'|) < \varepsilon_0.$$

Use the continuity of f^{-1} and upper semi-continuity of g^r to find $\varepsilon < n^{-1} - |x-x'|$, $\varepsilon > 0$ and such that $f^{-1}(\varepsilon + |x-x'|) < \varepsilon_0$ and $y \in (x-\varepsilon, x+\varepsilon)$ implies $y \in R_r^0$ and $g_y^r \le g_x^r + \varepsilon_0$. For such y take $h(y) = x - y + (x'-x) \in (-n^{-1}, n^{-1})$ and check that $g_{y+h}^r - g_y^r = g_{x'}^r - g_y^r > \varepsilon_0 > f^{-1}(\varepsilon + |x-x'|) \ge f^{-1}(|h|)$. Thus $x \in G_n^r \cap (a,b)$ and it follows that G_n^r is dense in R_1^0. Since R_1^0 is locally compact, the Baire category theorem implies that

$$S = \bigcap_{\substack{r \in Q \cap (0,1] \\ n \in \mathbb{N}}} G_n^r \quad \text{is a dense } G_\delta \text{ in } R_1^0.$$

Now let $x \in S$ and $t \in \Lambda_x(\psi)$. Then there are $r_n \in Q$ such that $r_n \downarrow t$ and $t = g_x^{r_n}$. But $x \in S \cap \bar{R}_{r_n}$, so we can find a sequence h_n with $h_n \to 0$ such that

$$g_{x+h_n}^{r_n} - g_x^{r_n} > f^{-1}(|h_n|).$$

Since $t_n = g_{x+h_n}^{r_n} \in (t, r_n]$, we have

$$|\psi(t_n) - \psi(t)| \wedge |\psi(t_{n-}) - \psi(t)| \le |x + h_n - x| < f(t_n - t).$$

By slightly moving the t_n's, if necessary, we get

$$|\psi(t_n) - \psi(t)| < f(t_n - t),\quad t_n \downarrow t,$$

and the proof is complete.

REMARK 1. The theorem is proved in [3] under the additional hypothesis that $\psi(s)$ is continuous.

REMARK 2. The theorem is false if (14) is omitted. Suppose $\psi(\cdot)$ is cadlag, nowhere monotone, such that for each x there is at most one value of t for which $x = \psi(t)$ or $x = \psi(t-)$. One could easily construct such a function directly, but note that any symmetric stable process of index $\alpha \le \frac{1}{2}$ has sample paths which a.s. have this property (see [28]). If we define $g_\cdot^r$ as in the proof of the Theorem, one can check that it is continuous, and hence uniformly continuous on $\bar{R}_1$. Therefore there is a continuous strictly increasing function ϕ such that

$$x, y \in \bar{R}_1 \Rightarrow |g_x' - g_y'| < \phi(|x-y|).$$

If $f = \phi^{-1}$, it follows that

$$|\psi(t) - \psi(s)| > f(|t-s|) \quad \text{for all } s,t \in [0,1)$$

and the theorem fails for f.

COROLLARY 5.3. *Suppose* X *is a Lévy process which a.s. has a continuous local time, then the conclusion of Theorem* 5.2 *a.s. holds for* $\psi(s) = X_s$.

PROOF: We need only check that X_s a.s. satisfies the hypothesis of the Theorem. The existence of a continuous local time L_t^x implies that the sample path X_s is nowhere monotone by a real variable argument (see Example 1 in Geman, Horowitz [12]). To prove (14) note that a.s.

$$\int_0^t I(X_s \in B)ds = \int_B L_t^x dx \quad \text{for all Borel } B,\ t \geq 0.$$

It follows that $\{x: L_r^x > 0\} \subset R_r^0$ and is dense in R_r for all $r > 0$.

EXAMPLE 5.4. Let X be a fixed Lévy process with a jointly continuous local time L_t^x. If V denotes an excursion of X from x, we put $\tau^-(V)$ and $\tau^+(V)$ for the start and end of V. There exists a continuous, strictly increasing function f with $f(0) = 0$ which grows slowly enough to ensure

$$(17) \qquad \liminf_{h \downarrow 0} |X_{\tau^-(V)+h} - x|/f(h) \geq 1$$

holds a.s. for fixed X, V and so for fixed x it holds a.s. for every excursion V from x. To find such an f, let μ denote the characteristic measure of the Poisson point process of excursions V from x (see Itô [16] or Fristedt, Taylor [11] for details), fix $u \in (0,1)$, and choose $\varepsilon_n \downarrow 0$ such that

$$\mu\{V: \inf\{|V_s|;\ s \in [u^{n+1}, u^n]\} < \varepsilon_n \text{ and } l(V) > u^n\} < 2^{-n}.$$

Here $\ell(V) = \tau^+(V) - \tau^-(V)$ is the length of the excursion and $V_s = X_{s-\tau^-(V)}$ for $0 \leq s < l(V)$. Define $f(u^n) = \varepsilon_n$ and by linear interpolation in (u^{n+1}, u^n), $n \in \mathbb{N}$. Then a standard Borel-Cantelli argument shows that

$$\mu\{V: \inf(V_s/f(s): u^{n+1} \leq s \leq u^n) < 1 \text{ i.o.}\} = 0.$$

This establishes (17) for this function f.

Let $N_\varepsilon(t,x)$ be the number of excursions from x exceeding ε in length and completed by time t, and let $N'_\varepsilon(t,x)$ be the number of these excursions that satisfy (17). For fixed x, $N_\varepsilon(t,x) = N'_\varepsilon(t,x)$ a.s.

By Maisonneuve [21, Theorem X.4], if E_ε denotes the subset of excursion space consisting of excursions of length greater than ε, we have, for each $x \in R$,

$$(18) \qquad \lim_{\varepsilon \downarrow 0} N'_\varepsilon(t,x)\mu(E_\varepsilon)^{-1} = L_t^x \quad t \geq 0 \text{ a.s.}$$

But Corollary 5.3 tells us that a.s. there is a dense G_δ in R_1^0, $S = S(\omega)$ such that, for $x \in S$ and $t \in \Lambda_x(X)$,

$$\liminf_{h \downarrow 0} |X_{t+h} - X_t|/f(h) = 0.$$

This shows that, for $x \in S$, (17) fails for every excursion which starts from x. There are only countably many levels x at which some excursion from x begins with a jump. Thus, if $x \in S'(\omega) = S(\omega)\backslash\{X(t-): X(t) \neq X(t-)\}$, (17) fails for every excursion starting from x, so that $N'_\varepsilon(t,x) = 0$ for all $0 \leq t \leq 1$. Thus we have found a random set $S'(\omega)$ that is a dense G_δ in R_1^0, and for which (18) fails for $x \in S'(\omega)$.

Note that if $t_0(x)$ is the first time $t \in \Lambda_x(B)$ for which (17) fails with $f(h) = h$ and $Q_t(\omega) = \{x: t_0(x) \leq t\}$, then Q_t is of the form considered in (6).

EXAMPLE 5.5. The construction above gives a counterexample to Question II, but it is not obvious that $N'_\varepsilon(t,x)\mu(E_\varepsilon)^{-1}$ fails to converge for some levels x. We now give such an example. Suppose the process is Brownian motion: then Perkins [23] showed that outside a fixed null set N,

(19) $$\lim_{\varepsilon\downarrow 0} (\pi/2)^{\frac{1}{2}} N_\varepsilon(t,x)\varepsilon^{\frac{1}{2}} = L_t^x \quad \text{for all } (x,t).$$

We now define $N''_\varepsilon(t,x)$ by counting all excursions from x completed by time t with $\ell(V) > \varepsilon$, and for which either $2^{-2k} \le \ell(V) < 2^{-2k+1}$ for some $k \in \mathbb{N}$, or $2^{-2k-1} \le \ell(V) < 2^{-2k}$ for some $k \in \mathbb{N}$ and (17) holds. If we now look at any point $x \in S \subset R_t$, (19) shows that $\varepsilon^{\frac{1}{2}} N''(t,x)$ cannot converge as $\varepsilon \downarrow 0$, for $0 \le t \le 1$. Clearly $N''_\varepsilon(t,x) = N_\varepsilon(t,x)$ a.s. for a fixed level x, so we have a construction for Brownian local time which fails to converge at some levels, giving a counterexample to Question I.

Both the constructions above depend on the behaviour of X_s away from the level x. A construction of L_t^x which depends only on the level set $\{s \le t: X_s = x\}$ is called intrinsic. One could ask whether negative answers to I and II are possible for an intrinsic construction. We will obtain such a counterexample again based on Brownian local time.

EXAMPLE 5.6. Let $f(t) = e^{-t^2}$, $t > 0$; consider the set of ending points of Brownian excursions from x

$$\Gamma_x(B) = \{t > 0: B_t = x,\ x \ne B_{t-h} \quad \text{for } h \in (0,\delta), \text{ some } \delta > 0\}.$$

Since the points t in $\Gamma_x(B)$ are stopping times we can apply the usual integral test for the lower asymptotic growth rate of L_{t+h}^x for small $h > 0$ to see that, for fixed $t \in \Gamma_x(B)$.

(20) $$(L_{t+h}^x - L_t^x) f(h)^{-1} \to \infty \quad \text{as } h \downarrow 0 \text{ a.s.}$$

and therefore, for each fixed x, a.s. (20) is true for all excursions from x. Hence if we put $N'''_\varepsilon(t,x)$ for the number of excursions from x which satisfy (20), then for fixed x,

$$\lim_{\varepsilon\downarrow 0} (\pi/2)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} N'''_\varepsilon(t,x) = L_t^x \quad \text{for all } t \ge 0, \text{ a.s.}$$

The condition (20) is intrinsic to the level set at x because of the uniform result (19). However we claim that there is a dense G_δ set $S_1 = S_1(\omega)$ such that for all $x \in S_1$, $t \in \Gamma_x(B)$ we have

(21) $$\liminf_{h\downarrow 0} (L_{t+h}^x - L_t^x) f(h)^{-1} \le 1;$$

so that for such levels x, $N'''_\varepsilon(t,x) = 0$ for all t.

For $r \ge 0$, $x \in \mathbb{R}$, let

$$T_r(x) = \inf\{t \ge r: B_t = x\},$$

$$G_n^r = \{x: \varepsilon > 0,\ h \in (0,n^{-1}) \text{ such that}$$
$$y \in (x-\varepsilon, x+\varepsilon) \Rightarrow L^y_{T_r(y)+h} - L^y_{T_r(y)} < f(h)\}.$$

If ω is chosen so that $B_t(\omega)$ is nowhere monotone and $L_t^x(\omega)$ is continuous, it is clear that G_n^r is open. We now show it is dense. For any open interval (a,b) let us assume $T_r(a) < T_r(b)$. As B is nowhere monotone we can find $t \in (T_r(a), T_r(b))$ such that

$$B_t < \sup(B_s: r \le s \le t) \equiv x_0 \in (a,b),$$

and hence a $\delta > 0$ such that $B_{T_r(x_0)+h} < x_0$ for $0 < h < \delta$. Fix $h < \delta \wedge n^{-1}$ and note that $\lim_{y\uparrow x_0} T_r(y) = T_r(x_0)$, so that

$$\lim_{y\uparrow x_0} L^y_{T_r(y)+h} - L^y_{T_r(y)} = L^{x_0}_{T_r(x_0)+h} - L^{x_0}_{T_r(x_0)} = 0.$$

It follows that, for some $\varepsilon > 0$, $(x_0-\varepsilon, x_0) \subset G_n^r$ and hence $G_n^r \cap (a,b) \ne \phi$.

We now take

$$S_1(\omega) = \bigcap_{\substack{n\in \mathbb{N} \\ r\in Q,\ r\geq 0}} G_n^r$$

and deduce that $S_1(\omega)$ is a dense G_δ by the Baire category theorem. Since for each $t \in \Gamma_x(B)$ we have $t = T_r(x)$ for some rational r, we have proved that for every $x \in S_1$, $t \in \Gamma_x(B)$ we have (21). This establishes the claim and completes the example.

6. Some constructions for L_t^x which converge at all levels

The following theorem extends a non-intrinsic construction of Getoor, Millar [14] to all levels simultaneously.

THEOREM 6.1. *Suppose* X_t *is a Lévy process with measure* ν *as defined in* (1) *satisfying* $\int(|x|\wedge 1)\nu(dx)=\infty$, *and assume* X *has a jointly continuous local time* L_t^x. *Define*

$$f_\varepsilon^a(x,y) = I_{\{x<a-\varepsilon,\ y>a+\varepsilon\}} + I_{\{x>a+\varepsilon,\ y<a-\varepsilon\}},$$

$$Q_\varepsilon^a(t) = \sum_{0\leq s\leq t} f_\varepsilon^a(X_{s-},X_s),$$

$$b_\varepsilon = \int_{-1}^{1} (|x| - 2\varepsilon)\vee 0\ \nu(dx).$$

If there exist sequences $\varepsilon_n \downarrow 0$, $\delta_n \downarrow 0$ *with* $0<\delta_n<\varepsilon_n$ *such that, as* $n\to\infty$

(22) $$\frac{b_{\varepsilon_n-\delta_n}}{b_{\varepsilon_n+\delta_n}} \to 1,$$

(23) $$b_{\varepsilon_n} / |\log \delta_n| \to +\infty,$$

then, for each $\eta, t_0 > 0$,

$$\lim_{n\to\infty} P\Big(\sup_{a\in\mathbb{R},\ s\leq t_0} |b_{\varepsilon_n}^{-1} Q_{\varepsilon_n}^a(s) - L_s^a| > \eta\Big) = 0.$$

If, in addition, as $n\to\infty$

(24) $$b_{\varepsilon_n}\cdot b_{\varepsilon_{n+1}}^{-1} \to 1,$$

(25) $$\sum_n e^{-\theta b_{\varepsilon_n}} \text{ converges for all } \theta>0,$$

then, for each t_0, *a.s. as* $\varepsilon\downarrow 0$

$$b_\varepsilon^{-1} Q_\varepsilon^a(s) \to L_s^a \text{ uniformly for } a\in\mathbb{R},\ 0\leq s\leq t_0.$$

REMARK. We know of no Lévy process with a continuous local time and satisfying $\int(|x|\wedge 1)\nu(dx)=\infty$, that fails to satisfy all the hypotheses of the above theorem.

PROOF: Let $L_t^* = \sup\{L_t^x,\ x\in\mathbb{R}\}$, and $T=\inf\{t: L_t^*=1\}$. It is clearly sufficient to prove the theorem with T in place of t_0.

We introduce the notation

$$N(x,dy) = \nu(dy-x)$$

$$Nf_\varepsilon^a(x) = \int N(x,dy)\, f_\varepsilon^a(x,y) = \begin{cases} \nu[a+\varepsilon-x,\infty), & \text{if } x\leq a-\varepsilon \\ \nu(-\infty,a-\varepsilon-x], & \text{if } x\geq a+\varepsilon \end{cases}$$

$$V_\varepsilon^a(t) = \int_0^t Nf_\varepsilon^a(X_{s-})ds = \int_{-\infty}^{\infty} Nf_\varepsilon^a(y)L_t^y dy, \quad \text{by (5).}$$

We will first use the continuity of L_t^x to show

(26) $$b_\varepsilon^{-1}V_\varepsilon^a(t) \to L_t^a \text{ uniformly in } (t,a)\in[0,T]\times\mathbb{R} \text{ as } \varepsilon\to 0 \text{ a.s.},$$

and then consider the difference

$$M_\varepsilon^a(t) = Q_\varepsilon^a(t) - V_\varepsilon^a(t).$$

By considering the process

$$X_t' = X_t - \sum_{0\le s\le t} (X_s - X_{s-})\, I_{\{|X_s - X_{s-}| \ge 1\}}$$

and working between consecutive jumps of X of magnitude exceeding 1, we may assume without loss of generality that

$$\nu(-\infty,-1] = \nu[1,\infty) = 0.$$

Now

$$(27)\qquad \int_{-\infty}^{\infty} Nf_\varepsilon^0(x)dx = \int_{-\infty}^{a-\varepsilon} \nu[a+\varepsilon-x,\infty)dx + \int_{a+\varepsilon}^{\infty} \nu(-\infty,a-\varepsilon-x]dx$$

$$= \int_{-\infty}^{\infty} (|u| - 2\varepsilon) \vee 0 \; \nu(du) = b_\varepsilon,$$

where we have used support $(\nu) \subset [-1,1]$ in the last line. As $\int(|x| \wedge 1)\nu(dx) = \infty$ and $\int(x^2 \wedge 1)\nu(dx) < \infty$, b_ε is finite for all $\varepsilon > 0$ and $b_\varepsilon \uparrow \infty$ as $\varepsilon \downarrow 0$. We now write

$$(28)\qquad b_\varepsilon^{-1} V_\varepsilon^a(t) - L_t^a = b_\varepsilon^{-1} \int_{-\infty}^{\infty} Nf_\varepsilon^a(y)(L_t^y - L_t^a)dy.$$

For any $\eta > 0$ we can first choose $\delta > 0$ such that

$$\sup\{|L_t^x - L_t^y| : |x-y| \le \delta,\ t \le T\} < \eta,$$

since L_t^x is uniformly continuous in $(x,t) \in \mathbb{R} \times [0,T]$. Also $\int_{|y-a|<\delta} Nf_\varepsilon^a(y)dy \le \int_{-\infty}^{\infty} Nf_\delta^a(x)dx = b_\delta$, whenever $\varepsilon < \delta$, so we can split the integral in (28) into two parts, $|y-a| \le \delta$ and $|y-a| > \delta$, to give

$$\sup_{t\le T} |b_\varepsilon^{-1} V_\varepsilon^a(t) - L_t^a| \le \eta + b_\varepsilon^{-1} b_\delta L_T^* .$$

Now let $\varepsilon \downarrow 0$, and we have a uniform bound which establishes (26).

We now consider $M_\varepsilon^a(t)$. As $(N(x,dy),t)$ is a Lévy system for X, $M_\varepsilon^a(t)$ is a martingale (see Benveniste-Jacod [5]) with $\langle M_\varepsilon^a, M_\varepsilon^a\rangle_t = V_\varepsilon^a(t)$.

LEMMA 6.2. *For all* $y \in (0,1)$, $\varepsilon > 0$, *and* $a \in \mathbb{R}$,

$$P(\sup_{t\ge 0} b_\varepsilon^{-1} |M_\varepsilon^a(t \wedge T)| > y) \le 2e^{-\frac{1}{4}b_\varepsilon y^2}.$$

PROOF: The definition of T and (27) imply $\langle M_\varepsilon^a, M_\varepsilon^a\rangle_T \le b_\varepsilon$. If the supremum is taken over $t \in \{i/n : i;0,\ldots,n^2\}$, the result is an easy consequence of Theorem 1.6 of Freedman [9]. Now let $n \to \infty$.

Fix $K > 0$ and let

$$A_n = \{i\delta_n,\ i \in \mathbb{Z}\} \cap [-K,K].$$

If $y \in (0,1)$, the above lemma gives

$$(29)\qquad P(\sup\{b_{\varepsilon_n}^{-1} |M_{\varepsilon_n}^a(t)|,\ a \in A_n,\ 0 \le t \le T\} > y)$$

$$\le 2\exp\{\log\left(\frac{2K}{\delta_n} + 1\right) - \tfrac{1}{4} b_{\varepsilon_n} y^2\}$$

which converges to zero by (23). If $x \in [-K,K]$ and $a_n(x)$ is the (suitably defined) "nearest" point to x in A_n, then

$$Q_{\varepsilon_n+\delta_n}^{a_n(x)}(t) \le Q_{\varepsilon_n}^x(t) \le Q_{\varepsilon_n-\delta_n}^{a_n(x)}(t).$$

Hence

$$(30)\qquad |b_{\varepsilon_n}^{-1} Q_{\varepsilon_n}^x(t) - L_t^x| \le |b_{\varepsilon_n}^{-1} Q_{\varepsilon_n-\delta_n}^{a_n(x)}(t) - L_t^x| + |b_{\varepsilon_n}^{-1} Q_{\varepsilon_n+\delta_n}^{a_n(x)}(t) - L_t^x|.$$

Each of these two terms may be bounded in a similar fashion; taking the first we have

$$|b_{\varepsilon_n}^{-1} Q_{\varepsilon_n-\delta_n}^{a_n(x)}(t) - L_t^x| \le b_{\varepsilon_n}^{-1} |Q_{\varepsilon_n-\delta_n}^{a_n(x)}(t) - V_{\varepsilon_n-\delta_n}^{a_n(x)}(t)|$$

$$+ b_{\varepsilon_n-\delta_n} b_{\varepsilon_n}^{-1} |b_{\varepsilon_n-\delta_n}^{-1} V_{\varepsilon_n}^{a_n(x)}(t) - L_t^{a_n(x)}|$$

$$+ |b_{\varepsilon_n-\delta_n} b_{\varepsilon_n}^{-1} L_t^{a_n(x)} - L_t^x| .$$

Using (29), (22), (23) and (26), and the joint continuity of L_t^x we deduce that, for each $y \in (0,1)$

$$\lim_{n\to\infty} P(\sup\{|b_{\varepsilon_n}^{-1} Q_{\varepsilon_n}^x(t) - L_t^x| : x \in [-K,K],\ 0 \le t \le T\} > y) = 0,$$

proving the first assertion in the theorem.

Now condition (25), applied to (29) gives, for each $y \in (0,1)$, a convergent series. An application of Borel Cantelli now gives

$$\sup\{b_{\varepsilon_n}^{-1} |M_{\varepsilon_n}^x(s)| : x \in A_n,\ 0 \le s \le T\} \to 0 \quad \text{a.s.}$$

and hence by (30) and the argument following

$$\sup\{|b_{\varepsilon_n}^{-1} Q_{\varepsilon_n}^x(t) - L_t^x| : x \in [-K,K],\ 0 \le s \le T\} \to 0 \quad \text{a.s.}$$

However, if $\varepsilon_{n+1} \le \varepsilon \le \varepsilon_n$,

$$(b_{\varepsilon_n} b_{\varepsilon_{n+1}}^{-1}) b_{\varepsilon_n}^{-1} Q_{\varepsilon_n}^x \le b_\varepsilon^{-1} Q_\varepsilon^x \le b_{\varepsilon_{n+1}} Q_{\varepsilon_{n+1}}^x (b_{\varepsilon_{n+1}} b_{\varepsilon_n}^{-1})$$

and therefore, by (24) we have a.s.

$$b_\varepsilon^{-1} Q_\varepsilon^x(t) \to L_t^x \quad \text{uniformly in } x \in [-K,K],\ t \in [0,T]. \qquad \square$$

EXAMPLE 6.3. Symmetric stable process of index α, $1 < \alpha < 2$ has Lévy measure $\nu(dx) = x^{-1-\alpha}\,dx$ so that $b_\varepsilon \sim c\varepsilon^{1-\alpha}$ as $\varepsilon \downarrow 0$. Take $\varepsilon_n = \frac{1}{n}$, $\delta_n = \frac{1}{n^2}$ to satisfy conditions (22) to (25). Boylan [6] proved that $S^{\gamma,0}$ has a continuous local time so the conclusion of our theorem is valid.

EXAMPLE 6.4. Critical asymmetric process $A^{1,\alpha,0,p}$ has a continuous local time for $\alpha > 0$. In this case

$$b_\varepsilon \sim c(\log \tfrac{1}{\varepsilon})^{\alpha+1} \quad \text{as } \varepsilon \downarrow 0.$$

Take $\varepsilon_n = e^{-n}$, $\delta_n = e^{-n-1}$ and all the conditions (22) to (25) are satisfied for $\alpha > 0$. Again we a.s. get uniform convergence for the construction.

EXAMPLE 6.5. Critical symmetric process $S^{1,\alpha,0}$ satisfies (22) to (25) if $\alpha > 0$, as in the asymmetric case. If $\alpha > 2$, $L_t^x(S^{1,\alpha,0})$ is jointly continuous and the theorem applies. If $\alpha \in (0,2)$, $L_t^x(S^{1,\alpha,0})$ is discontinuous. Nevertheless, the L^1-continuity of local time (in space) may be used in (23) to show $b_\varepsilon^{-1} V_\varepsilon^a(t) \xrightarrow{L^1} L_t^a$ as $\varepsilon \to 0$ for each a, t, and the rest of the proof goes through to show

$$\lim_{\varepsilon\to 0} \sup_{a,t \le t_0} P(|b_\varepsilon^{-1} Q_\varepsilon^a(t) - L_t^a| > \eta) = 0 \quad \text{for } \eta > 0,\ t_0 > 0.$$

Choose $\varepsilon_n \downarrow 0$ such that $b_{\varepsilon_n}^{-1} Q_{\varepsilon_n}^a(t) \to L_t^a$ a.s. for each a, t. If $\alpha = 1$ and $\beta \in (1,2)$ then F_t is nowhere dense by Kesten [18] and $L_t^x(S^{1,\alpha,0})$ is unbounded on $D \cap (-\eta,\eta)$ $(t,\eta > 0)$ by Millar and Tran [22]. By Theorem 5.1 (and the subsequent Remark 2), a.s. there exists $a = a(\omega)$

such that $\{b_{\varepsilon_n}^{-1} Q_{\varepsilon_n}^{a}(t)\}$ fails to converge as $n \to \infty$.

We now consider two "intrinsic" constructions of local time.

The first is the characterization of ℓ_t^x as the appropriate Hausdorff measure of the level set

$$Z_X(x,t) = \{s \in [0,t]: X_s = x\}.$$

This construction, which is described in Taylor [29], depends on the fact that for fixed x, the inverse of $\ell_\cdot^x$ is a subordinator, with each jump corresponding to an excursion of X from the level x. Fristedt, Pruitt [10] showed that for each subordinator there is a Hausdorff measure function which makes the measure of the range up to the time t grow linearly with t. Using the method introduced by Taylor, Wendel [30] for B, $S^{\gamma,\delta}$, $1<\gamma<2$, they concluded that for any Lévy process, X, with a local time there is a ϕ such that

(31) $$\phi - m(Z_X(x,t)) = \ell_t^x \quad t \ge 0 \text{ a.s. for each } x.$$

Here $\phi - m$ denotes Hausdorff ϕ-measure. Note that the "convergence" of this construction is built into the definition of ϕ-measure, so that there is trivially an affirmative answer to Question I of section 5. Turning now to Question II we have the following result, that will be proved in [3]:

THEOREM 6.6. (a) *If* $\phi(s) = (2s \log\log \frac{1}{s})^{\frac{1}{2}}$, $s < e^{-1}$, *then*

$$\phi - m(Z_B(x,t)) = L_t^x(B) \text{ for all } (x,t) \text{ a.s.}$$

(b) *If* $\phi_\alpha(s) = \rho^{-\gamma}\gamma^{-\gamma}(1-\gamma)^{\gamma-1}s^\gamma(\log\log \frac{1}{s})^{1-\gamma}$, $s < e^{-1}$, *where* $\gamma = 1 - \frac{1}{\alpha}$ *and* $\rho^{-\gamma}$ *is given by* (1) *of* [15], *then for* $1 < \alpha < 2$,

$$\phi_\alpha - m(Z_{S^{\alpha,\delta}}(x,t)) = L_t^x(S^{\alpha,\delta}) \text{ for all } (x,t) \text{ a.s.}$$

(c) *If* $\psi_\alpha(s) = n^{-2}(\alpha-1)^{-1}(\log \frac{1}{s})^{1-\alpha} \log(\log(\log \frac{1}{s}))$, $s < e^{-e}$, *then for* $\alpha > 2+\sqrt{2}$,

$$\psi_\alpha - m(Z_{S^{1,\alpha,0}}(x,t) = L_t^x(S^{1,\alpha,0}) \text{ for all } (x,t) \text{ a.s.}$$

REMARKS. (a) was proved by Perkins [24], but the argument used the Ray-Knight theorems on Brownian local time and hence does not extend to other Lévy processes.

(c) indicates that the scaling properties of the stable processes are not needed in the proofs. Indeed we feel confident that our methods will apply to a large class of Lévy processes. However, as the conditions on α are stronger than those needed to ensure continuity of $L(S^{1,\alpha,0})$, we are clearly unable to prove the theorem for every Lévy process with a continuous local time.

CONJECTURE 6.6. *If* X *is a Lévy process with a continuous local time* L_t^x, *and* ϕ *is the Hausdorff measure function giving* (31), *then*

$$\phi - m(Z_X(x,t)) = L_t^x \text{ for all } (x,t) \text{ a.s.}$$

There is another interesting intrinsic construction due to Kingman [19]. He showed that for any X_t with a local time, there is a suitable monotone function $\psi(h)$ such that, for fixed x, a.s.

(32) $$\psi(\varepsilon)|Z(x,t)(\varepsilon)| \to \ell_t^x \quad \text{for all } t > 0,$$

where $|E|$ denotes the Lebesgue measure of E, and $E(\varepsilon) = \{s: \exists\, u \in E \text{ with } |u-s| < \frac{\varepsilon}{2}\}$. There is no a priori reason why (32) should converge simultaneously at all levels, but again we can obtain a positive result if we keep away from the critical cases.

22. P.W. MILLAR, L.T. TRAN. Unbounded local times. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 30*, (1974), 87-92.

23. E. PERKINS. A global intrinsic characterisation of Brownian local time. *Annals Prob. 9*, (1981), 800-817.

24. E. PERKINS. The exact Hausdorff measure of the level sets of Brownian motion. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 58*, (1981), 373-388.

25. E. PERKINS. On the continuity of the local time of stable processes. To appear.

26. W.E. PRUITT, S.J. TAYLOR. The local structure of the sample paths of asymmetric Cauchy processes. To appear.

27. D.B. RAY. Sojourn times of a diffusion process. *Ill. J. Math. 1*, (1963), 615-630.

28. S.J. TAYLOR. Multiple points for the sample paths of the symmetric stable process. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 5*, (1966), 247-264.

29. S.J. TAYLOR. Sample path properties of processes with stationary independent increments, in *Stochastic Analysis*, 387-414. Wiley, London, 1973.

30. S.J. TAYLOR, J.G. WENDEL. The exact Hausdorff measure of the zero set of a stable process. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 6*, (1966), 170-180.

31. H.F. TROTTER. A property of Brownian motion paths. *Ill. J. Math. 2*, (1958), 425-433.

M.T. BARLOW
Statistical Laboratory
16 Mill Lane,
Cambridge
CB2 1SB
U.K.

S.J. TAYLOR
Dept. of Mathematics
University of Virginia,
Charlottesville,
VA 22903
U.S.A.

E.A. PERKINS
Dept. of Mathematics
University of
British Columbia,
Vancouver,
B.C.
Canada V6T 1Y4

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NOTES ON THE INHOMOGENEOUS SCHRÖDINGER EQUATION

by

K. L. CHUNG*

In [1] and [2] we discussed the solution of the homogeneous Schrödinger equation $\left(\frac{\Delta}{2} + q\right)u = 0$ with boundary condition. It is customary in classical analysis to treat this problem as equivalent to the solution of the corresponding inhomogeneous equation $\left(\frac{\Delta}{2} + q\right)u = \phi$ with vanishing boundary condition, by a simple substitution. However, sufficient smoothness of the given data is required for this method. It turns out that the probabilistic approach is easily adapted to the inhomogeneous case, via the potentials. Relatively mild assumptions are sufficient for the purpose. Whereas it is possible to treat the problem in a "purely analytic" setting based on old and new Green's functions, we follow a different route and carry out the calculations by integrations over time rather than over space.

Let D be a domain in R^d, $d \geq 1$, with $m(D) < \infty$, where m is the Lebesgue measure in R^d. No regularity assumption is imposed on ∂D. Define a class of functions, to be denoted by $L^*(D)$, as follows: $\phi \in L^*(D)$ iff ϕ is locally bounded in D and $\phi \in L^1(D,m)$. Then $L^*(D)$ is a linear space which admits the operation $\phi \to |\phi|$, and multiplication by a bounded measurable function.

Let q be a bounded Borel measurable function on R^d, $Q = \sup_{x \in R^d} |q(x)|$;

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THEOREM 6.7. (a) $\lim_{\varepsilon\to0^+} \sup_{x\in\mathbb{R},0\le t\le T} \left| \varepsilon^{-\frac{1}{2}} |Z_B(x,t)(\varepsilon)| - L_t^x(B) \right| = 0$ $\forall\, T > 0$ a.s.

(b) *for each* $\alpha \in (1,2)$ *there is a* $c_{\alpha,\delta}$ *such that*

$$\lim_{\varepsilon\to0^+} \sup_{x\in\mathbb{R},0\le t\le T} \left| c_{\alpha,\delta}\cdot\varepsilon^{-1/\alpha} |Z_{S^{\alpha,\delta}}(x,t)(\varepsilon)| - L_t^x(S^{\alpha,\delta}) \right| = 0$$

for all $T > 0$ *a.s.*

(c) *For each* $\alpha > 3$, *there is a* c_α *such that*

$$\lim_{\varepsilon\to0^+} \sup_{x\in\mathbb{R},0\le t\le T} \left| c_\alpha (\log \tfrac{1}{\varepsilon})^{1-\alpha}\varepsilon^{-1} |Z_{S^{1,\alpha,0}}(x,t)(\varepsilon)| - L_t^x(S^{1,\alpha,0}) \right| = 0$$

for all $T > 0$ *a.s.*

REMARK. (a) is proved in Perkins [23]. The proof of (b) and (c) is given in [3], where the interested reader may find the values of $c_{\alpha,\delta}$ and c_α.

References

1. M.T. BARLOW. Zero-one laws for the excursions and range of a Lévy process. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 55* (1981), 149-163.
2. M.T. BARLOW. Continuity of local times for Lévy processes. To appear.
3. M.T. BARLOW, E. PERKINS. Levels at which every Brownian excursion is exceptional. *Séminaire de Probabilités* XVIII, 1-28 (1984).
4. M.T. BARLOW, E. PERKINS, S.J. TAYLOR. Two uniform intrinsic constructions for the local time of a class of Lévy processes. To appear in *Ill. J. Math.*
5. A. BENVENISTE, J. JACOD. Systèmes de Lévy de processus de Markov. *Invent. Math. 21,* (1973), 183-198.
6. E.S. BOYLAN. Local times for a class of Markov processes. *Ill. J. Math. 8,* (1964), 19-39.
7. J. BRETAGNOLLE. Résultats de Kesten sur les processes à acroissements independents. *Séminaire de Probabilités V,* (1971), 21-36.
8. C. DELLACHERIE, P.A. MEYER. *Probabilités et Potentiel.* Vol. I, Hermann, Paris, 1975.
9. D. FREEDMAN. On tail probabilities for martingales. *Annals of Prob. 3,* (1975), 100-118.
10. B.E. FRISTEDT, W.E. PRUITT. Lower functions for increasing random walks and subordinators. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 18,* (1971), 167-182.
11. B.E. FRISTEDT, S.J. TAYLOR. Constructions of local time for a Markov process. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 62,* (1983), 73-112.
12. D. GEMAN, J. HOROWITZ. Occupation densities. *Annals Prob. 8,* (1980), 1-67.
13. R.K. GETOOR, H. KESTEN. Continuity of local times for Markov processes. *Compos. Math. 24,* (1972), 277-303.
14. R.K. GETOOR, P.W. MILLAR. Some limit theorems for local time. *Compos. Math. 25,* (1972), 123-134.
15. J. HAWKES. A lower Lipschitz condition for the stable subordinator. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 17,* (1971), 23-32.
16. K. ITÔ. Poisson point processes attached to Markov processes. *Proc. of Sixth Berkeley Symposium,* (1970), 225-239.
17. H. KESTEN. Hitting probabilities of single points for processes with stationary independent increments. *Memoir 93 AMS* (1969).
18. H. KESTEN. Lévy processes with a nowhere dense range. *Indiana Univ. Math. J. 25,* (1976), 45-64.
19. J.F.C. KINGMAN. An intrinsic description of local time. *J. London Math. Soc. 6,* (1973), 725-731.
20. F. KNIGHT. Random walks and the sojourn density process of Brownian motion. *Trans. Amer. Math. Soc. 107,* (1963), 56-86.
21. B. MAISONNEUVE. Systèmes régeneratifs. *Astérisque 15,* (1974).