ON PATHWISE UNIQUENESS AND EXPANSION OF FILTRATIONS

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Abstract. Suppose that pathwise uniqueness holds for the SDE $X_t = x_0 + \int_0^t \sigma(X_s)dB_s$ where $|\sigma|$ is bounded and bounded away from 0, and B is a Brownian motion on a filtered probability space, $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_t,P)$. We give conditions under which pathwise uniqueness continues to hold in the enlarged filtration $(\underline{\underline{F}}_t^L)$, where L is the end of an $(\underline{\underline{F}}_t)$-optional set.

1. Introduction

Let $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_t,P)$ be a filtered probability space ($(\underline{\underline{F}}_t)$ satisfies the usual conditions) carrying a Brownian motion B, and let $\sigma\colon \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying

$$(1.1) \qquad K^{-1} \le |\sigma(x)| \le K, \qquad x \in \mathbf{R}$$

for some constant $K \in (0,\infty)$. We consider the stochastic differential equation (SDE)

$$(1.2)(x_0,\sigma,B) \qquad X_t = x_0 + \int_0^t \sigma(X_s)dB_s \ .$$

Let L be the end of an $(\underline{\underline{F}}_t)$-optional set, and $(\underline{\underline{F}}_t^L)$ be the smallest filtration containing $(\underline{\underline{F}}_t)$ which makes L a stopping time - see Jeulin (1980). In this paper we discuss the following question: Suppose pathwise uniqueness holds for (1.2). Then does it continue to hold for (1.2) in the enlarged filtration $(\underline{\underline{F}}_t^L)$?

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Note that B will be a semimartingle, but not in general a martingale, in the filtration $(\underline{\underline{F}}_t^L)$ (see Barlow (1979)). Thus the SDE (1.2) continues to make sense, and the stochastic integral has the same value in both filtrations (see Stricker (1977)). However, to explain what 'pathwise uniqueness' means in the enlarged filtration we need a few definitions.

As these will not involve any special structure of the SDE (1.2), we will consider the more general SDE

$$(1.3)(x_0,\sigma,Z) \qquad X_t = x_0 + \int_0^t \sigma(s,X)dZ_s + T_t(X,Z)$$

where Z is a d-dimensional semimartingale, $x_0 \in R^d$, σ: $R_+ \times D(R_+,R^d) \to R^{n\times d}$ is bounded and predictable with respect to the canonical filtration on $D(R_+,R^n)$, and $T_t(X,Z)$ is a jointly measurable adapted functional of X and Z. (An example of such a functional would be a version of the local time $L_t^0(X-Z)$).

<u>Definition 1.1</u> Let $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_t,P)$ be a probability space carrying an $(\underline{\underline{F}}_t)$-semimartingale Z. <u>Uniqueness of solutions</u> (UOS) holds for $(1.3)(x_0,\sigma,Z)$ in $(\underline{\underline{F}}_t)$ if there is at most one $(\underline{\underline{F}}_t)$-adapted process X_t satisfying (1.3).

In the case where Z is a Brownian motion, pathwise uniqueness holds if UOS holds for $(1.3)(x_0,\sigma,Z')$ in $(\underline{\underline{F}}'_t)$ for every $(\underline{\underline{F}}'_t)$-Brownian motion Z' on a probability space $(\Omega',\underline{\underline{F}}',\underline{\underline{F}}'_t,P')$.

To generalize this to semimartingales we need the concept of the adapted distribution of a semimartingale Z in a filtration $(\underline{\underline{F}}_t)$, which we denote $\mathrm{adsn}(Z,(\underline{\underline{F}}_t))$. For the definition we refer the reader to Hoover and Keisler

(1984, Def 2.6): here we just remark that if Z^1 is an $(\underline{\underline{F}}_t^1)$-semimartingale and Z^2 is cadlag then $\mathrm{adsn}(Z^1,(\underline{\underline{F}}_t^1)) = \mathrm{adsn}(Z^2,(\underline{\underline{F}}_t^2))$ implies not only that Z^1 and Z^2 have the same law, that Z^2 is an $(\underline{\underline{F}}_t^2)$-semimartingale (Hoover-Kiesler (1984), Thm 6.5) and that Z^1 and Z^2 have the same predictable characteristics, but that the whole 'information environment' of the Z^i in their filtrations $(\underline{\underline{F}}_t^i)$ are the same.

<u>Definition 1.2</u> Let $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_t,P)$ be a filtered probability space carrying an $(\underline{\underline{F}}_t)$ semimartingale Z. <u>Pathwise uniqueness</u> (PU) holds for (1.3)(x_0,σ,Z) in $(\underline{\underline{F}}_t)$ if whenever $(\Omega',\underline{\underline{F}}',\underline{\underline{F}}_t',P')$ is a filtered probability space carring an $(\underline{\underline{F}}_t')$ semimartingale Z' and $\mathrm{adsn}(Z,(\underline{\underline{F}}_t)) = \mathrm{adsn}(Z',(\underline{\underline{F}}_t'))$, then UOS holds for (1.3)(x_0,σ,Z') in $(\underline{\underline{F}}_t')$.

<u>Remark</u> While this definition may appear both clumsy and sophisticated, something of the kind seems essential. In much of the literature pathwise uniqueness is only discussed for SDEs driven by a Brownian motion, or functions of a BM. If B^i are $(\underline{\underline{F}}_t^i)$-Brownian motions for i = 1,2, then $\mathrm{adsn}(B^1,(\underline{\underline{F}}_t^1)) = \mathrm{adsn}(B^2,(\underline{\underline{F}}_t^2))$ (see Hoover and Keisler (1984, Thm 2.8)), so in this case the definition given above reduces to the standard one.

Jacod and Memin (1981, Def (2.24)), in a paper which predated the introduction of adapted distributions, gave a definition of pathwise uniqueness for a general SDE which involved product extensions. It follows from a recent result of Hoover (1989, Theorem 5.1) that their definition of 'very good pathwise uniqueness' is equivalent to our 'pathwise uniqueness'.

In the course of our proofs we will require the space $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_t,P)$ to be 'rich' enough to carry processes independent of B. This could be done by taking a suitable product extension of $(\Omega,\underline{\underline{F}},P)$ on each occasion. However we feel that it is technically easier to work on a saturated space, and we recall the definition of this interesting class of spaces from Hoover and Keisler (1984). A stochastic process on $(\Omega,\underline{\underline{A}},\underline{\underline{A}}_t,P)$ is a $\underline{\underline{B}}([0,\infty)) \times \underline{\underline{A}}$ measurable mapping X from $[0,\infty) \times \Omega$ to a Polish space.

Definition 1.3 A filtered probability space $(\Omega,\underline{\underline{A}},\underline{\underline{A}}_t,P)$ satisfying the usual conditions is saturated if for any process X_1 on $(\Omega,\underline{\underline{A}},\underline{\underline{A}}_t,P)$ and for any pair of stochastic processes (X_1',X_2') on a second space $(\Omega',\underline{\underline{A}}',\underline{\underline{A}}_t',P')$ such that $\text{adsn}(X_1,(\underline{\underline{A}}_t)) = \text{adsn}(X_1',(\underline{\underline{A}}_t'))$, there is a process X_2 on $(\Omega,\underline{\underline{A}},\underline{\underline{A}}_t,P)$ such that $\text{adsn}(X_1,X_2,(\underline{\underline{A}}_t)) = \text{adsn}(X_1',X_2',(\underline{\underline{A}}_t'))$.

Remarks 1.4 (a) Hoover and Keisler (1984, Cor 4.6, Thm 5.2) prove that saturated spaces exist, by showing that any adapted Loeb space $(\Omega,\underline{\underline{A}},\underline{\underline{A}}_t,P)$ which carries an $(\underline{\underline{A}}_t)$-Brownian motion is saturated. Henceforth all our adapted Loeb spaces will carry an $(\underline{\underline{A}}_t)$-Brownian motion, and so will be saturated. Adapted Loeb spaces are constructed using nonstandard analysis - see for example Hoover and Perkins (1983, Section 3). Hoover (1989, Section 5) sketches a direct model-theoretic construction of a saturated space.

(b) If the processes X_1,X_1',X_2' in Definition 1.3 are cadlag, then the process X_2 may also be taken to be cadlag (Hoover and Keisler (1984, Cor 5.8)).

The usefulness of saturated spaces in determining whether or not PU holds is exhibited in the next theorem.

Theorem 1.5 Let $(\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, P)$ be a filtered probability space carrying an $(\underline{\underline{F}}_t)$-semimartingale Z and let $(\Omega, \underline{\underline{A}}, \underline{\underline{A}}_t, P_A)$ be a saturated space carrying an $(\underline{\underline{A}}_t)$-semimartingale $\tilde{Z}$ such that $\mathrm{adsn}(Z, (\underline{\underline{F}}_t)) = \mathrm{adsn}(\tilde{Z}, (\underline{\underline{A}}_t))$. Then PU holds for (1.3)(x_0, σ, Z) in $(\underline{\underline{F}}_t)$ if and only if UOS holds for (1.3)$(x_0, \sigma, \tilde{Z})$ in $(\underline{\underline{A}}_t)$.

The proof is given in Section 2.

In this paper we obtain two main results on pathwise uniqueness (or lack thereof) in an enlarged filtration. The first (Theorem 1.6) characterizes PU in (1.2)(x_0, σ, B) in enlargements $(\underline{\underline{F}}_t^L)$ in terms of PU in a related equation in the original $(\underline{\underline{F}}_t)$. This immediately gives a sufficient condition on σ for PU to hold for (1.2) in any enlargement $(\underline{\underline{F}}_t^L)$ (Corollary 1.9). The proofs of Theorem 1.6 and Corollary 1.9 are given in Section 3.

Notation If X is a semimartingale let $L_t^a(X)$, $t \geq 0$, $a \in \mathbf{R}$ denote its local time - see Yor (1978, p 20).

Theorem 1.6 Let $(\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, P)$ satisfy the usual conditions, let B be an $(\underline{\underline{F}}_t)$-Brownian motion and σ satisfy (1.1). Consider the equations

(1.4) $$X_t = x_0 + \int_0^t \sigma(X_s)\,dB_s ,$$

(1.5a) $$Y_t = x_0 + \int_0^t \sigma(Y_s)\,dB_s + \frac{1}{2} L_t^0(Y-X) ,$$

(1.5b) $$Y'_t = x_0 + \int_0^t \sigma(Y'_s)\,dB_s - \frac{1}{2} L_t^0(X-Y') .$$

The following are equivalent:

(a) For any L which is the end of an $(\underline{\underline{F}}_t)$-optional set, pathwise uniqueness holds for (1.4) in $(\underline{\underline{F}}_t^L)$.

(b) Pathwise uniqueness holds for the system (1.4), (1.5a), (1.5b) in $(\underline{\underline{F}}_t)$.

Remarks 1.7 (a) Note that any solution X to (1.4) is also a solution to (1.5a) and (1.5b).

(b) If pathwise uniqueness does not hold for (1.4) in $(\underline{\underline{F}}_t)$ then, as any $(\underline{\underline{F}}_t)$-adapted solution of (1.4) is also an $(\underline{\underline{F}}_t^L)$-adapted solution, both (a) and (b) fail trivially. So the theorem has content only in the case when pathwise uniqueness does hold for (1.4) in $(\underline{\underline{F}}_t)$.

The implication (a) => (b) is easy. The plan of the converse argument is as follows. We suppose that PU holds for (1.4) in $(\underline{\underline{F}}_t)$ and let X denote the unique solution. Let X' be an $(\underline{\underline{F}}_t^L)$-adapted solution to (1.4). We first prove (Lemma 3.2) that X and X' can only separate at L. We then construct various "approximations" to X, which converge to the processes Y and Y' satisfying (1.5). We show that the paths of X' cannot cross the paths of these approximating processes. Hence, if X = Y = Y' then the paths of X' are trapped between the paths of processes which converge to X, and so X = X'.

The following condition was introduced in Barlow and Perkins (1984).

Definition 1.8 σ satisfies (LT) if whenever V_t^1 and V_t^2 are continuous adapted processes of bounded variation on some $(\Omega,\underline{F},\underline{\underline{F}}_t,P)$ and X_t^i (i = i,2) are adapted solutions of

$$X_t^i = x_i + \int_0^t \sigma(X_s^i)dB_s + V_t^i \qquad i = 1,2$$

$(x_1, x_2 \in \mathbf{R})$, then $L_t^0(X^1 - X^2) = 0$ for all $t \geq 0$.

This condition together with (1.1) implies PU for (1.4) in any $(\underline{\underline{F}}_t)$ (see the remarks following Theorem 2.1 in Barlow-Perkins (1984)). We do not know if (LT) is equivalent to PU, but all known conditions on σ sufficient to establish PU for (1.4) in $(\underline{\underline{F}}_t)$ (as in LeGall (1983)) also establish (LT) for σ. Explicit conditions on σ which imply (LT) may be found in Barlow and Perkins (1984, Thm 2.1).

<u>Corollary 1.9</u> If σ satisfies (LT) and (1.1), then conditions (a) and (b) of Theorem 1.6 hold.

Our second main result (Theorem 1.11) was used in Barlow-Perkins (1989, Thm. 5.1) to prove that for a large class of σ's, which satisfy (1.1) and change sign at 0, PU fails for (1.2)(x_0, σ, B) in $(\underline{\underline{F}}_t)$. In that paper we first constructed a second solution to (1.2) on an enlarged filtration. This solution exhibited a certain path property which allows us to apply Theorem 1.11 (stated in Barlow-Perkins (1989) as Theorem 5.B) to conclude that PU must fail for (1.2)(x_0, σ, B) in the original $(\underline{\underline{F}}_t)$.

We first state a preliminary result which shows that (on an adapted Loeb space) if PU holds for (1.4) in $(\underline{\underline{A}}_t)$ but not in $(\underline{\underline{A}}_t^L)$ then the new solutions in the enlargement must separate from the $(\underline{\underline{A}}_t)$-adapted solution in a rather implausible manner.

<u>Proposition 1.10</u> Let $(\Omega, \underline{\underline{A}}, \underline{\underline{A}}_t, P)$ be an adapted Loeb space, B an $(\underline{\underline{A}}_t)$-Brownian motion, and L be the end of an optional set. Suppose PU holds for (1.4) in $(\underline{\underline{A}}_t)$, and fails for (1.4) in $(\underline{\underline{A}}_t^L)$. Let X be the unique $(\underline{\underline{A}}_t)$ adapted

solution, and let X' be an $(\underline{\underline{A}}_t^L)$ adapted solution. Suppose that $P(L<\infty) = 1$, and that $P(X_t \neq X'_t$ for some $t) = 1$. Then w.p.1 $X_t = X'_t$ for $0 \le t \le L$, $X_{L+t} \neq X'_{L+t}$ for all sufficiently small $t > 0$, and the event $\{X'_{L+t} > X_{L+t}$ for all sufficiently small $t > 0\}$ is $\underline{\underline{A}}_L$ measurable.

(We recall that $\underline{\underline{A}}_L = \sigma(Y_L$: Y is an $(\underline{\underline{A}}_t)$-optional process).).

<u>Theorem 1.11</u> Let $(\Omega, \underline{\underline{A}}, \underline{\underline{A}}_t, P)$ be an adapted Loeb space carrying a Brownian motion B. Let X be an $(\underline{\underline{A}}_t)$ adapted solution to (1.4), and let

$$T = \inf\{s: |X_s - x_0| = 1\}, \quad L = \sup\ \{s < T: X_s = x_0\}.$$

Then if there exists an $(\underline{\underline{A}}_t^L)$ adapted solution Y to (1.4) with the property that $\text{sign}(Y_{L+t} - x_0) = - \text{sign}(X_{L+t} - x_0)$ for all sufficiently small $t > 0$, then pathwise uniqueness fails in (1.4) relative to $(\underline{\underline{A}}_t)$.

(Here $\text{sign}(x) = 1_{(x > 0)} - 1_{(x < 0)}$).

Proposition 1.10 and Theorem 1.11 are proved in Section 4: we use the same basic strategy as in the proof of Theorem 1.6.

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<u>2. Preliminary Results</u>

We begin this section with some elementary results on adapted distributions, required for the proof of Theorem 1.5.

<u>Lemma 2.1</u> Assume X^i is a cadlag process on $(\Omega^i, \underline{\underline{F}}^i, \underline{\underline{F}}_t^i, P^i)$ taking values in a Polish space M, Y^i is a stochastic process on $(\Omega^i, \underline{\underline{F}}^i, \underline{\underline{F}}_t^i, P^i)$ $(i = 1,2)$

and ψ: $\mathbf{R}_+ \times D(\mathbf{R}_+, M) \to M'$ (M' is another Polish space) is universally measurable. If $\text{adsn}(X^1, Y^1, (\underline{\underline{F}}_t^1)) = \text{adsn}(X^2, Y^2, (\underline{\underline{F}}_t^2))$, then $\text{adsn}(X^1, Y^1, \psi(\cdot, X^1), (\underline{\underline{F}}_t^1)) = \text{adsn}(X^2, Y^2, \psi(\cdot, X^2), (\underline{\underline{F}}_t^2))$.

Proof. If $\psi(t,x) = \phi(t, x(t_1), \ldots, x(t_n))$ where ϕ: $\mathbf{R}_+ \times M^n \to M'$ is continuous, the conclusion follows easily from the definition of adapted distribution. Proposition 2.19 of Hoover-Keisler (1984) shows that the class of ψ's for which the conclusion holds is closed under pointwise convergence. A monotone class argument gives the result for Borel ψ if $M' = \mathbf{R}$ and also for general M' if ψ is Borel and finite-valued. In general, however, a Borel ψ is the pointwise limit of a sequence of finite-valued ψ's and hence the result holds for Borel ψ. The extension to universally measurable ψ is trivial. ∎

The following result on stochastic integration follows easily from the above lemma and Theorem 7.5 of Hoover-Keisler (1984).

Proposition 2.2. Let Z^i be a d-dimensional semimartingale on $(\Omega^i, \underline{\underline{F}}^i, \underline{\underline{F}}_t^i, P^i)$, X^i be a cadlag $\mathbf{R}^n$-valued $(\underline{\underline{F}}_t^i)$-adapted process, and Y^i be a stochastic process on Ω^i $(i = 1,2)$. Let $\sigma: \mathbf{R}_+ \times D(\mathbf{R}_+, \mathbf{R}^n) \to \mathbf{R}^{n\times d}$ be bounded and predictable (use the canonical right-continuous filtration on $D(\mathbf{R}_+, \mathbf{R}^n)$). If $\text{adsn}(Y^1, X^1, Z^1, (\underline{\underline{F}}_t^1)) = \text{adsn}(Y^2, X^2, Z^2, (\underline{\underline{F}}_t^2))$ then

$$\text{adsn}(Y^1, X^1, Z^1, \int_0^{\cdot} \sigma(s, X^1)\,dZ_s^1, (\underline{\underline{F}}_t^1)) = \text{adsn}(Y^2, X^2, Z^2, \int_0^{\cdot} \sigma(s, X^2)\,dZ_s^2, (\underline{\underline{F}}_t^2)).$$

Proof of Theorem 1.5. The "only if" assertion is trivial. To prove the converse, suppose PU fails for (1.3)(x_0,σ,z) in $(\underline{\underline{F}}_t)$. Then there exists a filtered space $(\Omega',\underline{\underline{F}}',\underline{\underline{F}}'_t,P')$ carrying an $(\underline{\underline{F}}'_t)$-semimartingale Z' with $\text{adsn}(Z',(\underline{\underline{F}}'_t)) = \text{adsn}(Z,(\underline{\underline{F}}_t))$ such that (1.3)(x_0,σ,Z) has two distinct solutions, X' and Y' say. By saturation (see Remark 1.4(b)) there are cadlag $(\underline{\underline{A}}_t)$-adapted process $\tilde{X}$ and $\tilde{Y}$ such that $\text{adsn}(X',Y',Z',(\underline{\underline{F}}'_t)) = \text{adsn}(\tilde{X},\tilde{Y},\tilde{Z},(\underline{\underline{A}}_t))$. Lemma 2.1 and Proposition 2.2 imply that $\tilde{X}$ and $\tilde{Y}$ are distinct solutions of (1.3)$(x_0,\sigma,\tilde{Z})$ on $(\Omega_A,\underline{\underline{A}},\underline{\underline{A}}_t,P_A)$ and so UOS fails in $(\underline{\underline{A}}_t)$. ∎

Lemma 2.3. Let X be a stochastic process on the filtered space $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_t,P)$ and let $\tilde{X}$ be a stochastic process on the saturated space that $(\Omega_A,\underline{\underline{A}},\underline{\underline{A}}_t,P_A)$ such that $\text{adsn}(X,(\underline{\underline{F}}_t)) = \text{adsn}(\tilde{X},(\underline{\underline{A}}_t))$. Assume L is the end of an $(\underline{\underline{F}}_t)$-optional set.

(a) There is an $\tilde{L}$, which is the end of an $(\underline{\underline{A}}_t)$-optional set, such that $\text{adsn}(X,(\underline{\underline{F}}^L_t)) = \text{adsn}(\tilde{X},(\underline{\underline{A}}^{\tilde{L}}_t))$.

(b) If $(\Omega_A,\underline{\underline{A}},\underline{\underline{A}}_t,P_A)$ is an adapted Loeb space, then so is $(\Omega_A,\underline{\underline{A}},(\underline{\underline{A}}^{\tilde{L}}_t),P_A)$. In particular, $(\Omega_A,\underline{\underline{A}},(\underline{\underline{A}}^{\tilde{L}}_t),P_A)$ is saturated.

(c) If $(\Omega_A,\underline{\underline{A}},\underline{\underline{A}}_t,P_A)$ is an adapted Loeb space and T is an $(\underline{\underline{A}}_t)$-stopping time which is finite a.s., then $(\Omega_A,\underline{\underline{A}},\underline{\underline{A}}_{T+\cdot},P_A)$ is an adapted Loeb space, and so is also saturated.

Proof (a) Let Λ be an $(\underline{\underline{F}}_t)$-optional set such that $L = \sup\{t: t \in \Lambda\}$, let $g_t = \sup\{s \le t: s \in \Lambda\}$ and $V_t = t - g_t$: we have $L = \sup\{t: V_t = 0\}$. Set $A_t = 1_{[L,\infty)}(t)$, and let ${}^{\circ}A_t$ be the (cadlag) $(\underline{\underline{F}}_t)$-optional projection of A. (See Dellacherie and Meyer (1982), VI.47). By saturation there exist cadlag

$(\underline{\underline{A}}_t)$-adapted processes $\tilde{V}$, $°\tilde{A}$ such that $\text{adsn}(X,V,°A,(\underline{\underline{F}}_t)) = \text{adsn}(\tilde{X},\tilde{V},°\tilde{A},(\underline{\underline{A}}_t))$. Let $\tilde{L} = \sup\{t: \tilde{V}_t = 0\}$: by Lemma 2.1 $\text{adsn}(X,°A,L,(\underline{\underline{F}}_t)) = \text{adsn}(\tilde{X},°\tilde{A},\tilde{L},(\underline{\underline{A}}_t))$, and hence $°\tilde{A}_t = P_A(\tilde{L} \le t|\underline{\underline{A}}_t)$ P_A-a.s. for all $t \ge 0$. It follows that $°\tilde{A}$ is the $(\underline{\underline{A}}_t)$-optional projection of $1_{[\tilde{L},\infty)}$. If $\phi \in L^1(\underline{\underline{F}})$ then (see Barlow (1979, Lemma 2.2, 3.1)),

$E(\phi|\underline{\underline{F}}_t^L) = (A_t/°A_t)\ E(\phi A_t|\underline{\underline{F}}_t) + ((1-A_t)/(1-°A_t))\ E(\phi(1-A_t)|\underline{\underline{F}}_t)$ (here $0/0 = 0$), and a similar equation holds for $E(\cdot|\underline{\underline{A}}_t^{\tilde{L}})$. Thus conditional expectations relative to $(\underline{\underline{F}}_t^L)$ and $(\underline{\underline{A}}_t^{\tilde{L}})$ can be reduced to conditional expectations relative to $(\underline{\underline{F}}_t)$ and $(\underline{\underline{A}}_t)$. It follows easily that $\text{adsn}(X,V,°A,(\underline{\underline{F}}_t)) = \text{adsn}(\tilde{X},\tilde{V},°\tilde{A},(\underline{\underline{A}}_t))$ implies $\text{adsn}(X,V,°A,(\underline{\underline{F}}_t^L)) = \text{adsn}(\tilde{X},\tilde{V},°\tilde{A},(\underline{\underline{A}}_t^{\tilde{L}}))$.

(b) The first assertion is proved just as in Theorem 5.A in Barlow and Perkins (1989), where it is shown that $(\Omega,\underline{\underline{A}},(\underline{\underline{A}}_{L+t}^L),P_A)$ is an adapted Loeb space. The second assertion is then immediate from Remark 1.4(a).

(c) Any stopping time T is also an end-of-optional time, so this is immediate from Barlow and Perkins (1989, Theorem 5.A). ∎

<u>Remark</u>. We have been unable to decide whether or not (b) remains valid if we replace "adapted Loeb space" by "saturated space" in both hypothesis and conclusion. This is why we have used adapted Loeb spaces in this work.

We close this section with a result on the convergence of Itô integrals, which is required in the next section.

Notation. Given a process Y we define Y^t by $Y^t_s = Y_{t\wedge s}$.

Lemma 2.4 Let σ be a bounded measurable function satisfying $K^{-1} < |\sigma(x)| < K$ for $x \in \mathbf{R}$, let B be a Brownian motion and let $(Y^n)_{1\le n\le\infty}$ be a sequence of semimartingales with decomposition $Y^n_t = Y^n_0 + M^n + A^n$, where M^n is continuous. Suppose that

(a) $\lim_{n\to\infty} Y^n_t = Y^\infty_t$ a.s. for each t,

(b) $\langle M^n\rangle_t = \int_0^t H^n_s\, ds$, where $K_2^{-1} \le H^n_s \le K_2$ for each s, for $1\le n\le\infty$,

(c) $\sup_n \|(Y^n)^t\|_{H_1} = c(t) < \infty$ for each t.

Then

$$E\left[\int_0^t \sigma(Y^n_{s-})dB_s - \int_0^t \sigma(Y^\infty_{s-})dB_s\right]^2 \to 0 \text{ for each } t \ge 0, \tag{2.1}$$

and so in particular there exists a subsequence (n_j) such that

$$\int_0^t \sigma(Y^{n_j}_{s-})dB_s \to \int_0^t \sigma(Y^\infty_{s-})dB_s \quad \text{a.s. uniformly on compacts.} \tag{2.2}$$

Proof. To prove (2.1) it is enough to prove

$$\lim_{n\to\infty} E\int_0^t (\sigma(Y^n_s) - \sigma(Y^\infty_s))^2\, ds = 0\ . \tag{2.3}$$

(As $\{s: Y^n_{s-} \ne Y^n_s\}$ is countable, we can replace Y^n_{s-} by Y^n_s).

If σ is continuous, (2.3) is immediate from dominated convergence. From El-Karoui (1978, Proposition 1.2, Remarque 3) and Barlow (1983, Theorem 5.5) there exists a universal constant c_1 such that

$$E\, L^a_t(Y^n) \le c_1\|(Y^n)^t\|_{H^1} \le c_1 c(t), \text{ for } t \ge 0,\ a \in \mathbf{R}\ . \tag{2.4}$$

So, if g is any bounded continuous function, and $1 \le n \le \infty$,

$$(2.5) \qquad E\int_0^t (g(Y_s^n) - \sigma(Y_s^n))^2\, ds = E\int_0^t (H_s^n)^{-1}(g(Y_s^n) - \sigma(Y_s^n))^2\, d\langle M^n\rangle_s$$

$$\leq K_2\, E\int_{-\infty}^{\infty} (g(a) - \sigma(a))^2\, L_t^a(Y^n)\, da$$

$$\leq c\, \|g - \sigma\|_2^2 .$$

Here c depends on K_2 and $c(t)$, but not on n.

Using (2.5) we have

$$E\int_0^t (\sigma(Y_s^n) - \sigma(Y_s^\infty))^2 ds \leq 3E\int_0^t (\sigma(Y_s^n) - g(Y_s^n))^2 ds + 3E\int_0^t (\sigma(Y_s^\infty) - g(Y_s^\infty))^2 ds$$

$$+ 3E\int_0^t (g(Y_s^n) - g(Y_s^\infty))^2 ds$$

$$\leq 6c\, \|g - \sigma\|_2^2 + 3E\int_0^t (g(Y_s^n) - g(Y_s^\infty))^2 ds.$$

The second term converges to 0, so (2.3) follows on approximating σ in L^2 by a continuous g (which is possible even though σ is not in L^2). Passing to a subsequence $n_j(t)$, and using Doob's inequality and a diagonalization argument we obtain (2.2). ∎

3. A Characterization of Pathwise Uniqueness in an Enlargement

We now fix a Loeb filtration $(\Omega, \underline{\underline{A}}, \underline{\underline{A}}_t P)$, an $(\underline{\underline{A}}_t)$ Brownian motion B, and an $(\underline{\underline{A}}_t)$-end of optional time L. We begin with a technical result on pathwise uniqueness.

<u>Lemma 3.1</u> Suppose pathwise uniqueness holds for (1.4) in $(\underline{\underline{A}}_t)$ for some initial point x_0. Then pathwise uniqueness holds for all initial points x.

<u>Proof</u>. Let X be the unique solution with $X_0 = x_0$. Suppose pathwise

uniqueness fails for some initial point x_1, and let $T = \inf\{t \geq 0: X_t = x_1\}$. As X is a time-changed Brownian motion and $\langle X \rangle_\infty = \infty$, $P(T < \infty) = 1$. The filtration $(\underline{\underline{A}}_{T+\cdot})$ is saturated by Lemma 2.3(c). By Theorem 1.5 there exist distinct solutions Y^1, Y^2 to

$$Y_t^i = x_1 + \int_0^t \sigma(Y_s^i)\,dB_{T+s} \ .$$

Let $Z_t^i = X_t 1_{(t < T)} + Y_{t-T}^i 1_{(t \geq T)}$: the Z^i are distinct $(\underline{\underline{A}}_t)$ adapted solutions to (1.4) with $Z_0^i = x_0$, giving a contradiction.

From now on we will assume pathwise uniqueness holds in (1.4) (relative to $(\underline{\underline{A}}_t)$). Let $\Lambda_t = \Lambda_t(x,B,s)$ be the unique $(\underline{\underline{A}}_t)$-adapted process such that

$$\Lambda_t = x \qquad 0 \leq t \leq s$$

$$\Lambda_t = x + \int_s^t \sigma(\Lambda_u)\,dB_u \qquad t > s \ .$$

From the continuity of paths, and the pathwise uniqueness, it is clear that if $x_1 > x_2$, then $\Lambda_t(x_1,B,s) \geq \Lambda_t(x_2,B,s)$ for all t. (These solutions may meet, however).

Let ${}^\circ A_t$ be the $(\underline{\underline{A}}_t)$-optional projection of $1_{[L,\infty)}$, so that for every $(\underline{\underline{A}}_t)$-stopping time T we have ${}^\circ A_T = P(L \leq T \mid \underline{\underline{A}}_T)$. Set

$$R = \inf\{s \geq 0: \ {}^\circ A_s = 1\} \ ;$$

R is "the time at which the enlargement comes to an end", and we have $\underline{\underline{A}}_R^L = \underline{\underline{A}}_R$, by Barlow (1979, Lemma 2.2). ∎

<u>Lemma 3.2</u> Let (Y_t^1), (Y_t^2) be $(\underline{\underline{A}}_t^L)$ adapted solutions to (1.4), and let $K = \inf\{t \geq 0: \ Y_t^1 \neq Y_t^2\}$. Then $[K] \subseteq [L]$.

<u>Proof</u> By Jeulin (1980, Prop 5.3, p. 75) there exist $(\underline{\underline{A}}_t)$ previsible processes Y^{ij} such that $Y^i = Y^{i1} 1_{[0,L]} + Y^{i2} 1_{(L,\infty)}$.

Let $T = \inf\{t: Y_t^{11} \neq \Lambda_t(x_0, B, 0)\}$. As Y^1 solves (1.4) we must have $T \geq L$ a.s., and so ${}^\circ A_T = 1$. Thus $T \geq R$, so that Y^{11} is a solution of (1.4) on $[0,R]$. Similarly, Y^{21} is a solution of (1.4) on $[0,R]$, and so, by the pathwise uniqueness, $Y^{11} = Y^{21}$ on $[0,R]$. Thus, as $L \leq R$, we have $K \geq L$.

It remains to show that, for each $\epsilon > 0$, $K = \infty$ on $\{K > L + \epsilon\}$. Let $\epsilon > 0$ be fixed: by Barlow (1979, Theorem 4.5) there exists a sequence (S_n) of $(\underline{\underline{A}}_t)$-stopping times such that $[L + \epsilon] \subseteq \bigcup_{n=1}^{\infty} [S_n]$. Let $T_n^i = \inf\{t \geq S_n: Y_t^{i2} \neq \Lambda_t(Y_{S_n}^{i2}, B, S_n)\}$. As Y^i is a solution to (1.4), and equals Y^{i2} on (L,∞), $T_n^i = \infty$ on $\{S_n = L + \epsilon\}$. Also, on $\{K > L + \epsilon\}$, we have $Y^{12} = Y^1 = Y^2 = Y^{22}$ on (L,K). Hence, on $\{S_n = L + \epsilon, K > L + \epsilon\}$, $Y_{S_n}^{12} = Y_{S_n}^{22}$ and $T_n^1 = T_n^2 = \infty$, so that $Y_t^{12} = Y_t^{22} = \Lambda_t(Y_{S_n}^{12}, B, S_n)$ for all $t \geq S_n$, and hence $K = \infty$.

<u>Corollary 3.3</u> With the notation of Lemma 3.2 let

$$S = \inf\{t > L: Y_t^1 = Y_t^2\}.$$

Then $Y^1 = Y^2$ on $[S,\infty]$.

<u>Proof</u>. It is enough to show that $Y_t^1 = Y_t^2$ for all $t \geq S_n$, where $S_n = \inf\{t > L + n^{-1}: Y_t^1 = Y_t^2\}$, for each n. Let $n \geq 1$ be fixed, and let $Y^3 = Y^1 1_{[0,S_n)} + Y^2 1_{[S_n,\infty)}$. Since $[S_n] \cap [L] = \emptyset$, by Lemma 3.2 we have

$Y^3 = Y^1$, so that $(Y^1 - Y^2)\, 1_{[S_n,\infty)} = 0$. ∎

Now fix $x_0 \in \mathbf{R}$, and let $X_t = \Lambda_t(x_0, B, 0)$. Let $\epsilon_n \downarrow 0$, and define processes Y^n as follows:

$$Y^0 \equiv +\infty,$$

and for $n \geq 1$, $k \geq 0$,

(3.1)(a) $$Y_0^n = x_0 + \epsilon_n\,, \qquad T_0^n = 0\,,$$

$$T_{k+1}^n = \inf\{t > T_k^n : \Lambda_t(Y^n(T_k^n), B, T_k^n) = X_t\}$$

$$Y_t^n = \Lambda_t(Y^n(T_k^n), B, T_k^n) \text{ on } [T_k^n, T_{k+1}^n)$$

(3.1)(b) $$Y^n(T_{k+1}^n) = \min(Y^{n-1}(T_{k+1}^n), X(T_{k+1}^n) + \epsilon_n).$$

<u>Proposition 3.4</u> (a) $Y_t^n > X_t$ for all t.

(b) $Y_t^n \downarrow Y_t^\infty$, where Y^∞ is a continuous semimartingale satisfying the equation

(3.2)(a) $$Y_t^\infty = x_0 + \int_0^t \sigma(Y_s^\infty)\,dB_s + \frac{1}{2} L_t^0(Y^\infty - X)$$

(3.2)(b) $$Y_t^\infty \geq X_t\,.$$

<u>Proof</u>. We may take $x_0 = 0$. We begin by showing that Y^n is well defined. Note that Y^n can only fail to be well defined if $\sup_k T_k^n < +\infty$, and that $Y_t^n > X_t$ for $0 \leq t < \sup_k T_k^n$. Since the jumps of Y^1 are all of size ϵ_1, and as $|\sigma| \leq K$, the times T_k^1 cannot accumulate, so $\sup_k T_k^1 = +\infty$. Suppose that Y^{n-1}

is well defined. If $\Delta Y^n(T_i^n) < \epsilon_n$ for some i, then $Y_t^n = Y_t^{n-1}$ on $[T_i^n, S_i^n]$, where $S_i^n = \inf\{t>T_i^n\colon \Delta Y_t^{n-1} > \epsilon_n\}$, and so Y^n is well defined on $[T_i^n, S_i^n]$. As the T_k^n cannot accumulate outside an interval of this form, we have $\sup_k T_k^n = +\infty$.

From the definition of Y^n we may write

$$Y_t^n = \int_0^t \sigma(Y_s^n)\,dB_s + A_t^n ,$$

where A^n is increasing and $\Delta A_t^n \le \epsilon_n$. Now $\Delta A^n(T_i^n) = Y^n(T_i^n) - X(T_i^n)$, and $Y^n(T_{i+1}^n-) = X(T_{i+1}^n)$ on $\{T_{i+1}^n < \infty\}$. So, setting $H_s^n = \sigma(X_s) - \sigma(Y_s^n)$, we have

$$\Delta A^n(T_i^n) = Y^n(T_i^n) - Y^n(T_{i+1}^n-) - (X(T_i^n) - X(T_{i+1}^n)) \tag{3.4}$$

$$= \int_{T_i^n}^{T_{i+1}^n} H_s^n\, dB_s \quad , \qquad \text{on } \{T_{i+1}^n < \infty\} .$$

Let $S_1 < S_2$ be stopping times, and let N,M be such that $T_{N-1}^n \le S_1 < T_N^n$, $T_M^n \le S_2 < T_{M+1}^n$. If $A_{S_2}^n > A_{S_1}^n$ then $M \ge N$. From (3.4) we have

$$A_{S_2}^n - A_{S_1}^n = \sum_{i:\ S_1<T_i^n\le S_2} \Delta A^n(T_i^n)$$

$$\le 1_{(M\ge N)} \left[\epsilon_n + \int_{T_N^n}^{T_M^n} H_s^n\, dB_s\right]$$

$$\le \epsilon_n + 1_{(T_N^n < S_2)} \sup_{T_N^n<t<S_2} \int_{T_N^n}^t H_s^n\, dB_s .$$

So, by the Burkholder-Davis-Gundy inequalities, for $p \ge 1$

$$E(A_{S_2}^n - A_{S_1}^n)^p \le c_p\, \epsilon_n^p + c_p K^p\, E(S_2-S_1)^{p/2}, \tag{3.5}$$

and

$$E|Y^n_{S_2}-Y^n_{S_1}|^p \le c_p\, \epsilon^p_n + c_p K^p\, E(S_2-S_1)^{p/2} . \tag{3.6}$$

By the definition of (Y^n), Y^n is decreasing, and we have $Y^n_t > X_t$. Let $Y^\infty_t = \lim_n Y^n_t$: $Y^\infty_t \ge X_t$. Using dominated convergence and the estimate (3.6) we see that $Y^n_t \to Y^\infty_t$ in L^p. Let $n \to \infty$ in (3.6): we have

$$E|Y^\infty_{S_2} - Y^\infty_{S_1}|^p \le c_p\, K^p\, E(S_2-S_1)^{p/2} . \tag{3.7}$$

By Dellacherie and Meyer (1982, VI.48), Y^∞ is right continuous. But taking $p > 2$ in (3.7) and applying Kolmogorov's continuity theorem, we also have that Y^∞ has a continuous modification. Hence Y^∞ is continuous.

Now Y^n_t is l.s.c., and so Y^∞ is a limit of a decreasing sequence of l.s.c. processes. Hence $Y^n \downarrow Y^\infty$ uniformly on compacts, and by dominated convergence

$$\lim_{n\to\infty} \| \sup_{s\le t} |Y^n_s - Y^\infty_s| \|_1 = 0 \quad \text{for each } t. \tag{3.8}$$

Thus (Y^n) satisfies the conditions of Barlow and Protter (1990, Theorem 1), and so Y^∞ is a semimartingale with decomposition $Y^\infty = M + A$, and

$$\lim_{n\to\infty} \|(M^n - M)^t\|_{H_1} = 0, \qquad \lim_{n\to\infty} \|A^n_t - A_t\|_1 = 0 . \tag{3.8}$$

Thus $\langle M\rangle_t = \int_0^t h_s ds$, where $K^{-1} \le |h_s| \le K$. So (Y^n) satisfy the hypotheses of Lemma 2.4, and (passing to a subsequence and relabelling) we deduce that

$$M_t = \lim_{n\to\infty} \int_0^t \sigma(Y^n_s)dB_s = \int_0^t \sigma(Y^\infty_s)dB_s .$$

As A^n_t are increasing, A_t must also be increasing. So we have proved that

$$Y^\infty_t = \int_0^t \sigma(Y^\infty_s)dB_s + A_t,$$

where A is increasing.

Let $[S_1,S_2]$ be an interval on which $Y^\infty > X$: then $Y^n_- > X$ on $[S_1,S_2]$ for each n, so A^n is constant on $[S_1,S_2]$, and hence A is constant on $[S_1,S_2]$. Thus dA is supported by $\{t:X_t = Y^\infty_t\}$.

By Tanaka's formula

$$
\begin{aligned}
(Y^\infty_t - X_t) &= (Y^\infty_t - X_t)^+ \\
&= \int_0^t 1_{(Y^\infty_s > X_s)} (\sigma(Y^\infty_s) - \sigma(X_s))\,dB_s + \int_0^t 1_{(Y^\infty_s > X_s)}\,dA_s + \frac{1}{2} L^0_t(Y^\infty - X) \\
&= \int_0^t \sigma(Y^\infty_s)\,dB_s - \int_0^t \sigma(X_s)\,dB_s + \frac{1}{2} L^0_t(Y^\infty - X) \\
&= Y^\infty_t - A_t - X_t + \frac{1}{2} L^0_t(Y^\infty - X).
\end{aligned}
$$

So $A_t = \frac{1}{2} L^0(Y^\infty - X)$, and the proposition is proved. ∎

We may define a similar sequence of processes $Y^{n'}$ which approximate X from below, by replacing (3.1)(a) by $Y^{n'}_0 = x_0 - \epsilon_n$, and (3.1)(b) by $Y^{n'}(T^{n'}_{k+1}) = \max\,(Y^{(n-1)'}(T^{n'}_{k+1}),\ X(T^{n'}_{k+1}) - \epsilon_n)$. Then an almost identical proof shows that $Y^{n'}$ increase to a limiting process $Y^{\infty'} \le X$, which satisfies

(3.9)(a) $$Y^{\infty'}_t = x_0 + \int_0^t \sigma(Y^{\infty'}_s)\,dB_s - \frac{1}{2} L^0(X - Y^{\infty'})$$

(3.9)(b) $$Y^{\infty'}_t \le X_t\ .$$

<u>Proof of Theorem 1.6</u>. By Remark 1.7(b) it suffices to consider the case when PU holds for (1.4).

(b) => (a). By Theorem 1.5 and Lemma 2.3 we may take our filtered space to be the adapted Loeb space $(\Omega, \underline{\underline{A}}, \underline{\underline{A}}_t, P)$. That is, we will assume UOS in (1.4),(1.5a) and (1.5b) in $(\underline{\underline{A}}_t)$ and show UOS for (1.4) in $(\underline{\underline{A}}_t^L)$ where L is a fixed end-of-optional time for $(\underline{\underline{A}}_t)$. (Lemma 2.3(a) shows that any end-of-optional time on a filtered space can be modelled on a Loeb space, and Lemma 2.3(b) and Theorem 1.5 would then give PU in (1.4) for $(\underline{\underline{A}}_t^L)$.)

Let Y^n, $Y^{n'}$ be the processes defined by (3.1). By Proposition 3.4, and our hypothesis that pathwise uniqueness holds in (1.5), $Y^\infty = Y^{\infty'} = X$, and

(3.10) $$Y_t^n \downarrow X_t, \qquad Y^{n'} \uparrow X_t \qquad \text{for all } t \ge 0, \text{ a.s.}$$

Let X' be an $(\underline{\underline{A}}_t^L)$-adapted solution of (1.4). We now show that

(3.11) $$X_t' \le Y_t^n \qquad \text{for all } t \ge 0.$$

By Lemma 3.2 $X = X'$ on $[0,L]$, so (3.11) holds for $0 \le t \le L$. Let $S = \inf\{t > L: X_t' \ge Y_t^n\}$: as X' is continuous and $\Delta Y^n \ge 0$, we must have $X_S' = Y_S^n$. Further, as $Y_L^n > X_L$, $S > L$. By Corollary 3.3 if $S_2 = \inf\{t > S: Y_t^n = X_t\}$, we must have $X_t' = Y_t^n = \Lambda_t(Y_S^n, B, S)$ for $S \le t < S_2$. Similarly, $X' = X$ on $[S_2, \infty)$. Thus $X' \le Y^n$ on each of the intervals $[0,L]$, $[L,S]$, $[S,S_2]$ and $[S_2,\infty]$, proving (3.11).

Letting $n \to \infty$ in (3.11), and using (3.10) we deduce that $X' \le X$. Similarly, using $Y^{n'}$ instead of Y^n we have $X' \ge X$ and so $X' = X$.

(a) => (b). Suppose that pathwise uniqueness fails for either (1.5a) or (1.5b): let us assume it fails for (1.5a), and let $Y \ne X$ be a solution. Let T be a stopping time such that $P(Y_T \ne X_T) > 0$, and let

$$L = \sup\{t < T: Y_t = X_t\}, \quad R = \inf\{t \ge T: Y_t = X_t\},$$

and set $X_t' = X_t 1_{[0,L)}(t) + Y_t 1_{[L,R)}(t) + X_t 1_{[R,\infty)}(t)$. Then, since $Y \ne X$ on $[L,R)$ we have $L_R^0(Y-X) - L_L^0(Y-X) = 0$. It is now easily checked that X' is an $(\underline{\underline{F}}_t^L)$ adapted solution to (1.4), and that $X' \neq X$. So pathwise uniqueness fails for (1.4) in $(\underline{\underline{F}}_t^L)$, and we are done. ∎

Proof of Corollary 1.9 The condition (LT) implies pathwise uniqueness for (1.4), and that if Y, Y' are solutions of (1.5a) and (1.5b) then $L^0(Y-X) = L^0(X-Y') = 0$. Thus Y and Y' are also solutions of (1.4), and so X=Y=Y', so that (b) holds. ∎

4. Consequences of Non-Uniqueness

To prove Proposition 1.10 and Theorem 1.11 we will need a different approximation to X, where the jump of $+\epsilon_n$ by Y^n at the times T_k^n is replaced by a jump with a random sign.

We continue with the notation and hypotheses of Section 3. In particular, we continue to assume PU holds in (1.4) in $(\underline{\underline{A}}_t)$. Let

$$\underline{\underline{G}} = \sigma(X_t, B_t, {}^{\circ}A_t, L, t \geq 0).$$

Let $\epsilon > 0$: we define a process Z^ϵ, a sequence of stopping times, T_r^ϵ, and a sequence, ξ_r^ϵ, of $\underline{\underline{A}}_{T_r^\epsilon}$- measurable random variables as follows:

(4.1) $T_0^\epsilon = 0, \quad Z_0^\epsilon = x_0 + \epsilon\xi_0^\epsilon$

$$T_{r+1}^\epsilon = \inf\{t \geq T_r^\epsilon : \Lambda_t(Z^\epsilon(T_r^\epsilon), B, T_r^\epsilon) = X_t\},$$

$$Z_t^\epsilon = \Lambda_t(Z^\epsilon(T_r^\epsilon), B, T_r^\epsilon) \quad \text{on} \quad [T_r^\epsilon, T_{r+1}^\epsilon),$$

$$Z^\epsilon(T_r^\epsilon) = Z^\epsilon(T_r^\epsilon -) + \epsilon\xi_r^\epsilon,$$

$$P(\xi_r^\epsilon = +1 \,|\, \underline{\underline{A}}_{T_r^\epsilon -} \vee \underline{\underline{G}}) = P(\xi_r^\epsilon = -1 \,|\, \underline{\underline{A}}_{T_r^\epsilon -} \vee \underline{\underline{G}}) = 1/2 .$$

As $(\Omega, \underline{\underline{A}}, \underline{\underline{A}}_t, P)$ is saturated, random variables (ξ_r^ϵ) with these properties can be found. As in the proof of Proposition 3.4 we can check that Z^ϵ is well-defined. Let

$$A_t^\epsilon = \sum_{r=0}^{\infty} 1_{[T_r^\epsilon,\infty)}(t), \qquad N_t^\epsilon = \sum_{r=0}^{\infty} \xi_r^\epsilon 1_{[T_r^\epsilon,\infty)}(t) .$$

Then

$$Z_t^\epsilon = x_0 + \int_0^t \sigma(Z_s^\epsilon)\,dB_s + \epsilon N_t^\epsilon, \tag{4.2}$$

so that Z^ϵ is a perturbation of a solution to (1.4).

We wish to show the term ϵN_t^ϵ is small. Applying Tanaka's formula to $Z_t^\epsilon - X_t$ we have

$$|Z_t^\epsilon - X_t| = \int_0^t (\sigma(Z_s^\epsilon)-\sigma(X_s))\,\mathrm{sgn}(Z_{s-}^\epsilon - X_s)\,dB_s + \epsilon A_t^\epsilon + L_t^0(Z^\epsilon - X). \tag{4.3}$$

Since $\{t: Z_{t-}^\epsilon = X_t\}$ is countable, and $\{t: Z_t^\epsilon = X_t\} = \emptyset$, $L^0(Z^\epsilon - X) = 0$, and so $|Z_t^\epsilon - X_t| = V_t + \epsilon A_t^\epsilon$, where V is a continuous martingale satisfying $\langle V\rangle_t \le 4K^2 t$. Let $R = \max\{r: T_r^\epsilon \le t\}$: then $\epsilon = |Z_{T_R}^\epsilon - X_{T_R}| = V_{T_R} + \epsilon A_{T_R}^\epsilon$, so that $A_t^\epsilon = A_{T_R}^\epsilon \le \epsilon^{-1}(1 + \sup_{s\le t} |V_s|)$. Hence, for each $t \ge 0$,

$$\begin{aligned} E\, A_t^\epsilon &\le \epsilon^{-1} + \epsilon^{-1} E(\sup_{s\le t} |V_s|) \\ &\le \epsilon^{-1} + \epsilon^{-1} cKt^{1/2}, \end{aligned} \tag{4.4}$$

by the Burkholder-Davis-Gundy inequalities. So, as $\langle N^\epsilon\rangle = A^\epsilon$,

$$E(\epsilon N_t^\epsilon)^2 = \epsilon^2 E A_t^\epsilon \le \epsilon + \epsilon cKt^{1/2}. \tag{4.5}$$

Set $U_t^\epsilon = Z_t^\epsilon - \epsilon N_t^\epsilon$; (4.5) implies that

$$E(\sup_{0\le s\le t} |U_s^\epsilon - Z_s^\epsilon|^2) \to 0 \text{ as } \epsilon \to 0, \text{ for each } t \ge 0. \tag{4.6}$$

Let $(\overline{U}_t, \overline{X}_t, \overline{B}_t)$ denote the co-ordinate process on $C=C(\mathbf{R}_+,\mathbf{R}^3)$, and let Q^ϵ be the probability law on C induced by (U^ϵ,X,B). The estimate $\langle U^\epsilon\rangle_t - \langle U^\epsilon\rangle_s \le K^2(t-s)$, and the similar estimates for $\langle X\rangle$ and $\langle B\rangle$ imply that $\{Q^\epsilon, 0<\epsilon<1\}$ is tight. Let $\epsilon_k \downarrow 0$ be a subsequence such that Q^{ϵ_k} converges, and let $Q = \lim Q^{\epsilon_k}$.

<u>Lemma 4.1</u>. On the space $(C,\underline{\underline{B}}(C),Q)$,

(i) $\overline{B}$ is a Brownian motion,

(ii) $\overline{X}_t = x_0 + \int_0^t \sigma(\overline{X}_s)d\overline{B}_s$ and $\overline{U}_t = x_0 + \int_0^t \sigma(\overline{U}_s)d\overline{B}_s$.

The proof is as in Barlow (1982). The additional problems which arise when σ is discontinuous can be handled using the methods of Lemma 2.4.

<u>Proposition 4.2</u> With notation as above we have

(4.7) $$E(\sup_{s\le t} |Z_s^{\epsilon_k} - X_s|) \to 0 \text{ as } k \to \infty \text{ for each } t \ge 0.$$

<u>Proof</u>. By the assumption of pathwise uniqueness for (1.4) we must have $\overline{X} = \overline{U}$ under Q. Thus using the uniform bounds on $\overline{X}^*$ and $\overline{U}^*$ given by the Burkholder-Davis-Gundy inequalities we have, for any $t \ge 0$,

(4.8) $$0 = \lim_{k\to\infty} \int \sup_{s\le t} |\overline{U}_s - \overline{X}_s|dQ^{\epsilon_k}$$

$$= \lim_{k\to\infty} E\ (\sup_{s\le t} |U_s^{\epsilon_k} - X_s|).$$

Combining (4.8) and (4.6) we obtain (4.7). ∎

Proof of Proposition 1.10. Let $S = \inf\{t > L: X'_t = X_t\}$. The assertions that $X=X'$ on $[0,L]$ and $X_{L+t} \neq X'_{L+t}$ for small t follow from Lemma 3.2 and Corollary 3.3, respectively. For the latter note that $X = X'$ on $[S,\infty)$ and $X \neq X'$ a.s. implies $S > L$, and so $\operatorname{sgn}(X'_t - X_t)$ is constant on (L,S).

Let Z^{ϵ_n} be the processes constructed above, let $\xi_n = \operatorname{sgn}(Z_L^{\epsilon_n} - X_L)$, $T_n = \inf\{t > L: Z_{t-}^{\epsilon_n} = X_{t-}\}$ and set $V_t^n = Z_t^{\epsilon_n} 1\{t < T_n\} + X_t 1\{t \geq T_n\}$. As $Z_t^{\epsilon_n} \to X_t$ a.s., we have $V_t^n \to X_t$ a.s.

If $\xi_n = 1$ then, as in the proof of Theorem 1.6, we have $X' \leq Z^{\epsilon_n}$ on $[L,T_n]$, and hence $X' \leq V^n$ on $[L,\infty)$. Thus $X'_t \leq \inf\{V_t^n: \xi_n = 1\}$ on $[L,\infty)$, so that on $G^+ = \{\xi_n = +1$ for infinitely many $n\}$ we have $X' \leq X$ on $[L,\infty)$. Similarly, on $G^- = \{\xi_n = -1$ for infinitely many $n\}$ we have $X' \geq X$ on $[L,\infty)$. As Z^{ϵ_n} are $(\underline{\underline{A}}_t)$-optional processes, G^+ and G^- are $\underline{\underline{A}}_L$ measurable, and the result follows. ∎

Proof of Theorem 1.11 Note first that $P(L<\infty) = 1$. Suppose that PU does hold for (1.4), and let η_X (respectively, η_Y) be the common value of $\operatorname{sign}(X_{L+t} - x_0)$ (respectively, $\operatorname{sign}(Y_{L+t} - x_0)$) for small $t > 0$. By hypothesis $\eta_Y = -\eta_X$. However, by Proposition 1.10 η_Y is $\underline{\underline{A}}_L$ measurable, and so η_X is $\underline{\underline{A}}_L$-measurable. But by Yor (1979, Proposition 10)

$$E(X_T - x_0 | \underline{\underline{A}}_L) = E(\eta_X | \underline{\underline{A}}_L) = 0.$$

This implies $\eta_X = 0$, which gives a contradiction. ∎

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