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ON WALSH'S BROWNIAN MOTIONS †

by

Martin BARLOW
Trinity College
Cambridge CB2 1TQ
England

Jim PITMAN
Department of Statistics
University of California
Berkeley, California 94720
United States

Marc YOR
Laboratoire de Probabilités
Université Pierre et Marie Curie
4, Place Jussieu - Tour 56
75252 Paris Cedex 05, France

§1. Introduction.

This paper is concerned with the singular diffusions in the plane introduced by Walsh in the epilogue of [W]. Started at a point z in the plane away from the origin 0, such a process moves like a one dimensional Brownian motion along the ray joining z and 0 until it reaches 0. Then the process is kicked away from 0 by an entrance law which makes the radial part (R_t) of the diffusion a reflecting Brownian motion, while randomizing the angular part. For an intuitive description of how this happens we cannot better Walsh's account, which we now quote (with slight changes of notation) from [W, p. 44]:

"The idea is to take each excursion of R_t and, instead of giving it a random sign, to assign it a random variable Θ with a given distribution in $[0,2\pi)$, and to do this independently for each excursion. That is, if the excursion occurs during the interval (u,v), we replace R_t by the pair (R_t,Θ) for $u \le t < v$, Θ being a random variable with values in $[0,2\pi)$. This provides a process $\{(R_t,\Theta_t),\ t \ge 0\}$, where Θ_t is constant during each excursion from 0, has the same distribution as Θ, and is independent for different excursions. We then consider $Z_t = (R_t,\Theta_t)$ as a process in the plane, expressed in polar coordinates. It is a diffusion which, when away from the origin, is a Brownian motion along a ray, but which has what might be called a *round-house singularity* at the origin: when the process enters it, it, like Stephen Leacock's hero, immediately rides off in all directions at once."

Our interest in these processes arose from several sources.

(a) Let (F_t^Z) be the usual filtration of Z (i.e. the usual completion of the natural filtration of Z). Let $W_t = R_t - \frac{1}{2}L_t^0(R)$, where $L_t^0(R)$ stands for the local time at zero of the reflecting Brownian motion R up to time t. Then W is an (F_t^Z) Brownian motion, and, according to Theorem 4.1 below, every (F_t^Z) local martingale M is of the form

† Research supported in part by National Science Foundation Grant DMS88-01808

$$M_t = \int_0^t H_s \, dW_s,$$

for some $(\mathbf{F}_t^Z)$ previsible process H. An old problem of Yor [Y2, pp. 195-196, Questions 2 and 2'], in which there has recently been renewed interest, is whether every filtration of this type is generated by a Brownian motion. The filtration $(\mathbf{F}_t^Z)$ seems to us a good test case; we have been unable to find an $(\mathbf{F}_t^Z)$ Brownian motion U_t such that $\mathbf{F}_t^Z = \mathbf{F}_t^U$, but nor can we prove that no such Brownian motion exists. Moreover, the structure of $(\mathbf{F}_t^Z)$ at the random times when R leaves 0 makes the construction of any such Brownian motion hard to imagine. We discuss these questions in greater detail in Section 4.

(b) Let $G = (V, E)$ be a locally finite undirected graph. A random walk on G is a discrete time Markov chain Y_n with transition probabilities $P_{xy} = P(Y_n = y \mid Y_{n-1} = x)$ given by

$$P_{xy} = \begin{cases} 1/N(x) & \text{if } \{x,y\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Here $N(x)$ is the number of neighbours of x. It is sometimes useful to embed Y_n in a continuous process X_t. A natural way to do this is to use the 'cable system' of Varopoulos [V] (see also Frank and Durham [FD], Baxter and Chacon [BC]): each pair of vertices x, y with $\{x,y\} \in E$ is joined by a line of length 1. Let $\bar{G}$ be the resulting set. A diffusion X_t is defined on $\bar{G}$ as follows. On the interior of each cable, X performs Brownian motion until it reaches an end point. When X is at a vertex x, it makes excursions along each of the cables joining x to other vertices with equal probability. Thus X has the same behaviour as Walsh's process in a neighborhood of each vertex.

(c) Walsh's process on a finite number of rays also emerges from study of the asymptotic behaviour of windings and crossing numbers of ordinary planar Brownian motion. This is indicated briefly in Section 5.

(d) Study of the joint distribution of the time spent by a Walsh process up to time t in various sectors of the plane leads to some interesting extensions of Lévy's arcsine law for the time spent positive up to t by a one dimensional Brownian motion. We refer to [BwPY1,2] for these developments.

Due to our interest in Walsh's process in these various settings, we thought it worthwhile to record here some basic facts about the process. Section 2 briefly surveys some approaches to construction of a Walsh process Z, then goes into details of one construction in particular, via the semigroup. In Section 3 we look at the martingale problem description of Z in the case when the process lives on a finite number of rays. In Section 4 we study the filtration of Z. Using general martingale representation theorems we deduce from the results of Section 3 that every $(\mathbf{F}_t^Z)$ martingale is a stochastic integral of $W_t = R_t - \frac{1}{2}L_t^0(R)$. We conclude Section 4 with a discussion of some open problems.

§2. Construction.

Constructions of Walsh's Brownian motions have been given by Rogers [R] (using resolvents), Baxter and Chacon [BC] (from the infinitesimal generator), Varopoulos [V] (using Dirichlet space techniques), and Salisbury [Sa] (using excursion theory). These approaches all rely on a fair amount of background machinery. A more elementary approach is to use the intuitive description of the process to write down the semigroup, then check that this semigroup gives rise to a diffusion. This was suggested by Walsh [W] in the case of skew Brownian motion, when the process lives on just two rays, and sketched also by Frank and Durham [FD] for 3 rays and Van der Weide [VW] for n rays. This section goes into this construction in detail.

Let $E = \mathbf{R}^2$; we will use polar co-ordinates (r, θ) to denote points in E. Let μ be a fixed probability measure on $[0, 2\pi)$. This μ will be the distribution of angles Θ in Walsh's description. To see what the semigroup must be, accept for a moment the existence of Walsh's process $Z_t = (R_t, \Theta_t)$, started at (r, θ). According to Walsh's description, R_t must be a reflecting Brownian motion starting at r. Starting with $r = 0$, angle Θ_t must have distribution μ independent of R_t, for each $t > 0$. Starting at (r, θ) with $r > 0$ however, Θ_t must equal θ if $\tau_0 > t$, where $\tau_0 = \inf\{t \ge 0 : R_t = 0\}$; and given $\tau_0 < t$, Θ_t must be randomized according to μ independently of R_t. This describes the distribution of $Z_t = (R_t, \Theta_t)$ given $Z_0 = (r, \theta)$ for any $t > 0$, hence the semigroup $(P_t, t \ge 0)$ of Walsh's process with angular distribution μ.

To describe the semigroup more compactly, we introduce the following notation. For a function $f \in C(E)$ we define functions $\bar{f}, f_\theta$ in $C(\mathbf{R}_+)$ by

$$\text{(2.1)} \qquad \begin{aligned} \bar{f}(r) &= \int f(r, \theta)\, \mu(d\theta), && r \ge 0 \\ f_\theta(r) &= f(r, \theta), && r \ge 0, \quad \theta \in [0, 2\pi). \end{aligned}$$

Let T_t^+, $t \ge 0$ be the semigroup of reflecting Brownian motion on $\mathbf{R}_+$, and T_t^0, $t \ge 0$ be the semigroup of Brownian motion on $\mathbf{R}_+$ killed at 0. Let $t > 0$. Then it is easy to see that P_t described above must act as follows on $f \in C_0(E)$:

$$\text{(2.2)} \qquad \begin{aligned} P_t f(0, \theta) &= T_t^+ \bar{f}(0). \\ P_t f(r, \theta) &= T_t^+ \bar{f}(r) + T_t^0 (f_\theta - \bar{f})(r), && r > 0, \ \theta \in [0, 2\pi). \end{aligned}$$

We take (2.2) as the formal definition of P_t. We now mention two possible methods to rigorously establish the existence of a continuous path strong Markov process satisfying Walsh's description.

Either

(i) take the process created by Walsh's sample path construction, starting with a reflecting Brownian motion and an infinite sequence of independent angles with common distribution μ, and show it has continuous paths and the strong Markov property with semigroup (P_t).

Or

(ii) establish credentials of the semigroup (P_t) which ensure by general theory that the canonical presentation of the process with that semigroup has the strong Markov property. Then show this canonical process has continuous paths fitting Walsh's description for the radial and angular parts.

Method (i) is at first quite attractive. It is elementary that Z made this way has continuous paths. And, by a variation of the argument used already to derive P_t; it is easy to show that this Z is Markov at fixed times with semigroup (P_t). But it seems hard to make a convincing argument for the strong Markov property of Z without first establishing credentials of P_t and proceeding more or less according to method (ii).

So in the end it seems simplest just to employ method (ii) from the start, which we now proceed to do.

Theorem 2.1 *$(P_t, t \geq 0)$ is a Feller semigroup on $C_0(E)$.*

Proof. We will check:

(i) $P_t : C_0(E) \to C_0(E)$.

(ii) If $f \in C_0(E)$ and $0 \leq f \leq 1$ then $0 \leq P_t f \leq 1$.

(iii) P_0 is the identity on $C_0(E)$, and $P_t P_s = P_{t+s}$ for $s, t \geq 0$.

(iv) $\lim_{t \downarrow 0} \| P_t f - f \|_\infty = 0$ for all $f \in C_0(E)$.

(i). Let $f \in C_0(E)$, and note that $\bar{f}, f_\theta \in C_0(\mathbb{R}_+)$. Let $(r,\theta) \in E$. Then, if $r > 0$, we have

$$|P_t f(r,\theta) - P_t f(r',\theta')| \leq |T_t^+ \bar{f}(r) - T_t^+ \bar{f}(r')| + |T_t^0 \bar{f}(r) + T_t^0 \bar{f}(r')| + |T_t^0 f_\theta(r) - T_t^0 f_\theta(r')| + |T_t^0(f_\theta - f_{\theta'})(r')|.$$

The first three terms converge to 0 as $(r',\theta') \to (r,\theta)$ by the Feller property of (T_t^+) and (T_t^0). For the final term, we have

$$|T_t^0(f_\theta - f_{\theta'})(r')| \leq \|f_\theta - f_{\theta'}\|_\infty,$$

and as $f \in C_0(E)$ this last term converges to 0 as $\theta' \to \theta$. If $r = 0$ then

$$|P_t f(0,\theta) - P_t f(r',\theta')| \leq |T_t^+ \bar{f}(0) - T_t^+ \bar{f}(r')| + |T_t^0(f_{\theta'} - \bar{f})(r')|, \leq |T_t^+ \bar{f}(0) - T_t^+ \bar{f}(r')| + 2\|f\|_\infty T_t^0 1(r').$$

As $T_t^0 1(r') \to 0$ as $r' \to 0$, we deduce that $P_t f$ is continuous at 0. We have proved that $P_t f \in C(E)$. However as $\bar{f}$ and $\sup_\theta f_\theta$ both vanish at infinity, $T_t^+ \bar{f}$, $T_t^0 \bar{f}$ and $\sup_\theta T_t^0 f_\theta$ all vanish at infinity, and so $P_t f \in C_0(E)$.

(ii). If $f \geq 0$ then $T_t^+ \bar{f} - T_t^0 \bar{f} \geq 0$, and so $P_t f \geq 0$. It is easily checked that $P_t 1 = 1$, and by the linearity of P_t it follows that $\|P_t f\|_\infty \leq \|f\|_\infty$, proving (ii).

(iii). That P_0 is the identity is immediate from the corresponding property of T_0^+ and T_0^0. By the linearity of T_t^+, T_t^0 we have

$$\overline{P_t f}(r) = T_t^+ \bar{f}(r) - T_t^0 \bar{f}(r) + \int (T_t^0 f_\theta(r))\mu(d\theta) \tag{2.3}$$

$$= T_t^+ \bar{f}(r),$$

and

$$(P_t f)_\theta(r) = \overline{P_t f}(r) = T_t^0(f_\theta - \bar{f})(r). \tag{2.4}$$

So, using the semigroup property for (T_t^0), (T_t^+),

$$\begin{aligned} P_s P_t f(r,\theta) &= T_s^+ \overline{P_t f}(r) + T_s^0((P_t f)_\theta - \overline{P_t f})(r) \\ &= T_s^+ T_t^+ \bar{f}(r) + T_s^0 T_t^0(f_\theta - \bar{f})(r) \\ &= T_{s+t}^+ \bar{f}(r) + T_{s+t}^0(f_\theta - \bar{f})(r) = P_{s+t} f(r,\theta). \end{aligned}$$

(iv). Since (i)-(iii) hold, it is sufficient to check that

$$\lim_{t \downarrow 0} P_t f(r,\theta) = f(r,\theta) \text{ for each } f \in C_0(E), (r,\theta) \in E.$$

So let $f \in C_0(E)$, $(r,\theta) \in E$. Then, using the corresponding property for T_t^+ and T_t^0, we have, if $r > 0$,

$$\begin{aligned} \lim_{t \downarrow 0} P_t f(r,\theta) &= \lim_{t \downarrow 0} T_t^+ \bar{f}(r) + \lim_{t \downarrow 0} T_t^0 f_\theta(r) \\ &= \bar{f}(r) + f_\theta(r) \\ &= f(r,\theta). \end{aligned}$$

Similarly, $\lim_{t \downarrow 0} P_t f(0,\theta) = \lim_{t \downarrow 0} T_t^+ \bar{f}(0) = \bar{f}(0) = f(0,\theta)$. □

Remark. Note that P_t is not strong Feller. For example, if $f(r,\theta) = 1_{(0<r<1)} 1_{(\theta=0)}$, then $P_t f$ is always discontinuous away from the origin on the line $\theta = 0$. But if μ is concentrated on a finite set $\{\theta_1, \ldots, \theta_n\}$ then we may define P_t on the reduced state space

$$E_n = \{(r,\theta_i), r \geq 0, 1 \leq i \leq n\},$$

and on this space P_t is strong Feller.

Using the general theory we may now define a strong Markov process $(\Omega, \mathrm{F}, \mathrm{F}_t, Z_t, \theta_t, P^z)$ with state space E and semigroup P_t, and such that Z is cadlag. We write $Z_t = (R_t, \Theta_t)$, and set $\Theta_t = 0$ if $R_t = 0$. We now proceed to argue that Z_t is a diffusion with the features of Walsh's process described in the introduction.

Lemma 2.2. *R_t is a reflecting Brownian motion $/((\mathrm{F}_t), P^z)$, for any $z \in E$.*

Proof. Fix $z \in E$. Let $g \in C_0(\mathbb{R}_+)$, and set $f(r,\theta) = g(r)$. Thus $f \in C_0(E)$, $\bar{f} = g$, $f_\theta - \bar{f} = 0$, and $f(Z_t) = g(R_t)$. Let S be any (F_t)-stopping time. As Z is strong Markov

$$E^z(g(R_{S+t}) | \mathrm{F}_S) = E^z(f(Z_{S+t}) | \mathrm{F}_S)$$

$$= P_t f(Z_S)$$
$$= T_t^+ g(R_S).$$

Thus R is strong Markov $/((F_t), P^z)$, and has semigroup T_t^+. Thus R is a reflecting Brownian motion $/((F_t), P^z)$. □

Lemma 2.2 shows that R is a.s. continuous. A little more work shows that so is Z.

Given any process X on E or a subset of E, we set

$$\tau_0(X) = \inf\{t \geq 0 : X_t = 0\},$$

and we write $\tau_0 = \tau_0(Z)$.

Lemma 2.3.

(a) For $g \in C_0(\mathbf{R}_+)$,

$$E^{(r,\theta)} 1_{(\tau_0 \leq t)} T^+_{t-\tau_0} g(0) = T_t^+ g(r) - T_t^0 g(r).$$

(b). For $f \in C_0(E)$,

$$E^{(r,\theta)} 1_{(\tau_0 > t)} f(Z_t) = T_t^0 f_\theta(r)$$

(c). Θ_t *is constant on* $[0, \tau_0)$, P^z *a.s. for each* $z \in E$.

Proof. Since $\tau_0 = \tau_0(R)$, and R is a reflecting Brownian motion, (a) is evident.

For (b) we have

$$E^{(r,\theta)} f(Z_t) 1_{(\tau_0 \leq t)} = E^{(r,\theta)} 1_{(\tau_0 \leq t)} P_{t-\tau_0} f(Z_{\tau_0})$$
$$= E^{(r,\theta)} 1_{(\tau_0 \leq t)} T^+_{t-\tau_0} \bar{f}(0)$$
$$= T_t^+ \bar{f}(r) - T_t^0 \bar{f}(r),$$

by (a). Subtracting this last equality from (2.2), we deduce (b).

(c). If $z = 0$ then $\tau_0 = 0$, P^z a.s., and there is nothing to prove. So let $z_0 = (r, \theta_0) \neq 0$ be fixed. Let $\tau_\varepsilon = \inf\{t \geq 0 : R_t \leq \varepsilon\}$. As Z is cadlag and R is continuous, Θ_t is cadlag on $[0, \tau_\varepsilon]$, and since $\tau_0 = \lim_{\varepsilon \downarrow 0} \tau_\varepsilon$, it is enough to show that Θ_t is constant on $[0, \tau_\varepsilon]$ for each $\varepsilon > 0$. As Θ_t is cadlag on $[0, \tau_\varepsilon]$, it is therefore sufficient to prove that, for each $t > 0$, $P^{z_0}(\Theta_t \neq \theta, \tau_0 > t) = 0$. Let $f \in C_0(E)$, with $f > 0$ on $E - \{(r, \theta_0), r > 0\}$ and $f(r, \theta_0) = 0$ for $r \in (0, \infty)$. Then, by (b), as $f_{\theta_0} = 0$,

$$E^{(r_0, \theta_0)} 1_{(\tau_0 > t)} f(Z_t) = 0,$$

and hence $P^{z_0}(\Theta_t \neq \theta, \tau_0 > t) = 0$.

Theorem 2.4. Z_t, $t \geq 0$ *is* P^z *a.s. continuous.*

Proof. Using the strong Markov property of Z, Lemma 2.3 (c) implies Θ is a.s. constant on each excursion of R from 0. This, together with the continuity of R, implies that Z is continuous. □

Putting Theorems 2.1 and 2.4 together we deduce

Corollary 2.5. Z_t, $t \geq 0$, *is a Feller diffusion on* E.

To finish the job of matching this process Z with Walsh's description, it should be argued that the angles associated with different excursions of R away from zero are independent with common distribution μ. But now that the strong Markov property of Z has been established, this follows from the excursion theory of Itô [I], after using Lemmas 2.2 and 2.3(c) to show that for the excursions of Z away from 0, Itô's characteristic measure of excursions is

(2.5)
$$n = \int_0^{2\pi} n_\theta \mu(d\theta)$$

where n_θ is Itô's law for excursions away from 0 of a reflecting Brownian motion on the ray at angle θ. We leave details of this argument to the reader.

Remark. In the construction of Z given above we have not used any special properties of reflecting Brownian motion. The whole argument carries over to the case where T_t^+ is the semigroup of a Feller diffusion X on $\mathbf{R}^+$, and T_t^0 is the semigroup for X killed at 0. In particular, the discussion applies in case T_t^+ is the semigroup of a Bessel process of dimension $\delta \in (0, 2)$, which is our setup in [BwPY1]. Moreover, apart from the continuity results of Lemma 2.2 (b) and Theorem 2.4, the arguments and results are valid for any Feller process X on $\mathbf{R}^+$.

§3. The Martingale Problem for Z.

In this section we restrict our attention to the case

(3.1) μ assigns probabilities $p_1, \cdots, p_n$ to distinct angles $\theta_1, \ldots, \theta_n$

where $\Sigma_i p_i = 1$ and $p_i > 0$ for $i = 1, \cdots, n$. We take as state space for Z the set $E_n = \{(r, \theta_i), r > 0, 1 \leq i \leq n\}$. Let $q_i = 1 - p_i$, and define

$$h_i(r, \theta) = 1_{(r>0)}[\, q_i 1_{(\theta = \theta_i)} - p_i 1_{(\theta \neq \theta_i)} \,],$$
$$g_i(r, \theta) = r\, h_i(r, \theta).$$

Set $W_t = R_t - \frac{1}{2} L_t^0(R)$, so W is a Brownian motion.

Proposition 3.1. *Let $z \in E_n$. For $1 \le i \le n$, $g_i(Z_t)$ is a P^z martingale, with*

$$g_i(Z_t) = g_i(z) + \int_0^t h_i(Z_s)\,dW_s. \tag{3.2}$$

Furthermore,

$$g_i(Z_t)^2 - \int_0^t h_i(Z_s)^2 ds \quad \text{is a } P^z \text{ martingale}.$$

Proof. Write $f(r) = r$, $r \ge 0$, and note that $T_t^0 f(r) = r$. So, using the Markov property of Z at time s with $s < t$,

$$\begin{aligned} E^z(g_i(Z_t)\mid \mathbf{F}_s) &= P_{t-s}\,g_i(Z_s) \\ &= h_i(Z_s)\,T_{t-s}^0 f(R_s) \qquad \text{by (2.2), since } \bar{g}_i = 0, \\ &= R_s\,h_i(Z_s) = g_i(Z_s). \end{aligned}$$

Thus $g_i(Z_t)$ is a martingale.

Note that $h_i(Z)$ is constant on the excursions of R from 0. So, by the formula for balayage of semimartingales [MSY Théorème 1], [E, Théorème 2] we can write

$$g_i(Z_t) = h_i(Z_t)R_t = g_i(z) + \int_0^t h_i(Z_s)\,1_{(R_s>0)}\,dR_s + A_t,$$

where A_t is a previsible finite variation process constant on the excursions of R from 0. Now, $g_i(Z_t)$ is a martingale, $1_{(R_s>0)}\,dR_s = 1_{(R_s>0)}dW_s$, and all these processes are continuous, so $A = 0$, and (3.2) follows on noting that $h_i(0) = 0$. The final part of the proposition is immediate from (3.2). □

We now present the law of Z as the solution of a martingale problem. Let $\Omega_0 = C(\mathbb{R}_+, E_n)$, Z be the co-ordinate maps on Ω_0, and $(\mathbf{F}_t^0)$ be the natural filtration of Z. We write $(P^z, z \in E_n)$ for the family of probability measures on Ω_0 corresponding to the semigroup P_t defined by (2.2).

Let $z_0 \in E_n$, and consider the following martingale problem for a probability Q on $(\Omega_0, \mathbf{F}_\infty^0)$:

$$\begin{cases} Q(Z_0 = z_0) = 1 \\ g_i(Z_t) \text{ and } g_i(Z_t)^2 - \int_0^t h_i(Z_s)^2 ds \quad \text{are } (Q, (\mathbf{F}_t^0)) \text{ martingales}, \; i = 1, \cdots, n. \end{cases} \tag{3.3}$$

Theorem 3.2. *The martingale problem (3.3) has exactly one solution, which is P^{z_0}.*

Proof. By Proposition 3.1, the probability P^{z_0} is a solution to (3.3).

Now let Q be any solution. As Z and g_i are continuous, the martingales $g_i(Z_t)$, $g_i(Z_t)^2 - \int_0^t h_i(Z_s)^2 ds$ are continuous. Set $Y_t^i = g_i(Z_t)$, and let

$$U_t^i = \int_0^t (q_i^{-1} 1_{(Y_s^i>0)} + p_i^{-1} 1_{(Y_s^i \le 0)})\,dY_s^i. \tag{3.4}$$

Then U^i is a martingale $/Q$, and

$$< U^i >_t = \int_0^t (q_i^{-2} 1_{(Y_s^i>0)} + p_i^{-2} 1_{(Y_s^i \le 0)})\,h_i(Z_s)^2 ds = t.$$

So U^i is a Brownian motion $/(Q, (\mathbf{F}_t^0))$. Set $\phi_i(x) = q_i 1_{(x>0)} + p_i 1_{(x \le 0)}$. From (3.4) we have, for $0 < s \le t$,

$$Y_t^i = Y_s^i + \int_s^t \phi_i(Y_v^i)\,dU_v^i. \tag{3.5}$$

This SDE has a pathwise unique solution ([N], [L]), and hence a solution which is unique in law. Write $r_i(s, t, y_0, dy)$ for the law of Y_t^i obtained by solving (3.5) with $Y_s^i = y_0$; this r_i does not depend on Q. Let $f \in C_0(E_n)$; we can easily check that

$$f(z) = \sum_{i=1}^n f_{\theta_i}(q_i^{-1} g_i(z))\,1_{(g_i(z)>0)} + f(0)\,1_{(g_1(z)=0)}.$$

So

$$f(Z_t) = \sum_{i=1}^n \psi_i(Y_t^i),$$

where $\psi_1, \ldots, \psi_n$ are bounded measurable functions. Then

$$\begin{aligned} E^Q(f(Z_t)\mid \mathbf{F}_s^0) &= \sum_{i=1}^n E^Q(\psi_i(Y_t^i)\mid \mathbf{F}_s^0) \\ &= \sum_{i=1}^n \int r_i(s, t, g_i(Z_s), dy_i)\,\psi_i(y). \end{aligned}$$

So, if Q and Q' are both solutions to (3.3), then for any $f \in C_0(E_n)$ and $0 < s < t$ we have $E^Q(f(Z_t)\mid \mathbf{F}_s^0) = E^{Q'}(f(Z_t)\mid \mathbf{F}_s^0)$. A standard argument, considering products of the form $\prod_{i=1}^n f_i(Z_{t_i})$ now shows that $Q = Q'$.

§4. The Filtration of Z.

We continue in this section to consider a Walsh process Z on n rays. Let $(\Omega, \mathbf{F}, \mathbf{F}_t, P)$ be a probability space carrying the process Z with $Z_0 = 0$. Given a process X we write $(\mathbf{F}_t^X)$ for the usual filtration of X. Write $\mathbf{M}_{loc}$ for the space of $(\mathbf{F}_t^Z)$ local martingales null at 0, and let $W_t = R_t - \frac{1}{2}L_t^0(R)$ be as in Section 3. Given $K \subseteq \mathbf{M}_{loc}^2$, let $\mathbf{L}(K)$ be the stable subspace of $\mathbf{M}_{loc}^2$ generated by K.

Theorem 4.1. *The Brownian motion W has the martingale representation property for $(\mathbf{F}_t^Z)$. That is, for each $M \in \mathbf{M}_{loc}$ there exists an $(\mathbf{F}_t^Z)$ previsible process H such that*

$$M_t = \int_0^t H_s \, dW_s .$$

Proof. Set $K_1 = \{g_i(Z_t),\, g_i(Z_t)^2 - \int_0^t h_i(Z_s)^2 ds,\ i = 1, \cdots, n\}$, $K_2 = \{W\}$. Theorem 3.2 shows that the martingale problem corresponding to K_1 has a unique solution, and hence, by Jacod [J], we have $\mathbf{L}(K_1) = \mathbf{M}_{loc}^2$.

On the other hand, by Proposition 3.1 the processes $g_i(Z_t)$ and $g_i(Z_t)^2 - \int_0^t h_i(Z_s)^2 ds$ are both stochastic integrals of W. Thus $K_1 \subseteq \mathbf{L}(K_2)$, and so $\mathbf{L}(K_2) = \mathbf{M}_{loc}^2$. The result now follows from the general theory presented in Yor [Y0]. □

Remarks.

1. Note that $\mathbf{F}_t^R = \mathbf{F}_t^W$, a classical result of Skorokhod.

2. Although W has the martingale representation property for $(\mathbf{F}_t^Z)$, it is clear that $\mathbf{F}_t^W \neq \mathbf{F}_t^Z$. For example, the random variable Θ_1 is $\mathbf{F}_1^Z$ measurable, but not $\mathbf{F}_1^W$ measurable. (In fact, Θ_1 is independent of $\mathbf{F}_1^W$).

In Theorem 4.1 we showed that the filtration $(\mathbf{F}_t) = (\mathbf{F}_t^Z)$ has the following property: there is an $(\mathbf{F}_t)$ Brownian motion W_t such that every $(\mathbf{F}_t)$ martingale is a stochastic integral of W_t. We say such a filtration has the *Brownian representation property*. This property (and, more generally, the multiplicity of a filtration in the sense of Davis and Varaiya [DV]) may be thought of as 'invariants' of that filtration. That is, they are intrinsic to the filtration (and the probability measure on it), and do not depend on any representation of the filtration as the natural filtration of some process. (An example of quantities which are not 'intrinsic' are the θ shifts in Markov process theory).

To introduce another 'invariant' of a filtration, we consider first a notion of relative multiplicity of two σ-fields. Let $(\Omega, \mathbf{G}, P)$ be a probability space, and $\mathbf{F} \subseteq \mathbf{G}$ be a sub-σ-field of $\mathbf{G}$ which contains all $\mathbf{G}$ measurable P null sets. For $g_1, \ldots, g_k$ in $L^2(\mathbf{G})$ define

$$< g_1, g_2, \ldots, g_k > \; = \; \text{closure}\,\{\sum_{i=1}^k f_i g_i,\ f_i \in L^\infty(\mathbf{F})\},$$

where the closure is taken in $L^2(\mathbf{G})$. With the convention $\inf\{\phi\} = +\infty$, let

$$m_1(\mathbf{G}\,|\,\mathbf{F}) = \inf\{n : L^2(\mathbf{G}) = < g_1, \ldots, g_n > \text{ for } g_1, \ldots, g_n \in L^2(\mathbf{G})\};$$

$$m_2(\mathbf{G}\,|\,\mathbf{F}) = \inf\{n : L^2(\mathbf{G}) = < g_1, \ldots, g_n > \text{ for } g_1, \ldots, g_n \in L^2(\mathbf{G}) \text{ with } E(g_i g_j\,|\,\mathbf{F}) = 0 \text{ for } i \neq j\};$$

$$m_3(\mathbf{G}\,|\,\mathbf{F}) = \inf\{n : \text{there exists a partition } A_1, \ldots, A_n \text{ of } \Omega \text{ such that } \mathbf{G} = \mathbf{F} \vee \sigma(A_1, \cdots, A_n)\};$$

$$m_4(\mathbf{G}\,|\,\mathbf{F}) = \sup\{n : \text{there exists a } \mathbf{G} \text{ measurable partition } A_1, \ldots, A_n \text{ of } \Omega \text{ such that } P(\,P(A_i\,|\,\mathbf{F}) > 0 \text{ for } i = 1, 2, \cdots, n) > 0\}.$$

Elementary but tedious arguments show that $m_1 = m_2 = m_3 = m_4$. We call the common value of the m_i the multiplicity of $\mathbf{G}$ over $\mathbf{F}$, and write it as *mult* $(\mathbf{G}\,|\,\mathbf{F})$.

Now given a filtration $(\mathbf{F}_t)$, for an $(\mathbf{F}_t)$ optional set Γ, denote

$$L_\Gamma = \sup\{t : (\omega, t) \in \Gamma\},$$

the 'end of Γ'. Recall the definitions, for a random time L,

$$\mathbf{F}_L = \sigma(X_L : X \text{ is a bounded } (\mathbf{F}_t) \text{ optional process}),$$

$$\mathbf{F}_{L+} = \sigma(X_L : X \text{ is a bounded } (\mathbf{F}_t) \text{ progressive process}).$$

Definition 4.2. The *splitting multiplicity* of a filtration $(\mathbf{F}_t)$ is defined by

$$sp\ mult\ ((\mathbf{F}_t)) = \sup_\Gamma mult\ (\mathbf{F}_{L_\Gamma +}\,|\,\mathbf{F}_{L_\Gamma}),$$

where the supremum is taken over all $(\mathbf{F}_t)$ optional sets Γ.

Proposition 4.3. *Let Z be the Walsh process on E_n as in Sections 3 and 4. Then*

$$sp\ mult\ ((\mathbf{F}_t^Z)) \geq n .$$

Proof. Let $T = \inf\{s \geq 0 : R_s = 1\}$, and set

$$L = \sup\{s \leq T : R_s = 0\}, \quad A_i = \{\Theta_T = \theta_i\}, \quad 1 \leq i \leq n .$$

Then, as $\Theta_t = \Theta_T$ for $L < t < T$, $A_i \in \mathbf{F}_{L+}$. Fix i, let g_i be as for Proposition 3.1, and let

$$M_t = g_i(Z_{t \wedge T}).$$

Then M is a uniformly integrable $(\mathbf{F}_t^Z)$ martingale, and $M_L = 0$. So, by Yor [Y1],

$$E(M_T\,|\,\mathbf{F}_L) = 0 .$$

It follows that $P(A_i\,|\,\mathbf{F}_L) = p_i > 0$ for $1 \leq i \leq n$. So $m_4(\mathbf{F}_{L+}\,|\,\mathbf{F}_L) \geq n$. □

Remark. In fact, if L is as above, we have that $\mathbf{F}_{L+} = \mathbf{F}_L \vee \sigma(A_1, \ldots, A_n)$, so that *mult* $(\mathbf{F}_{L+}\,|\,\mathbf{F}_L)$ is exactly n. While this seems intuitively obvious, a formal proof needs some care (see appendix).

We conclude this section with some open problems.

Problem 1. Let B_t be a Brownian motion. What is *sp mult* $((\mathbf{F}_t^B))$?

This problem seems to us very hard. The trivial bound *sp mult* $((\mathbf{F}_t^B)) \geq 2$ implied by Proposition 4.3 uses the very simplest last exit times. But none of the various classes of ends of optional sets we have considered does any better. In particular, Millar [M, Corollary 4.2] shows that for L the last zero before an arbitrary random time, *mult* $(\mathbf{F}_{L+}^B | \mathbf{F}_L^B)$ is at most 2.

We also remark that, for a Brownian motion X_t in $\mathbb{R}^d$, we only know that *sp mult* $((\mathbf{F}_t^X)) \geq 2$.

Problem 2. For the Walsh process Z on n rays, does there exist an $(\mathbf{F}_t^Z)$ Brownian motion B such that $(\mathbf{F}_t^Z) = (\mathbf{F}_t^B)$?

This is a special case of a problem posed by Yor [Y2]:

Problem 3. Given a filtration $(\mathbf{G}_t)$ with the Brownian representation property and with $\mathbf{G}_0$ trivial, does there exist a $(\mathbf{G}_t)$ Brownian motion B_t such that $(\mathbf{G}_t) = (\mathbf{F}_t^B)$?

Problems 1 and 2 are clearly related: if *sp mult* $((\mathbf{F}_t^B)) = 2$ then $(\mathbf{F}_t^Z)$ cannot be a Brownian filtration whenever $n > 2$. (For $n = 2$, $(\mathbf{F}_t^Z)$ is Brownian, by the result of Harrison and Shepp [HS]). On the other hand, if the answer to Problem 2 is 'yes', then the last exits from 0 of Z would be ends of optional sets for B with *mult* $(\mathbf{F}_{L+} | \mathbf{F}_L) = n$. Thus a positive answer to Yor's problem for the filtration $(\mathbf{F}_t^Z)$ would give rise to an interesting class of random times for the Brownian filtration.

On the other hand, consideration of splitting multiplicities may not be essential for resolution of Problem 3. If we consider a process whose law is locally equivalent to that of Brownian motion, then the Brownian motion found by Girsanov's formula has the representation property. The splitting multiplicity of this process for its own filtration will be the same as for Brownian motion. But even in this case, for instance in Tsirelson's example, we have no affirmative solution to problem 3; (see Stroock and Yor [SY] for further discussion). Related problems are discussed by Skorokhod [Sk], where an affirmative solution to Problem 3 is announced.

§5. A Walsh process associated with planar Brownian motion.

We sketch in this section how a Walsh process turns up in the study of windings and crossings of planar Brownian motion, undertaken in [PY] and [ByPY]. To match notation with [PY], $(Z_t, t \geq 0)$ will now denote a Brownian motion in the complex plane, rather than a Walsh process. Let $z_1, \ldots, z_n$ be n distinct points in the plane, distinct also from the starting point z_0 of Z. Take numbers $r_1, \ldots, r_n > 0$, and $r_\infty > |z_0|$. And consider the $2n$ additive functionals

$$U_\pm^j(t) = \int_0^t \frac{1(|Z_s - z_j| \in I_\pm^j)}{|Z_s - z_j|^2}\, ds,$$

where $\pm$ is $+$ or $-$, $j = 1, 2, \cdots, n$, and $I_-^j = (0, r_j)$, $I_+^j = (r_j, \infty)$. These are the increasing processes (or *clocks*) associated with the $2n$ conformal martingales

$$G_\pm^j(t) + i\Phi_\pm^j(t) = \int_0^t \frac{1(|Z_s - z_j| \in I_\pm^j)}{(Z_s - z_j)}\, dZ_s.$$

As argued in Section 6 of [PY], the U_+^j are asymptotically equivalent a.s. as $t \to \infty$. Indeed,

$$\frac{U_+^j(t)}{U^\infty(t)} \xrightarrow{a.s.} 1,$$

where

$$U^\infty(t) = \int_0^t \frac{1(|Z_s| > r_\infty)}{|Z_s|^2}\, ds.$$

The joint asymptotic limit behaviour as $t \to \infty$ of the $U_\pm^j(t)$ was discussed in [ByPY], in the framework of log scaling laws for planar Brownian motion developed in Section 8 of [PY]. Now write just $U^j(t)$ for $U_-^j(t)$, $j = 1, \ldots, n$, and let $L(t)$ be any additive functional of the planar Brownian motion Z with $\|L\| = 2\pi$, and let $h(t) = 1/(2\log t)$. Then

$$\left[\frac{U^j(t)}{h^2(t)}, j = 1, \ldots, n;\ \infty;\ \frac{L(t)}{h(t)}\right] \underset{t\to\infty}{\xrightarrow{d}} \left[A_j(\sigma_*), j = 1, \ldots, n, \infty;\ \frac{\lambda(\sigma_*)}{n+1}\right]$$

where the random variables appearing on the right may be defined as follows, in terms of a Walsh process started at zero which moves with equal probabilities along each of $n + 1$ rays labelled $j = 1, \ldots, n, \infty$: the time when the Walsh process first reaches modulus 1 on ray ∞ is σ_*; the occupation time of ray j by the Walsh process up to time σ_* is $A_j(\sigma_*)$; and $\lambda(\sigma_*)$ is the local time at 0 of the radial part of the Walsh process up to time σ_*. This is just a paraphrase of Lemma 4.3 of [ByPY], due to the following consequences of the excursion theory of Itô [I], (or the method of Section 5 of [PY]), applied to the Walsh process. Firstly, for each j the joint distribution of $A_j(\sigma_*)$, $A_\infty(\sigma_*)$, and $\lambda(\sigma_*)/(n+1)$ can be described, in terms of a standard Brownian motion B up to the time σ when B first hits 1, as that of the time B spent negative before σ, the time B spent positive before σ, and the local time of B at zero before σ. And secondly, given $\lambda(\sigma_*)$, the variables $A_j(\sigma_*)$ are mutually independent.

Similarly, if the processes U_j are evaluated at

$$\tau_h = \inf\{t : L_t = h\},$$

for any additive functional L of Z with $\|L\| = 2\pi$, then we get

$$\left[\frac{U^j(\tau_h)}{h^2}, j = 1, \ldots, n, \infty\right] \xrightarrow{d} (A_j(\tau_*), j = 1, \ldots, n, \infty)$$

where $\tau_* = \inf\{u : \lambda(u) = n+1\}$. The right side is now a vector of $n + 1$ independent stable (1/2) random variables. As in [ByPY], we could replace the U^j by suitable processes counting crossing numbers, and the asymptotics would be the same.

The question now arises: what time should we look at to get say $(A_j(1), j = 1, \ldots, n, \infty)$ as the limit? The answer would seem to be

$$\alpha_u = \inf\{t : U^{\Sigma}(t) = u\}$$

where

$$U^{\Sigma}(t) = \sum_{j \in \{1,\ldots,n,\infty\}} U^j(t).$$

And we should expect something like

$$\left(\frac{U^j(\alpha_u)}{u^2}\right) \underset{u\to\infty}{\overset{d}{\to}} (A_j(1)).$$

This seems to take us beyond the framework of log-scaling laws, because it does not seem reasonable for $1/h^2 U^{\Sigma}(\alpha_{u(h)})$ to have a log-scaling limit for any $u(h)$, as required in Th 8.4 of [PY]. This invites creation in the limit of a full Walsh process on $n+1$ rays (for the log radial parts) or even a process on $n+1$ copies of a half plane stuck together along the imaginary axis to tell the winding story as well.

We now sketch such a development, just for the log radial parts. Assume for simplicity that $Z_0 = 0$, and that the n discs centered at z_j with radius r_j, and the complement of the disc of radius r_∞ centered at 0, form $n+1$ disjoint regions, say $R_1, R_2, \ldots, R_n, R_\infty$. Pick $n+1$ different rays in the plane, at arbitrary angles. Define a process $\omega(u)$, $u \geq 0$ on the $n+1$ rays by declaring that $\omega(u)$ is in ray j at radial distance

$$\left|\log\left[\frac{|Z(t) - z_j|}{r_j}\right]\right| \text{ if } Z(t) \text{ is in } R_j \text{ at time } \alpha_u$$

where α_u is the inverse of the total clock U^{Σ}. Notice that ω watched only when in ray j is just a reflecting BM in ray j. However the switching of ω between rays is not instantaneous, as in the Walsh process, but with delays while Z sticks in one region before switching over to another. But these delays will vanish in the scaling limit. Thus if we let

$$\omega^{(h)}(u) = h^{-1}\omega(h^2 u), \quad u \geq 0,$$

then we should expect that as $h \to \infty$

$$\omega^{(h)} \underset{h\to\infty}{\overset{d}{\to}} \omega^{\infty}$$

where ω^{∞} is the Walsh process on the $n+1$ rays. Here ω^{∞} could be constructed from the excursions of the ζ^{∞} process in Theorem 6.2 of [PY]. But the above conjecture is a bit more delicate than that theorem. The time scales of the different $\zeta^{j,h}$ processes are being riffled together, and matching up well due to the universality of the asymptotic local time which appears as the limit of all good additive functionals. As a final remark, we note that skew Brownian motion appears in a similar setting in [LY].

Appendix : The aim of this appendix is to present an improvement of Proposition 4.3. which may eventually shed some light on Problem 1.

Proposition : Let Z be the Walsh process on E_n as in Sections 3 and 4. Then, for $g = \sup\{s \leq 1 : R_s = 0\}$, we have :

(a) $\text{mult}(F_{g+}|F_g) = n$.

Consequently, $\text{sp mult}((F_t^Z)) \geq n$.

The following result will play a crucial rôle in our proof of (a).

Lemma (Lindvall-Rogers [LR], lemma 2) : Let C and $(D_\varepsilon, 0 < \varepsilon < 1)$ be σ-fields on (Ω, F, P) such that :

i) $(D_\varepsilon, 0 < \varepsilon < 1)$ increases with ε ;

ii) C and D_1 are independent.

Then : (b) $\bigcap_{\varepsilon>0} (C \vee D_\varepsilon) = C \vee (\bigcap_{\varepsilon>0} D_\varepsilon)$, up to P-negligible sets.

Proof of the Lemma : It is obvious that $C \vee (\bigcap_{\varepsilon>0} D_\varepsilon) \subseteq \bigcap_{\varepsilon>0} (C \vee D_\varepsilon)$.

In order to prove the converse inclusion (up to P-negligible sets), we need only show that :

$$E\left[H \middle| \bigcap_{\varepsilon>0} (C \vee D_\varepsilon)\right]$$

is measurable with respect to $C \vee (\bigcap_{\varepsilon>0} D_\varepsilon)$ (mod P), when H belongs to a family $\underline{\underline{H}}$ of r.v's which is total in $L^2(C \vee D_1)$.

This is certainly the case for $\underline{\underline{H}} = \{CD ; C \in L^2(C), D \in L^2(D_1)\}$ thanks to the independence of C and D_1.

Moreover, for such a variable $H = CD$, we have :

$$E\left[H \middle| \bigcap_{\varepsilon>0} (C \vee D_\varepsilon)\right] = \lim_{\varepsilon\downarrow 0} E[H|C \vee D_\varepsilon]$$

$$= C \lim_{\varepsilon\downarrow 0} E[D|D_\varepsilon]$$

$$= C\, E\left[D \middle| \bigcap_{\varepsilon>0} D_\varepsilon\right] = E\left[H \middle| C \vee (\bigcap_{\varepsilon>0} D_\varepsilon)\right],$$

and the lemma is proved.

Remark : H. von Weizsäcker [Wei] gives a necessary and sufficient condition which ensures that (b) holds.

Proof of (a) : Let (H_t) be the smallest right-continuous enlargement of (F_t) such that g becomes a stopping time.

Then, we have : $H_g = F_{g+}$ (see Jeulin [Je], p. 77).

Define, for $\varepsilon \in (0,1)$, $g_\varepsilon = g + \varepsilon(1-g)$; this is an increasing family of (H_t) stopping times, such that : $H_{g_\varepsilon} = F_{g_\varepsilon}$ (see Jeulin [Je], Lemme 5.7, p. 78).

Moreover, since (H_t) is right-continuous, we have :

$$H_g = \bigcap_{\varepsilon>0} F_{g_\varepsilon}.$$

Thanks to the Lemma, the property (a) shall be established once we have proved :

(c) $F_{g_\varepsilon} = F_g \vee \sigma\{\theta_1\} \vee M_\varepsilon \quad (\text{mod } P)$

where $M_\varepsilon = \sigma\{m_u \; ; \; u \le \varepsilon\}$, and $m_u = \frac{1}{\sqrt{1-g}} R_{g+u(1-g)}$, $u \le 1$, is the so-called Brownian meander associated with R ;

(d) F_g, θ_1, M_1 are independent ;

(e) $M_{0+} = \bigcap_{(\varepsilon>0)} M_\varepsilon$ is P-trivial.

The equality (c) follows easily from $F_{g_\varepsilon} = \sigma\{Z_{u\wedge g_\varepsilon} \; ; \; u \ge 0\} \vee \sigma(g_\varepsilon)$ which is a consequence of Stricker [St].

To prove (d), we begin to show that M_1 is independent from $H_g = F_{g+}$, hence from $F_g \vee \sigma\{\theta_1\}$. We first remark that the (F_t) submartingale $P(g < t|F_t)$ $(t < 1)$ can be computed explicitly. We easily find :

$$P(g < t|F_t) = \Phi\left(\frac{R_t}{\sqrt{1-t}}\right),$$

where $\Phi(y) = \sqrt{\frac{2}{\pi}} \int_0^y dx \exp\left(-\frac{x^2}{2}\right)$ (see, for instance, [Je], p. 124), and we deduce from this, using the explicit enlargement formulae (see [Je] again) that :

$$\text{(f)} \qquad R_{g+u} = \beta_u + \int_0^u \frac{ds}{\sqrt{1-(g+s)}} \left(\frac{\Phi'}{\Phi}\right)\left(\frac{R_{g+s}}{\sqrt{1-(g+s)}}\right), \text{ for } u < 1-g,$$

where $(\beta_u, u \ge 0)$ is a $(H_{g+u}, u \ge 0)$ Brownian motion.

In particular, $(\beta_u, u \ge 0)$ is independent from H_g.

Now, using Brownian scaling, we deduce from (f) that :

$$\text{(g)} \qquad m_v = \gamma_v + \int_0^v \frac{dh}{\sqrt{1-h}} \left(\frac{\Phi'}{\Phi}\right)\left(\frac{m_h}{\sqrt{1-h}}\right) \qquad (v < 1)$$

where $\gamma_v = \frac{1}{\sqrt{1-g}} \beta_{(1-g)v}$ is again a Brownian motion which is independent from H_g.

Then, from (g), we deduce that the filtrations of m and γ are identical, hence m is independent from H_g.

Furthermore, since the filtrations of m and γ are identical, and the germ σ-field of γ is trivial, so is M_{0+}, which proves (e).

In order to prove (d) fully, it remains to show that F_g and θ_1 are independent.

However, if we define $A_i = \{\theta_1 = \theta_i\}$, $1 \le i \le n$, and let $M_t = g_i(Z_{t\wedge 1})$, where g_i is defined as in Proposition 3.1, then M is a uniformly integrable (F_t) martingale, and $M_g = 0$.

So, by Yor [Y1], $E[M_1|F_g] = 0$.

It follows that $P(A_i|F_g) = p_i$, for $1 \le i \le n$, which proves the independence of θ_1 and F_g. □

Remark : A simple modification of the arguments used above to show (a) allows to prove

(a') $\text{mult}(F_{L+}|F_L) = n$,

where L is the random time considered in the Proof of Proposition 4.3.

The proof of (a') is in fact simpler than that of (a) since, instead of having to consider the Brownian meander (as above), all we need is to remark that $(R_{L+u} \; ; \; u \le T-L)$ is a BES(3) process up to its first hitting time of 1, and that this process is independent from F_L and θ_T.

References

[BwPY1] Barlow, M.T., Pitman, J.W. and Yor, M. (1989). Une extension multidimensionelle de la loi de l'arc sinus. In this volume.

[BwPY2] Barlow, M.T., Pitman, J.W. and Yor, M. (1989). Some extensions of the arcsine law. Tech. Report # 189, Dept. Statistics, U.C. Berkeley.

[BC] Baxter, J.R. and Chacon, R.V. (1984). The equivalence of diffusions on networks to Brownian motion. *Contemp. Math.* **26**, 33-47.

[ByPY] Burdzy, K., Pitman, J.W. and Yor, M. (1989). Some asymptotic laws for crossings and excursions. *Colloque Paul Lévy sur les Processus Stochastiques, Astérisque* **157-158**, 59-74.

[DV] Davis, M.H.A. and Varaiya, P. (1974). The multiplicity of an increasing family of σ-fields. *Ann. Prob.* **2**, 958-963.

[E] El Karoui, N. A propos de la formule d'Azéma-Yor. *Sém. Prob. XIII.* Lecture Notes in Math. **721**, Springer, 443-452.

[FD] Frank. H.F. and Durham, S. (1984). Random motion on binary trees. *J. Appl. Prob.* **21** 58-69.

[HS] Harrison, J.M. and Shepp, L.A. (1981). On skew Brownian motion. *Ann. Prob.* **9**, 309-313.

[I] Itô, K. (1970): Poisson point processes attached to Markov processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* University of California Press, Berkeley, pp. 225-239.

[J] Jacod, J. (1976). A general theorem of representation for martingales. *Z. Wahrscheinlichkeitstheorie verw. Gebeite* **34**, 225-244.

[Je] Jeulin, T. (1980). *Semi-Martingales et Grossissement d'une Filtration.* Lecture Notes in Math. **833**, Springer-Verlag.

[L] LeGall, J-F (1983). Applications du temps local aux équations différentielles stochastiques unidimensionelles. *Sém. Prob. XVII.* Lecture Notes in Math. **986**, Springer-Verlag, 15-31.

[LY] Le Gall J.F. and Yor M. (1986). Etude asymptotique de certains mouvements browniens complexes avec drift. *Probability and Related Fields* , **71**, 183-229

[LR] Lindvall, T. and Rogers, L.C.G. (1986). Coupling of multidimensional diffusions by reflection. *Ann. Prob.* 14, 860-872.

[MSY] Meyer, P.A., Stricker, C. and Yor, M. (1979). Sur une formule de la théorie du balayage. *Sém. Prob. XIII.* Lecture Notes in Math. **721**, Springer-Verlag, 478-487.

[M] Millar, P.W. (1977). Germ sigma fields and the natural state space of a Markov process. *Z. Wahrscheinlichkeitstheorie verw. Gebeite* **39**, 85-101.

[N] Nakao, S. (1972). On the pathwise uniqueness of solutions of stochastic differential equations. *Osaka J. Math.* **9**, 513-518.

[PY] Pitman, J. and Yor, M. (1986a). Asymptotic laws of planar Brownian motion. *Ann. Prob.* **14**, 733-779.

[R] Rogers, L.C.G. (1983). Itô excursion theory via resolvents. *Z. Wahrscheinlichkeitstheorie verw. Gebeite* **63**, 237-255.

[S] Salisbury, T.S. (1986). Construction of right processes from excursions. *Z. Wahrscheinlichkeitstheorie verw. Gebeite* **73**, 351-367.

[Sk] Skorokhod, A. V. (1987). Random processes in infinite dimensional spaces (in Russian). *Proceedings of the International Congress of Mathematicians: August 3-11, 1986, Berkeley* [edited by Andrew M. Gleason]. American Mathematical Society, Providence, R.I., 163-171.

[St] Stricker, C. (1981). Sur un théorème de H. J. Engelbert et J. Hess. *Stochastics* **6**, 73-77.

[SY] Stroock, D.W. and Yor, M. (1980). On extremal solutions of martingale problems. *Ann. Scient. E.N.S., 4ième série,* **13**, *95-164.*

[V] Varopoulos, N. Th. (1985). Long range estimates for Markov Chains. *Bull. Sci. Math.* **109**, 225-252.

[VW] Van der Weide, J.A.M. (1987). Stochastic processes and point processes of excursions. Ph.D. Thesis, Technische Universiteit Delft.

[W] Walsh, J. (1978). A diffusion with a discontinuous local time. In: *Temps Locaux, Astérisque* **52-53**, *37-45.*

[Wei] Weizsäcker, H.v. (1983). Exchanging the order of taking suprema and countable intersection of sigma algebras. *Ann. Inst. H. Poincaré* **19**, *91-100.*

[Y0] Yor, M. (1978). Sous-espaces denses dans L^1 ou H^1 et représentation des martingales. *Sém. Prob. XII,* Lecture Notes in Math. **649**, Springer, 265-309.

[Y1] Yor, M. (1979). Sur le balayage des semi-martingales continues. *Sém. Prob. XIII.* Lecture Notes in Math. **721**, Springer-Verlag, 453-171.

[Y2] Yor, M. (1979). Sur les martingales continues extrémales. *Stochastics* **2**, 191-196.