Divergence form operators on fractal-like domains

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Abstract. We consider elliptic operators $\mathcal{L}$ in divergence form on certain domains in $\mathbb{R}^d$ with fractal volume growth. The domains we look at are pre-Sierpinski carpets, which are derived from higher dimensional Sierpinski carpets. We prove a Harnack inequality for non-negative $\mathcal{L}$-harmonic functions on these domains and establish upper and lower bounds for the corresponding heat equation.

Keywords. Harnack inequality, divergence form, Sierpinski carpet, fractals, heat equation, elliptic operators, fundamental solution, diffusions

AMS Subject Classifications. Primary: 35J15; Secondary: 60J60, 60J35

[1] Research partially supported by a NSERC (Canada) grant.
[2] Research partially supported by NSF Grant DMS 9700721.

1. Introduction.

The purpose of this paper is to consider divergence form operators on pre-Sierpinski carpets, to prove a Harnack inequality of Moser type, and to obtain estimates on the fundamental solution of the corresponding heat equation.

Let us begin by describing Sierpinski carpets, which are the class of fractal subsets of $\mathbb{R}^d$ formed by the following procedure. Let $d \geq 2$ and let $F_0 = [0,1]^d$. Let $l_{\mathcal{F}} \geq 3$ be an integer and divide F_0 into $(l_{\mathcal{F}})^d$ equal subcubes. Next remove a symmetric pattern of subcubes from F_0 and call what remains F_1. Now repeat the procedure: divide each subcube that is contained in F_1 into $l_{\mathcal{F}}^d$ equal parts, remove the same symmetric pattern from each as was done to obtain F_1 from F_0, and call what remains F_2. Continuing in this way we obtain a decreasing sequence of (closed) subsets of $[0,1]^d$. Let $\mathcal{F} = \cap_{n=0}^{\infty} F_n$; we call $\mathcal{F}$ a *Sierpinski carpet* or simply, a *carpet*. The standard Sierpinski carpet (see [Sie]) is the carpet for which $d = 2$, $l_{\mathcal{F}} = 3$, and F_1 consists of F_0 minus the central square. If $d = 3$, $l_{\mathcal{F}} = 3$, and F_1 consists of F_0 minus the 7 subcubes that do not share an edge with F_0, we obtain the Menger sponge; see [Man], p. 145 for a picture.

The domains we will consider are what are known as *pre-carpets* – see [O]. These are the sets $\mathcal{P} = \cup_{n=0}^{\infty} l_{\mathcal{F}}^n F_n$. (Here and throughout this paper we write $\lambda G = \{\lambda x : x \in G\}$). Note that $\mathcal{P} \subset \mathbb{R}_+^d$, and that $\mathcal{P} \cap [0, l_{\mathcal{F}}^n]^d$ consists of $[0, l_{\mathcal{F}}^n]^d$ with a number of (possibly adjacent) cubical holes removed, of sides varying from 1 to $l_{\mathcal{F}}^{n-1}$. If Γ is the interior of $\mathcal{P}$, then Γ is a (non-empty) domain in $\mathbb{R}^d$ with a piecewise linear boundary. We may regard pre-carpets as idealized models of a region with obstacles of many different sizes. The set $\mathcal{P}$ is not a fractal, since the interior of $\mathcal{P}$ is a non-empty domain in $\mathbb{R}^d$. However, if we write $V(x,R)$ for the volume of the intersection of $\mathcal{P}$ with the Euclidean ball of radius R centered at x, then $\mathcal{P}$ has 'fractal volume growth' in the sense that there exists $\alpha \in (1,d)$ such that

$$c_1 R^\alpha \leq V(x,R) \leq c_2 R^\alpha, \quad x \in \mathcal{P},\ R \geq 1.$$

Let $\mathcal{L}$ be the divergence form operator

$$\mathcal{L} f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j} f\Big)(x),$$

where the matrix $a_{ij}(x)$ is bounded, measurable, and symmetric for each x, and satisfies the uniform ellipticity condition

$$\lambda_1^2 |\xi|^2 \leq \sum_{i,j=1}^{d} \xi_i a_{ij}(x) \xi_j \leq \lambda_2^2 |\xi|^2, \quad x \in \mathcal{P}, \tag{1.1}$$

with $0 < \lambda_1 < \lambda_2 < \infty$. We will assume the a_{ij} are smooth, but our estimates will not depend on the smoothness of the a_{ij}. A function f is $\mathcal{L}$*-harmonic* on a subdomain D of $\mathcal{P}$ if $\mathcal{L}f = 0$ there and the conormal derivative of f is 0 almost everywhere on $D \cap \partial\mathcal{P}$. For further information on diffusions with conormal reflection, see [PW].

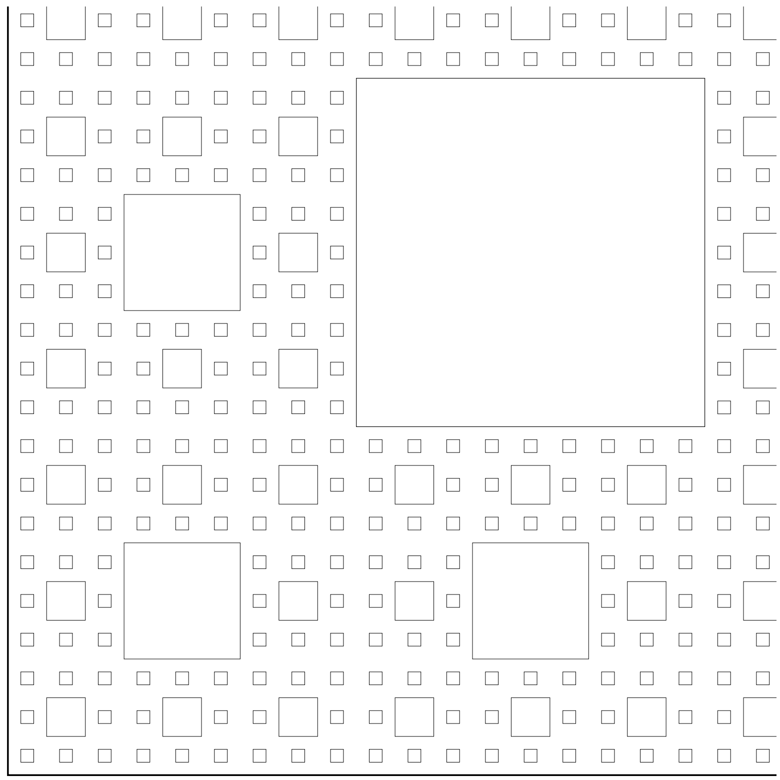

Figure 1: (Part of) a pre-carpet. The small squares have side length 1.

The main result of this paper is an elliptic Harnack inequality for $\mathcal{L}$-harmonic functions. For x, $y \in \mathcal{P}$ write $\gamma(x, y)$ for the (Euclidean) length of the shortest path in $\mathcal{P}$ connecting x and y, and let $B_\gamma(x, r) = \{y \in \mathcal{P} : \gamma(x, y) < r\}$.

Theorem 1.1. *Let $x \in \mathcal{P}$, $R > 0$ and suppose f is nonnegative and $\mathcal{L}$-harmonic in $B_\gamma(x, 2R)$. There exists c_1, not depending on x, R or f, such that*

$$f(x) \le c_1 f(y) \quad \text{for } x, y \in B_\gamma(x, R). \tag{1.2}$$

A crucial point is that c_1 does not depend on R for otherwise this result is an easy consequence of Moser's Harnack inequality. In a previous paper [BB3] we proved Theorem 1.1 in the case $\mathcal{L} = \frac{1}{2}\Delta$, using a probabilistic coupling argument. This Harnack inequality was then the key step in obtaining bounds on the fundamental solution to the heat equation on $\mathcal{P}$:

$$\frac{\partial u}{\partial t} = \Delta u, \tag{1.3}$$

where u has Neumann boundary conditions on $\partial\mathcal{P}$.

The arguments here do not replace those in [BB3]: we need various properties of the solutions to (1.3) (and the diffusion process W associated with them) to prove Theorem 1.1.

We should say a few words at this point about some of the difficulties in proving Theorem 1.1. Even a cursory glance at Moser's method [M1] for proving a Harnack inequality as well as the method of Nash-Davies-Fabes-Stroock (see [FS]) shows that a Sobolev or Nash inequality is a crucial ingredient. Sobolev inequalities exist in the context of Sierpinski carpets; see [BB3], Section 7. However, both approaches rely on the existence of suitable 'cut-off' functions with bounded gradient. The results of Kusuoka [K] for the Sierpinski gasket suggest that such functions do not exist in the carpet case, and that, while they do exist in the pre-carpet case, scaling of the right order will not hold. Thus it is not clear how to use the above methods to prove Theorem 1.1 with c_1 independent of R.

Our method is a variation of the Moser technique. The key step is to prove a weighted Sobolev inequality, where the L^p norm is with respect to an energy measure for the precarpet, rather than Lebesgue measure. We prove this inequality by first using Dirichlet form techniques to obtain a weighted Poincaré inequality, then to derive a weighted Nash inequality, and finally from this we obtain the weighted Sobolev inequality.

As in [BB3], once we have an elliptic Harnack inequality it is relatively straightforward to obtain bounds on the solutions of the associated heat equation $\partial u/\partial t = \mathcal{L}u$ on $\mathcal{P}$. The bounds are quite different from those for the heat equation on $\mathbb{R}^d$. In the latter case, Aronson's bounds tell us that the fundamental solution $p(t,x,y)$ to $\partial u/\partial t = \mathcal{L}u$ on $\mathbb{R}^d$ is comparable to

$$c_1 t^{-d/2} \exp(-c_2|x-y|^2/t).$$

More precisely, for all x,y,t we have that $p(t,x,y)$ is bounded above by $c_1 t^{-d/2}\exp(-c_2|x-y|^2/t)$ and bounded below by $c_3 t^{-d/2}\exp(-c_4|x-y|^2/t)$ for constants c_1,c_2,c_3,c_4. In contrast, for the heat equation on $\mathcal{P}$, there exist constants d_w and d_s depending on $\mathcal{P}$ such that for all $t\geq 1$ such that $|x-y|\leq t$ the fundamental solution $q(t,x,y)$ is comparable to

$$c_1 t^{-d_s/2} \exp\Big(-c_2\Big(\frac{|x-y|^{d_w}}{t}\Big)^{1/(d_w-1)}\Big);$$

moreover $d_w > 2$. See Theorem 5.3 for a precise statement.

Our results are also of interest in light of recent research concerning necessary and sufficient conditions for a parabolic Harnack inequality to hold (see Grigor'yan [G] and Saloff-Coste [SC]) and also the discussion in [BB3]). Although both a Poincaré inequality and a volume doubling condition hold, the proofs of [G] and [SC] are not sufficient to prove that a parabolic or elliptic Harnack inequality hold on a Sierpinski carpet; their methods again require the use of suitable functions with bounded gradient.

In a series of papers Sturm has studied Harnack inequalities on metric spaces - see for example [St]. However, the hypotheses imposed rule out spaces with the kind of large scale fractal structure that pre-carpets have.

The layout of this paper is as follows. Section 2 introduces the notation we will use together with a few basic facts. Section 3 contains the proof of the weighted Sobolev inequality. We prove Theorem 1.1 in Section 4. The heat kernel bounds are derived in Section 5.

2. Notation and preliminaries.

We begin by setting up our notation. We use the letter c with subscripts to denote constants which depend only on the dimension d and the carpet $\mathcal{F}$. We renumber the constants for each lemma, proposition, theorem, and corollary. We use the notation $A \asymp B$ to mean $c_1 A \le B \le c_2 A$, where c_i are as above.

Let $d \ge 2$, $F_0 = [0,1]^d$, and let $l_{\mathcal{F}} \in \mathbb{N}$, $l_{\mathcal{F}} \ge 3$ be fixed. For $n \in \mathbb{Z}$ let $\mathcal{S}_n$ be the collection of closed cubes of side length $l_{\mathcal{F}}^{-n}$ with vertices in $l_{\mathcal{F}}^{-n}\mathbb{Z}^d$. For $A \subseteq \mathbb{R}^d$, set

$$\mathcal{S}_n(A) = \{S \colon S \subset A, S \in \mathcal{S}_n\}.$$

For $S \in \mathcal{S}_n$, let Ψ_S be the orientation preserving affine map which maps F_0 onto S.

We now define a decreasing sequence (F_n) of closed subsets of F_0. Let $1 \le m_{\mathcal{F}} < l_F^d$ be an integer, and let F_1 be the union of $m_{\mathcal{F}}$ distinct elements of $\mathcal{S}_1(F_0)$. We impose the following conditions on F_1:

Hypotheses 2.1.

(H1) (Symmetry) F_1 is preserved by all the isometries of the unit cube F_0.

(H2) (Connectedness) The interior of F_1 is connected, and contains a path connecting the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$.

(H3) (Non-diagonality) Let B be a cube in F_0 which is the union of 2^d distinct elements of $\mathcal{S}_1$. (So B has side length $2l_{\mathcal{F}}^{-1}$). Then if the interior of $F_1 \cap B$ is non-empty, it is connected.

(H4) (Borders included) F_1 contains the line segment $\{x : 0 \le x_1 \le 1, x_2 = ... = x_d = 0\}$.

Of these, (H1) and (H2) are essential, while (H3) and (H4) could be weakened somewhat. See the discussion in [BB3].

We may think of F_1 as being derived from F_0 by removing the interiors of $l_{\mathcal{F}}^d - m_{\mathcal{F}}$ squares in $\mathcal{S}_1(F_0)$. Given F_1, F_2 is obtained by removing the same pattern from each of the squares in $\mathcal{S}_1(F_1)$. Iterating, we obtain a sequence (F_n), where F_n is the union of m_F^n squares in $\mathcal{S}_n(F_0)$. Formally, we define

$$F_{n+1} = \bigcup_{S \in \mathcal{S}_n(F_n)} \Psi_S(F_1) = \bigcup_{S \in \mathcal{S}_1(F_1)} \Psi_S(F_n), \quad n \ge 1.$$

We call the set $\mathcal{F} = \cap_{n=0}^{\infty} F_n$ a *Sierpinski carpet.*

Set

$$\mathcal{P} = \bigcup_{r=0}^{\infty} l_{\mathcal{F}}^r F_r.$$

We call $\mathcal{P}$ the *pre-carpet* (see [O]). We define the *unbounded scaled pre-carpet* $\mathcal{P}_N$ by

$$\mathcal{P}_N = l_{\mathcal{F}}^{-N} \mathcal{P} = \bigcup_{r=0}^{\infty} l_{\mathcal{F}}^{r-N} F_r, \quad N \geq 0.$$

Until the end of Section 4 we will fix $N \geq 0$, and work on the scaled pre-carpet $\mathcal{P}_N$. Any dependence of constants on N will be given explicitly.

We will require a certain amount of notation to describe various subsets of $\mathcal{P}_N$. Let $\mathcal{S}_n^*$ be the set of cubes in $\mathbb{R}^d$ of side length $2l_{\mathcal{F}}^{-n}$ which are unions of 2^d cubes in $\mathcal{S}_n$. For $x \in \mathcal{P}_N$ let $Q(x)$ be the cube in $\mathcal{S}_n^*$ with center closest to x. (We use some procedure to break ties.) Set $D_n(x) = Q(x) \cap \mathcal{P}_N$.

If $x = (x_1, \ldots, x_d)$ is a point in $\mathbb{R}^d$, write $\|x\|_{l^\infty} = \max_{1 \leq i \leq d} |x_i|$. For $x, y \in \mathcal{P}_N$ let $d(x, y)$ denote the length of the shortest path (i.e., geodesic) in $\mathcal{P}_N$ connecting x and y, where the length of the path is measured in terms of the l^∞ norm. We have

$$d(x,y) = \lim_{\delta \to 0} \Big(\inf \Big\{ \sum_{i=0}^{m} ||x_i - x_{i-1}||_{l^\infty} : x_0 = x, x_m = y, ||x_i - x_{i-1}||_{l^\infty} < \delta, x_i \in \mathcal{P}_N \Big\} \Big).$$

We write $B(x, r) = \{y \in \mathcal{P}_N : d(x, y) < r\}$. Note that the boundary of $B(x, r)$ is a finite union of flat surfaces orthogonal to the axes. We write $\operatorname{diam}(A)$ for the diameter of A in the metric d, and $\operatorname{dist}(A, B)$ for the distance between the sets A and B.

For $A \subset \mathbb{R}^d$ we write A^o, $\operatorname{cl}(A)$, A^c for the usual interior, closure and complement of A, respectively. Let D be a relatively open set in $\mathcal{P}_N$ (in the metric d). We write $\partial_a D$ for the relative boundary of D in $\mathcal{P}_N$, and $\partial_e D$ for the boundary of D in $\mathbb{R}^d$. Set $\partial_r D = \partial_e D - \partial_a D$. We use the subscripts a, e, r as mnemonics for 'absorbing', 'everywhere', and 'reflecting', respectively. If $S \in \mathcal{S}_n$ for some $n \in \mathbb{Z}$, and $S^o \cap \mathcal{P}_N$ is non-empty we call $S \cap \mathcal{P}_N$ a *special* $\mathcal{P}_N$ *cube*. Note that if Q is any special $\mathcal{P}_N$ cube of side length $l_{\mathcal{F}}^{-n}$ then Q is isomorphic to $l_{\mathcal{F}}^{-n} F_{N-n}$, where we write $F_{-k} = [0,1]^d$, $k \geq 1$. If $B = B(x, r)$ then we define $B^* = B(x, 2r)$. Let $I = S \cap \mathcal{P}_N$ be a special $\mathcal{P}_N$ cube. Write S' for the cube with side length 3 times that of S with the same center as S and faces parallel to those of S. Let $I^* = \operatorname{cl}((S' \cap \mathcal{P}_N)^o)$.

Let $Q \subset \mathbb{R}^d$ be a cube with edges parallel to the axes. We call any set of the form $Q \cap \mathcal{P}_N$ a $\mathcal{P}_N$*-cube*. Note that $\mathcal{P}_N$ cubes do not have to be connected.

We define the resistance constant R_n by

$$R_n^{-1} = \inf \Big\{ \int_{l_{\mathcal{F}}^n F_n} |\nabla f|^2 dx : f = 0 \text{ on } x_1 = 0, f = 1 \text{ on } x_1 = l_{\mathcal{F}}^n \Big\}.$$

Thus R_n is the resistance between two opposite faces of the set $l_{\mathcal{F}}^n F_n$. It is known (see [BB3], [McG]) that there exists a constant $\rho_{\mathcal{F}}$ and constants c_1, c_2 such that

$$c_1 \rho_{\mathcal{F}}^n \le R_n \le c_2 \rho_{\mathcal{F}}^n.$$

Let $t_{\mathcal{F}} = (m_{\mathcal{F}})(\rho_{\mathcal{F}})$. We define the fractal dimension, dimension of the walk, and spectral dimension of $\mathcal{F}$ by

$$\begin{aligned} d_f &= \log m_{\mathcal{F}} / \log l_{\mathcal{F}}, \\ d_w &= \log t_{\mathcal{F}} / \log l_{\mathcal{F}}, \\ d_s &= 2d_f/d_w = 2 \log m_{\mathcal{F}} / \log t_{\mathcal{F}}. \end{aligned}$$

d_f is the Hausdorff dimension (and also the packing dimension) of $\mathcal{F}$. We remark (see [BB3, Remark 5.4]) that we have $d_w > 2$. We will also use

$$\zeta = \frac{l_{\mathcal{F}}^2}{t_{\mathcal{F}}} = l_{\mathcal{F}}^{2-d_w}. \tag{2.1}$$

Since $d_w > 2$ we have $\zeta < 1$. Let

$$\kappa = \frac{m_{\mathcal{F}}}{l_{\mathcal{F}}^d} = l_{\mathcal{F}}^{d_f - d}.$$

This is the Lebesgue measure of F_1; we have $\kappa < 1$.

Let $|A|$ denote the Lebesgue measure of a Borel set A. If Q is a special $\mathcal{P}_N$ cube of side length $s = l_{\mathcal{F}}^{-n}$ then it is easy to check that

$$|Q| = \begin{cases} s^d, & \text{if } s \le l_{\mathcal{F}}^{-N}, \\ \kappa^N s^{d_f}, & \text{if s} \ge l_{\mathcal{F}}^{-N}. \end{cases}$$

As the ball $B(x,r)$ contains a special $\mathcal{P}_N$ cube of side length $s \ge c_1 r$, and can be covered by c_2 or fewer special $\mathcal{P}_N$ cubes of side length $s \in [r, r l_{\mathcal{F}})$ we deduce that

$$|B(x,r)| \asymp \begin{cases} r^d, & \text{if } r \le l_{\mathcal{F}}^{-N} \\ \kappa^N r^{d_f}, & \text{if } r \ge l_{\mathcal{F}}^{-N}. \end{cases} \tag{2.2}$$

Note that this implies that Lebesgue measure on $\mathcal{P}_N$ has the volume doubling property:

$$|B(x, 2r)| \le c_1 |B(x,r)|, \quad r > 0.$$

Let W_t be Brownian motion on $\mathcal{P}_N$ with normal reflection on the boundary of $\mathcal{P}_N$. Define $Y_t^N = W(\zeta^{-N} t)$. Then Y_t^N is a process on $\mathcal{P}_N$ with generator $\frac{1}{2}\zeta^{-N}\Delta$, and Green function that is ζ^N times that of W_t.

If D is a domain in $\mathcal{P}_N$ (so $D \subset \mathcal{P}_N$, D is connected and relatively open in $\mathcal{P}_N$) write $u_D(x,y)$ for the Green function of Y^N on D. Then u_D is symmetric, continuous except on the diagonal $\{x = y\}$ and satisfies

$$\tfrac{1}{2}\Delta u_D(x,y) = -\zeta^N \delta_x(y), \quad x, y \in D$$

in the distributional sense, where δ_x is point mass at x. If D is suitably regular (such as a $\mathcal{P}_N$-cube or a ball) then we have $u_D(x,y) \to 0$ as $y \to \partial_a D$; we extend u_D to $\mathcal{P}_N \times \mathcal{P}_N$ by taking it to be zero off $D \times D$.

Let D be a domain in $\mathcal{P}_N$, and $A \subset D$. Define

$$U(x, A, D) = \int_A u_D(x,y)dy = \mathbb{E}^x \int_0^{\tau_D} 1_A(Y_s^N)ds;$$

here $\tau_D = \inf\{s > 0 : Y_s^N \in D^c\}$. Note that U is monotone in A and D: if $A \subset A' \subset D \subset D'$ then

$$U(x, A, D) \le U(x, A', D) \le U(x, A', D'). \tag{2.3}$$

Define the function

$$\psi(r) = \begin{cases} r^{d_w}, & \text{if } r \ge l_{\mathcal{F}}^{-N}, \\ \zeta^N r^2 & \text{if } r \le l_{\mathcal{F}}^{-N}. \end{cases}$$

Note that we can also write $\psi(r) = r^{d_w} \vee \zeta^N r^2$.

Lemma 2.2. *Let B be either a special $\mathcal{P}_N$ cube of side length r or a ball $B(x_0, r)$. Then*

(a) $$U(x, B, B) \le c_1 \psi(r) \text{ for } x \in \mathcal{P}_N,$$

(b) $$U(x, B^*, B^*) \ge c_2 \psi(r) \text{ for } x \in B.$$

(c) $$U(x, B, B^*) \ge c_3 \psi(r) \text{ for } x \in B.$$

(d) If $\frac{1}{2}t \le s < t$ then

$$U(x, B(x_0, s), B(x_0, t)) \ge c_4 \psi(t - s) \text{ for } x \in B(x_0, s).$$

Proof. (a) and (b) follow from the estimates on hitting times of sets in [BB3], Proposition 5.5.
(c) We just do the case when $B = B(x_0, r)$; the result for special $\mathcal{P}_N$ cubes is very similar. Let $B_1 = B(x_0, r/2)$. Using the Markov property of Y^N and writing $T_1 = \inf\{t \ge 0 : Y_t^N \in B_1\}$, $\tau = \inf\{t \ge 0 : Y_t^N \notin B^*\}$, we have for $x \in B$,

$$\begin{aligned} U(x, B, B^*) &\ge \mathbb{P}^x(T_1 < \tau) \inf_{y \in B_1} U(y, B, B^*) \\ &\ge \mathbb{P}^x(T_1 < \tau) \inf_{y \in B_1} U\big(y, B(y, r/2), B(y, r/2)\big) \\ &\ge c_5 \mathbb{P}^x(T_1 < \tau)\psi(r/2). \end{aligned}$$

Here we used (2.3) and (b) to obtain the final line. It follows from the estimates on the transition density of Y^N given in [BB3], Section 6, that there exists $c_6 > 0$ such that $\mathbb{P}^x(T_1 < \tau) > c_6$. Since $\psi(r/2) \ge c_7 \psi(r)$, (c) follows.

(d) Let $y \in B(x_0, s)$. We can find a point z on the geodesic connecting y and x_0 such that $y \in B(z, t-s) \subset B(x_0, s)$. Then $B(z, 2(t-s)) \subset B(x_0, t)$, so, using (2.3),

$$c_8 \psi(t-s) \le U(y, B(z, t-s), B(z, 2(t-s)) \le U(y, B(x_0, s), B(x_0, t)).$$

□

The following result generalizes Theorem 5.3 of [BB1]. Since there is an error in the proof of that result, we give details of the proof.

Lemma 2.3. *There exist constants c_1, $\beta > 0$ such that if D is a $\mathcal{P}_N$-cube with side length less than $l_{\mathcal{F}}^2$ and A is a Borel subset of D*

$$|U(x, A, D) - U(y, A, D)| \le c_1 |x-y|^\beta, \quad x, y \in D. \tag{2.4}$$

Proof. Let $f = 1_A$ and write

$$U_D f(x) = \int_D u_D(x, y) f(y) dy = U(x, A, D).$$

For $B \subset \mathcal{P}_N$ let

$$\tau_B = \inf\{t \ge 0 : Y_t^N \in B^c\}$$

be the first exit time of Y^N from B.

By a proof almost identical with that of Theorem 5.2 of [BB1], there exist constants c_2, $\beta_1 > 0$ such that if D is a $\mathcal{P}_N$-cube with side length less than $l_{\mathcal{F}}^2$, then

$$\mathbb{E}^x \tau_D \le c_2 d(x, \partial_a D)^{\beta_1}, \quad x \in D. \tag{2.5}$$

Fix x, $y \in D$. If $4d(x, y)^{1/2} > \operatorname{dist}(x, \partial_a D)$, then

$$|U_D f(x) - U_D f(y)| \le E^x \tau_D + \mathbb{E}^y \tau_D \le c_3 d(x, y)^{\beta_1/2}$$

by (2.5), and the theorem is proved in this case..

Now look at the case where $4d(x, y)^{1/2} \le \operatorname{dist}(x, \partial_a D)$. Let $\delta = 4d(x, y)^{1/2}$ and let $B = B(x, \delta)$. Then $B \subset D$ and if $z \in B$, then $\tau_B \le \tau_D$. So by the strong Markov property

$$\begin{aligned} U_D f(z) &= \mathbb{E}^z \int_0^{\tau_B} f(Y_t^N) dt + \mathbb{E}^z \int_{\tau_B}^{\tau_D} f(Y_t^N) dt \\ &= \mathbb{E}^z \int_0^{\tau_B} f(Y_t^N) dt + \mathbb{E}^z U_D f(Y_{\tau_B}^N), \qquad z \in B. \end{aligned} \tag{2.6}$$

The function $z \to \mathbb{E}^z U_D f(Y^N_{\tau_B})$ is harmonic in B. The elliptic Harnack inequality for Y^N (see Theorems 4.2 and 4.3 of [BB3]) implies there exist constants c_4, β_2 such that for $x', y' \in B(x, \delta/2)$

$$|\mathbb{E}^{x'} U_D f(Y^N_{\tau_B}) - \mathbb{E}^{y'} U_D f(Y^N_{\tau_B})| \le c_4 \Big(\frac{d(x',y')}{\delta} \Big)^{\beta_2} \|U_D f\|_\infty. \tag{2.7}$$

By (2.5) we have

$$|U_D f(z)| \le \|f\|_\infty \mathbb{E}^z \tau_D \le c_5, \qquad z \in D,$$

and

$$\mathbb{E}^z \int_0^{\tau_B} f(Y^N_t) dt \le \|f\|_\infty \mathbb{E}^z \tau_B \le c_6 \delta^{\beta_1}. \tag{2.8}$$

Combining (2.6), (2.7) with $x' = x$ and $y' = y$, and (2.8) with z first equal to x and then equal to y, we obtain our result in this case also. □

There is a Poincaré inequality for $\mathcal{P}_N$ which may be stated in the following form.

Lemma 2.4. *Let B be either a special $\mathcal{P}_N$ cube of side length r or a ball of radius r, and let $I = B$ or $I = B^*$. Suppose the gradient of f is square integrable over I. Then, writing $f_I = |I|^{-1} \int_I f$,*

$$\int_I |f - f_I|^2 \le c_1 \psi(r) \zeta^{-N} \int_I |\nabla f|^2. \tag{2.9}$$

Proof. If $N = 0$ and $I = D_n(x)$ for some $n \le 0$ then this is Proposition 7.10 of [BB3]. The case $N = 0$ and $I = D_n(x)$ with $n > 0$ is the usual Poincaré inequality in $\mathbb{R}^d$. The same argument as in [BB3] also proves this for special $\mathcal{P}_N$ cubes. The case with $N \ge 1$ follows easily by scaling.

To obtain (2.9) when I is a ball we use the argument of Jerison [J], Section 5. We write $s(D_n(x))$ for the diameter (in the metric d) of the set $D_n(x)$: we have $s(D_n(x)) \asymp l_{\mathcal{F}}^{-n}$. Then it is quite straightforward to find a Whitney decomposition $\mathcal{F} = \{D_i = D_{n_i}(x_i), i \ge 1\}$ of I with the following properties:

The sets D_i^o are pairwise disjoint,
$I = \cup_i D_{n_i - 2}(x_i)$,
For each $y \in I$, $\quad \#\{i : y \in D_{n_i - 4}(x_i)\} \le c_2$,
$l_{\mathcal{F}}^6 \le \text{dist}\,(x_i, \partial_a I)/s(D_{n_i}(x_i)) \le l_{\mathcal{F}}^{12}, \quad i \ge 1.$

Then, working with the sets D_i rather than balls, the remainder of Jerison's argument follows, with only minor changes, to give (2.9) for balls. □

3. Sobolev and other inequalities.

We continue to work on $\mathcal{P}_N$, and will mention explicitly any dependence on N in our estimates.

For the remainder of this section we fix two $\mathcal{P}_N$-cubes $Q(h) \subset Q(k)$ such that $h = \operatorname{diam}(Q(h)) \le k = \operatorname{diam}(Q(k)) \le l_{\mathcal{F}}^2$. Set

$$r(x) = U(x, Q(h), Q(k)), \qquad x \in \mathcal{P}_N,$$
$$\gamma = 1 + \zeta^{-N} |\nabla r|^2.$$

Note that $r = 0$ off $Q(k)$, r is strictly positive on $Q(h)$, and $\frac{1}{2}\Delta r = -\zeta^N 1_{Q(h)}$.

Lemma 3.1. *(a) r satisfies the bound*

$$r(x) \le c_1, \qquad x \in \mathcal{P}_N.$$

(b) There exists c_2 such that

$$\int_{Q(k)} |\nabla r|^2 \le c_2 \zeta^N |Q(h)| \sup_{Q(k)} |r|.$$

Proof. (a) is immediate from Lemma 2.2, and the fact that $Q(k)$ is contained in a ball of radius $l_{\mathcal{F}}^2$. For (b) note that $r = 0$ on $\partial_a Q(k)$ and $\partial r / \partial n = 0$ on $\partial_r Q(k)$. So by Green's first identity in the domain $Q(k)$,

$$\int_{Q(k)} |\nabla r|^2 = -\int_{Q(k)} r \Delta r = 2\zeta^N \int_{Q(k)} r 1_{Q(h)}.$$

The result is now immediate. □

We will need the following elementary lemma.

Lemma 3.2. *Let $x, y, z \ge 0$. If $x \le c_1(x^{1/2} z^{1/2} + y)$, then*

$$x \le 2c_1 y + 4c_1^2 z.$$

Proof. If $x < 2c_1 y$, we are done. If $x \ge 2c_1 y$, then $c_1 x^{1/2} z^{1/2} \ge x - c_1 y \ge x/2$, so $2c_1 z^{1/2} \ge x^{1/2}$. □

We begin by proving a weighted Poincaré inequality.

Proposition 3.3. *Let $Q(h), Q(k), r$ be as in Lemma 3.1. Let $I \subset Q(k)$ be either a special $\mathcal{P}_N$ cube of side length s or a ball of radius s. Suppose f and its gradient are square integrable over I^*. There exists $c_1 > 0$ such that*

$$\int_I f^2 |\nabla r|^2 \le c_1 s^{2\beta} \Big(\int_{I^*} |\nabla f|^2 + \zeta^N \psi(s)^{-1} \int_{I^*} f^2 \Big).$$

Proof. Let $\varphi = U(\cdot, I, I^*)$, and write $\Phi_0 = \inf_I \varphi$, $\Phi_1 = \sup_{I^*} \varphi$. By Lemma 2.2 we have

$$c_2 \psi(s) \le \Phi_0 \le \Phi_1 \le c_3 \psi(s).$$

Write

$$\begin{aligned} A &= \int_{I^*} f^2 \varphi^2 |\nabla r|^2, \\ B &= \int_{I^*} \varphi^2 |\nabla f|^2, \\ C &= \int_{I^*} f^2, \\ D &= \int_{I^*} f^2 |\nabla \varphi|^2, \\ E &= \int_{I^*} |\nabla f|^2. \end{aligned}$$

Then

$$\int_I f^2 |\nabla r|^2 \le (\inf_I \varphi)^{-2} \int_I f^2 |\nabla r|^2 \varphi^2 \le \Phi_0^{-2} A.$$

We begin by bounding A. Choose $x_0 \in I$. If $I^* \not\subset Q(k)$, set $\widetilde{r} = r$. If $I^* \subset Q(k)$, set $\widetilde{r} = r - r(x_0)$. In either case we see that there exists a point in I^* at which $\widetilde{r}$ is zero. Also, $\nabla \widetilde{r} = \nabla r$, which is 0 off $Q(k)$. Set

$$R = \sup_{I^*} \widetilde{r}.$$

By Lemma 2.3 $|\widetilde{r}(x) - \widetilde{r}(y)| \le c_4 |x-y|^\beta$ if $x, y \in I^*$, and therefore $R \le c_5 s^\beta$.

We write

$$\begin{aligned} A = \int_{I^*} f^2 \varphi^2 |\nabla r|^2 &= \int_{I^*} f^2 \varphi^2 |\nabla \widetilde{r}|^2 = \int_{Q(k)} f^2 \varphi^2 |\nabla \widetilde{r}|^2 \\ &= \tfrac{1}{2} \int_{Q(k)} f^2 \varphi^2 \Delta(\widetilde{r}^2) - \int_{Q(k)} f^2 \varphi^2 \widetilde{r} \Delta \widetilde{r}, \end{aligned} \tag{3.1}$$

where in the last line we used the identity $|\nabla u|^2 = \frac{1}{2} \Delta u^2 - u \Delta u$.

Now consider the first term on the right hand side of (3.1). If $I^* \not\subset Q(k)$, then $\partial(\widetilde{r}^2)/\partial n = 2\widetilde{r}(\partial\widetilde{r}/\partial n) = 2r(\partial r/\partial n)$, which is 0 on $\partial_a Q(k)$ and $\partial_r Q(k)$. If $I^* \subset Q(k)$, then $\partial(\widetilde{r}^2)/\partial n = 2\widetilde{r}(\partial\widetilde{r}/\partial n) = 2\widetilde{r}(\partial r/\partial n)$ is 0 on $\partial_r Q(k)$ and $f^2\varphi^2$ is 0 on $\partial_a Q(k)$. So by Green's first identity,

$$\int_{Q(k)} f^2\varphi^2\Delta(\widetilde{r}^2) = -\int_{Q(k)} \nabla(f^2\varphi^2)\nabla(\widetilde{r}^2).$$

Thus

$$\begin{aligned} A &= -\tfrac{1}{2}\int_{Q(k)} \nabla(f^2\varphi^2)\nabla(\widetilde{r}^2) - \int_{Q(k)} f^2\varphi^2\widetilde{r}\Delta\widetilde{r} \\ &\le \Big|\int_{Q(k)} \nabla(f^2\varphi^2)\nabla(\widetilde{r}^2)\Big| + c_6 R\zeta^N \int_{I^*} f^2\varphi^2 \\ &\le 4\Big|\int_{Q(k)} f^2\varphi(\nabla\varphi)\widetilde{r}\nabla\widetilde{r}\Big| + 4\Big|\int_{Q(k)} f(\nabla f)(\varphi^2)\widetilde{r}\nabla\widetilde{r}\Big| + c_6 R\zeta^N\Phi_1^2\int_{I^*} f^2 \\ &\le 4\Big(\int f^2\varphi^2\widetilde{r}^2|\nabla\widetilde{r}|^2\Big)^{1/2}\Big(\int f^2|\nabla\varphi|^2\Big)^{1/2} \\ &\qquad + 4\Big(\int f^2\widetilde{r}^2|\nabla\widetilde{r}|^2\varphi^2\Big)^{1/2}\Big(\int \varphi^2|\nabla f|^2\Big)^{1/2} + c_6 R\zeta^N\Phi_1^2 C \\ &\le c_7(R^2A)^{1/2}(D^{1/2}+B^{1/2}) + c_6R\zeta^N\Phi_1^2C. \end{aligned}$$

As $D^{1/2}+B^{1/2} \le 2(B+D)^{1/2}$, by Lemma 3.2

$$A \le c_8R^2(B+D) + c_8\zeta^N R\Phi_1^2 C.$$

We now bound D. We have

$$D = \int_{I^*} f^2|\nabla\varphi|^2 = \tfrac{1}{2}\int_{I^*} f^2\Delta(\varphi^2) - \int_{I^*} f^2\varphi\Delta\varphi.$$

Since $\partial(\varphi^2)/\partial n = 2\varphi(\partial\varphi)/\partial n$ is 0 on $\partial_r I^*$ and 0 on $\partial_a I^*$, by Green's first identity

$$\int_{I^*} f^2\Delta(\varphi^2) = -\int_{I^*} \nabla(f^2)\nabla(\varphi^2).$$

Since $|\Delta\varphi|$ is bounded by $2\zeta^N$ on I^*, then $|\int_{I^*} f^2\varphi\Delta\varphi| \le c_9\zeta^N\Phi_1 C$. So

$$\begin{aligned} D &\le \Big|\int_{I^*} \nabla(f^2)\nabla(\varphi^2)\Big| + c_9\zeta^N\Phi_1C \\ &= 4\Big|\int_{I^*} f(\nabla f)\varphi\nabla\varphi\Big| + c_9\zeta^N\Phi_1C \\ &\le 4\Big(\int f^2|\nabla\varphi|^2\Big)^{1/2}\Big(\int \varphi^2|\nabla f|^2\Big)^{1/2} + c_9\zeta^N\Phi_1C \\ &\le c_{10}(D^{1/2}B^{1/2} + \zeta^N\Phi_1C). \end{aligned}$$

Using Lemma 3.2 again we conclude that

$$D \le c_{11}(B + \zeta^N \Phi_1 C).$$

Finally, as $B \le \Phi_1^2 E$, we deduce that

$$A \le c_{12} R^2 \Phi_1^2 E + c_{12} \zeta^N (R\Phi_1^2 + R^2 \Phi_1) C.$$

Since $\psi(s) \le c_{13} s^\beta$, and $R \le c_{14} s^\beta$ we have,

$$\int_I f^2 |\nabla r|^2 \le \Phi_0^{-2} A \le c_{15} (\Phi_1/\Phi_0)^2 s^{2\beta} E + c_{15} \zeta^N (\Phi_1/\Phi_0)^2 s^{2\beta} \Phi_1^{-1} C.$$

Using the bounds above on Φ_i the conclusion follows . □

Corollary 3.4. *Let f, I, and I^* be as in Proposition 3.3.*
(a) Then if $f_{I^*} = |I^*|^{-1} \int_{I^*} f$,

$$\int_I (f - f_{I^*})^2 \gamma \le c_1 \zeta^{-N} s^{2\beta} \int_{I^*} |\nabla f|^2. \tag{3.2}$$

(b) Further,

$$\int_I f^2 \gamma \le c_2 \zeta^{-N} s^{2\beta} \int_{I^*} |\nabla f|^2 + |I|^{-1} \Big(\int_I |f| \gamma \Big)^2.$$

Proof. Applying Proposition 3.3 to $f - f_{I^*}$ we deduce

$$\int_I (f - f_{I^*})^2 |\nabla r|^2 \le c_3 s^{2\beta} \Big(\int_{I^*} |\nabla f|^2 + \zeta^N \psi(s)^{-1} \int_{I^*} (f - f_{I^*})^2 \Big). \tag{3.3}$$

By the Poincaré inequality Lemma 2.4 we have

$$\int_I (f - f_{I^*})^2 \le \int_{I^*} (f - f_{I^*})^2 \le c_4 \zeta^{-N} \psi(s) \int_{I^*} |\nabla f|^2. \tag{3.4}$$

Substituting the second inequality of (3.4) into (3.3),

$$\int_I (f - f_{I^*})^2 |\nabla r|^2 \le c_5 s^{2\beta} \int_{I^*} |\nabla f|^2. \tag{3.5}$$

Since $\psi(s) \le c_6 s^\beta$ we obtain (a) by adding (3.4) and (3.5).

(b) Now let $b = \int_I f\gamma / \int_I \gamma$. Then

$$\begin{aligned} \int_I f^2 \gamma &= \int_I (f - b)^2 \gamma + b^2 \int_I \gamma \\ &= \int_I (f - b)^2 \gamma + \Big(\int_I \gamma \Big)^{-1} \Big(\int_I f\gamma \Big)^2. \\ &\le \int_I (f - f_{I^*})^2 \gamma + |I|^{-1} \Big(\int_I f\gamma \Big)^2. \end{aligned} \tag{3.6}$$

Combining (3.2) and (3.6) completes the proof. □

We can obtain a sharper result if we just consider special $\mathcal{P}_N$ cubes.

Corollary 3.5. *Let I be a special $\mathcal{P}_N$ cube. Then*

$$\int_I f^2|\nabla r|^2 \le c_1 s^{2\beta}\Big(\int_I |\nabla f|^2 + \zeta^N \psi(s)^{-1}\int_I f^2\Big), \tag{3.7}$$

$$\int_I f^2\gamma \le c_1\zeta^{-N}s^{2\beta}\int_I |\nabla f|^2 + |I|^{-1}\Big(\int_I |f|\gamma\Big)^2. \tag{3.8}$$

Proof. Note that the left-hand sides of (3.7) and (3.8) do not depend on the values of f outside of I. Recall that I^* is the union of the (3^d or fewer) special $\mathcal{P}_N$-cubes of side length s touching I; extend f to a function $\widetilde{f}$ on I^* by reflection. Then

$$\int_{I^*} \widetilde{f}^2 \le 3^d \int_I f^2, \quad \int_{I^*} |\nabla \widetilde{f}|^2 \le 3^d \int_I |\nabla f|^2,$$

and (3.7) and (3.8) now follow from Proposition 3.3 and Corollary 3.4(b) for $\widetilde{f}$. □

Next we proceed to a Nash inequality for special $\mathcal{P}_N$ cubes. Because the Laplacian is not a symmetric operator with respect to γ, we cannot use the method in [SC].

Proposition 3.6. *Let J be a special $\mathcal{P}_N$ cube with side length $s \le l_{\mathcal{F}}^2$. Suppose the gradient of f is square integrable over J and $\int_J f^2\gamma < \infty$. Then*

$$\int_J f^2\gamma \le c_1 \max\Big(A^{d_f/(2\beta+d_f)}B^{2\beta/(2\beta+d_f)}, A^{d/(2\beta+d)}(\kappa^N B)^{2\beta/(2\beta+d)}\Big)$$

where

$$A = \zeta^{-N}\int_J |\nabla f|^2 + s^{-2\beta}\int_J f^2\gamma, \qquad B = \kappa^{-N}\Big(\int_J |f|\gamma\Big)^2.$$

Proof. The result is trivial if $A = 0$, so we may assume $A > 0$. Let $t \in (0, s)$. We can find a covering of J by special $\mathcal{P}_N$ cubes I_i of side length between $t/l_{\mathcal{F}}$ and t such that $J = \cup I_i$, and the I_i^o are disjoint. Note that $|I_i| \asymp \kappa^N t^{d_f} \wedge t^d$. Set $\psi_0(t) = t^{-d_f} \vee (\kappa^N t^{-d})$. We apply Corollaries 3.4 and 3.5 and sum. So

$$\begin{aligned}\int_J f^2\gamma &= \sum_i \int_{I_i} f^2\gamma \\ &\le c_2 t^{2\beta}\zeta^{-N}\sum_i \int_{I_i}|\nabla f|^2 + c_2\sum_i |I_i|^{-1}\Big(\int_{I_i}|f|\gamma\Big)^2 \\ &\le c_2 t^{2\beta}c_3\zeta^{-N}\int_J |\nabla f|^2 + c_2\psi_0(t)\kappa^{-N}\Big(\sum_i\int_{I_i}|f|\gamma\Big)^2 \\ &\le c_4 t^{2\beta}A + c_5\psi_0(t)B. \end{aligned} \tag{3.9}$$

If $t > s$ then (possibly adjusting the constant c_4), the inequality (3.9) is trivial. If we now choose t_0 so that $t_0^{2\beta} A = \psi_0(t_0) B$, then we have that

$$t_0 = \begin{cases} (B/A)^{1/(2\beta+d_f)}, & \text{if } t_0 \geq l_{\mathcal{F}}^{-N}, \\ (\kappa^N B/A)^{1/(2\beta+d)}, & \text{if } t_0 \leq l_{\mathcal{F}}^{-N}. \end{cases}$$

Now let $t = t_0$ and substitute in (3.9) to conclude the proof. □

Next is a preliminary version of a weighted Sobolev inequality. Again the lack of symmetry of the Laplacian with respect to γ necessitates new methods.

Proposition 3.7. *Let J be a special $\mathcal{P}_N$ cube with side length $s \leq 1$. Let f be as above. Then for any $R \in (2, 2 + 2\beta/d)$ there exists $c_1(R) < \infty$ such that*

$$\Big(\kappa^{-N} \int_J |f|^R \gamma\Big)^{1/R} \leq c_1(R) \Big[\zeta^{-N} \kappa^{-N} \int_J |\nabla f|^2 + \kappa^{-N} s^{-2\beta} \int_J f^2 \gamma\Big]^{1/2}. \tag{3.10}$$

Proof. Since $|\nabla(f^+)| \leq |\nabla f|$ a.e. and $|f| \leq f^+ + f^-$, it suffices to consider nonnegative f. Write

$$A_0(f) = \zeta^{-N} \kappa^{-N} \int_J |\nabla f|^2 + \kappa^{-N} s^{-2\beta} \int_J f^2 \gamma, \qquad B_0(f) = \Big(\kappa^{-N} \int_J |f| \gamma\Big)^2.$$

Multiplying f by $A_0(f)^{-1/2}$, it is enough to prove

$$\kappa^{-N} \int_J |f|^R \gamma \leq c_1 \qquad \text{if } A_0(f) = 1. \tag{3.11}$$

Set

$$p_n = \kappa^{-N} \int_{\{f \geq 2^n\} \cap J} \gamma.$$

Then

$$p_n \leq p_0 \leq \kappa^{-N} \int_{\{f \geq 1\} \cap J} f^2 \gamma \leq \kappa^{-N} \int_J f^2 \gamma \leq s^{2\beta} A_0(f) \leq 1.$$

Let $f_n = (f \wedge 2^{n+1}) - (f \wedge 2^n)$; note that $f_n \leq 2^n$, that $f_n = 2^n$ on $J \cap \{f \geq 2^{n+1}\}$, and that $f_n = 0$ on $\{f < 2^n\}$. Therefore

$$\kappa^{-N} \int_J f_n \gamma = \kappa^{-N} \int_{\{f \geq 2^n\} \cap J} f_n \gamma \leq \kappa^{-N} \int_{\{f \geq 2^n\} \cap J} \gamma 2^n = 2^n p_n, \tag{3.12}$$

while

$$\kappa^{-N} \int_J f_n^2 \gamma \geq \kappa^{-N} \int_{\{f \geq 2^n\} \cap J} f_n^2 \gamma \geq \kappa^{-N} \int_{\{f \geq 2^{n+1}\} \cap J} f_n^2 \gamma = 2^{2n} p_{n+1}. \tag{3.13}$$

Since $\int_J f_n^2\gamma \le \int_J f^2\gamma$ and $\int_J |\nabla f_n|^2 \le \int_J |\nabla f|^2$, we have $A_0(f_n) \le A_0(f)$. So, from (3.13) we deduce $p_n \le 4\cdot 2^{-2n}$. Applying Proposition 3.6 to f_n we have, using the fact that $A_0(f) \le 1$,

$$\kappa^{-N}\int_J f_n^2\gamma \le c_2 \max\Big(B_0(f_n)^{2\beta/(2\beta+d_f)}, \kappa^N B_0(f_n)^{2\beta/(2\beta+d)}\Big).$$

Hence, we obtain

$$2^{2n}p_{n+1} \le c_3 \max\Big((2^n p_n)^{2\beta/(2\beta+d_f)}, (\kappa^N 2^n p_n)^{2\beta/(2\beta+d)}\Big). \tag{3.14}$$

Since $2^n p_n \le 4$ and $d_f < d$, both the terms on the right hand side of (3.14) are dominated by $c_4(2^n p_n)^{2\beta/(2\beta+d)}$. Therefore

$$p_{n+1} \le c_4(2^{-n})^{(2\beta+2d)/(2\beta+d)}(p_n)^{2\beta/(2\beta+d)}.$$

Elementary calculations now verify that $p_n \le a2^{-n\theta}$ where $\theta = 2(\beta+d)/d$ and $a = c_5 \ge 1$, is a constant depending only on c_4, β and d.

Since

$$\kappa^{-N}\int_J |f|^R\gamma \le c_6\sum_{n=0}^{\infty} 2^{nR}p_n,$$

we deduce (3.11) (with a constant depending on R) for any $R \in (2, 2+2\beta/d)$. □

We can modify slightly the final term in (3.10).

Corollary 3.8. *Let J be a special $\mathcal{P}_N$ cube of side $s \le l_{\mathcal{F}}^2$. Let f and its gradient be square integrable over J. Then for any $R \in (2, 2+2\beta/d)$ there exists $c_1(R) < \infty$ such that*

$$\Big(\kappa^{-N}\int_J |f|^R\gamma\Big)^{1/R} \le c_1(R)\Big[\zeta^{-N}\kappa^{-N}\int_J|\nabla f|^2 + \kappa^{-N}s^{-d_w}\int_J f^2\Big]^{1/2}. \tag{3.15}$$

Proof. We have, using Corollary 3.5, and the fact that $2\beta \le 2 < d_w$,

$$\begin{aligned}\int_J f^2\gamma &= \int_J f^2 + \zeta^{-N}\int_J f^2|\nabla r|^2\\ &\le \int_J f^2 + c_2\zeta^{-N}s^{2\beta}\int_J|\nabla f|^2 + c_3 s^{2\beta-d_w}\int_J f^2\\ &\le c_4\zeta^{-N}s^{2\beta}\int_J|\nabla f|^2 + c_5 s^{2\beta-d_w}\int_J f^2.\end{aligned}$$

The result now follows from substituting this in the last term of (3.10). □

We now fix $R \in (2, 2+2\beta/d)$.

Theorem 3.9. *Let $Q \subset \mathcal{P}_N$ be the union of a finite number of disjoint special $\mathcal{P}_N$ cubes each of side length $s \le l_{\mathcal{F}}^2$. Let f and its gradient be square integrable over Q. Then*

$$\Big(\kappa^{-N} \int_Q |f|^R \gamma\Big)^{1/R} \le c_1(R)\Big[\zeta^{-N}\kappa^{-N}\int_Q |\nabla f|^2 + \kappa^{-N} s^{-d_w} \int_Q f^2\Big]^{1/2}. \qquad (3.16)$$

Proof. If J_i are the cubes with $\cup J_i = Q$ then, applying Corollary 3.8 to each of the J_i,

$$\begin{aligned}\kappa^{-N}\int_Q |f|^R\gamma &= \sum_{i=1}^M \kappa^{-N}\int_{J_i}|f|^R\gamma\\ &\le c_2 \sum_{i=1}^M \Big(\zeta^{-N}\kappa^{-N}\int_{J_i}|\nabla f|^2 + s^{-d_w}\kappa^{-N}\int_{J_i} f^2\Big)^{R/2}.\end{aligned}$$

If $p > 1$ and $x_i > 0$ then $\sum x_i^p \le (\sum x_i)^p$. So

$$\begin{aligned}\kappa^{-N}\int_Q |f|^R\gamma \le&\Big(\sum_{i=1}^M \Big[\zeta^{-N}\kappa^{-N}\int_{J_i}|\nabla f|^2 + s^{-d_w}\kappa^{-N}\int_{J_i} f^2\Big]\Big)^{R/2}\\ =&\Big(\zeta^{-N}\kappa^{-N}\int_Q|\nabla f|^2 + s^{-d_w}\kappa^{-N}\int_Q f^2\Big)^{R/2}.\end{aligned}$$

□

4. Harnack inequality.

In this section we prove Theorem 1.1. We look at the operator $\mathcal{L} = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j)$, where the a_{ij} are bounded, strictly elliptic, and smooth. On the boundaries of $\mathcal{P}_N$ we impose conormal reflection. We will show the Harnack inequality for nonnegative $\mathcal{L}$-harmonic functions with bounds that do not depend on the smoothness of the a_{ij}.

The following result is proved exactly as in Moser [M1], Lemma 4.

Proposition 4.1. *Let $D \subset \mathcal{P}_N$ be a domain in $\mathcal{P}_N$, and suppose u is $\mathcal{L}$-harmonic in D, $v = u^k$, where $k \in \mathbb{R}$, $k \ne 1/2$, and η is supported in D^o. Suppose the gradient of η is square integrable over D. Then*

$$\int_D \eta^2 |\nabla v|^2 dx \le c_1 \Big(\frac{2k}{2k-1}\Big)^2 \int_D |\nabla \eta|^2 v^2 dx.$$

Now let u be $\mathcal{L}$-harmonic and non-negative in $\mathcal{P}_N \cap [0, l_{\mathcal{F}}]^d$; we assume $u \not\equiv 0$. The usual Harnack inequality in $\mathbb{R}^d$, combined with the connectedness hypothesis (H2) implies that u

and u^{-1} are continuous and bounded (by a constant $c(u,N,\varepsilon)$ depending on u, N and ε) on $\mathcal{P}_N \cap [0, l_{\mathcal{F}} - \varepsilon]^d$ for every $\varepsilon > 0$. By Proposition 4.1, the gradient of powers of u will be square integrable over bounded subsets of $\mathcal{P}_N$.

Let $y_0 = 1$ and $y_k = 1 - \sum_{i=1}^k l_{\mathcal{F}}^{-i}$ for $1 \le k \le \infty$, and set

$$Q_k = [0, y_k]^d \cap \mathcal{P}_N, \qquad 0 \le k \le \infty.$$

Since $l_{\mathcal{F}} \ge 3$, then $y_\infty \ge \frac{1}{2}$. Note that each Q_k is a $\mathcal{P}_N$-cube and is the union of at most $l_{\mathcal{F}}^{kd_f}$ special $\mathcal{P}_N$ cubes each of side length $l_{\mathcal{F}}^{-k}$. Note also that $\operatorname{dist}(Q_{k+1}, \mathcal{P}_N - Q_k) = l_{\mathcal{F}}^{-(k+1)}$.

Proposition 4.2. *Let v be either u or u^{-1}. There exists c_1 such that if $0 < q < 2$, then*

$$\sup_{Q_\infty} v^{2q} \le c_1 \kappa^{-N} \int_{Q_0} (\zeta^{-N} |\nabla v^q|^2 + v^{2q}).$$

Proof. For $0 \le k < \infty$ let

$$r_k = U(\cdot, Q_{k+1}, Q_k), \qquad \gamma_k = 1 + \zeta^{-N} |\nabla r_k|^2.$$

Let $f = u^p$, where $p \in \mathbb{R}$, $p \ne \frac{1}{2}$. Applying Theorem 3.9 we have, writing $S = R/2$,

$$\Big(\kappa^{-N} \int_{Q_{k+1}} f^{2S} \gamma_{k+1}\Big)^{1/S} \le c_2 \Big[\kappa^{-N} \int_{Q_{k+1}} \zeta^{-N} |\nabla f|^2 + l_{\mathcal{F}}^{(k+1)d_w} \kappa^{-N} \int_{Q_{k+1}} f^2\Big]. \tag{4.1}$$

We start with the first term on the right-hand side of (4.1). If $x \in Q_{k+1}$ then there exists a special $\mathcal{P}_N$ cube I of side $l_{\mathcal{F}}^{-k-1}$ such that $x \in I \subset Q_{k+1}$. Then $I^* \subset Q_k$, so by Lemma 2.2(c) we have $r_k \ge c_3 l_{\mathcal{F}}^{-kd_w}$ on Q_{k+1}. Hence

$$\begin{aligned}
\kappa^{-N} \int_{Q_{k+1}} \zeta^{-N} |\nabla f|^2 &\le c_4 \kappa^{-N} \zeta^{-N} l_{\mathcal{F}}^{2kd_w} \int_{Q_{k+1}} |\nabla f|^2 r_k^2 \\
&\le c_4 \kappa^{-N} \zeta^{-N} l_{\mathcal{F}}^{2kd_w} \int_{Q_k} |\nabla f|^2 r_k^2 \\
&\le c_5 \Big(\frac{2p}{(2p-1)}\Big)^2 \kappa^{-N} \zeta^{-N} l_{\mathcal{F}}^{2kd_w} \int_{Q_k} f^2 |\nabla r_k|^2 \\
&\le c_5 \Big(\frac{2p}{(2p-1)} \vee 1\Big)^2 \kappa^{-N} l_{\mathcal{F}}^{2kd_w} \int_{Q_k} f^2 \gamma_k.
\end{aligned} \tag{4.2}$$

Here we used Proposition 4.1 in the third line. If c_5 is taken large enough, the right hand term in (4.2) also dominates the final term in (4.1). Therefore,

$$\Big(\kappa^{-N} \int_{Q_{k+1}} f^{2S} \gamma_{k+1}\Big)^{1/S} \le c_6 \Big(\frac{2p}{(2p-1)} \vee 1\Big)^2 l_{\mathcal{F}}^{2kd_w} \Big(\kappa^{-N} \int_{Q_k} f^2 \gamma_k\Big). \tag{4.3}$$

Choose $q' > 0$ such that $\inf_{m\in\mathbb{Z}} |q'S^m - \frac{1}{2}| \geq c_7 > 0$. Suppose first that $q_0 = q'S^{-i}$ for some i. Let $p_n = 2q_0 S^n$ for $n \geq 0$, and write

$$\Psi_k = \Big[\kappa^{-N} \int_{Q_k} v^{p_k} \gamma_k\Big]^{1/p_k}.$$

Note that $p_{k+1}/2S = p_k/2$. Applying (4.3) to $f = v^{p_{k+1}/(2S)} = v^{p_k/2}$ we have

$$\begin{aligned}\Psi_{k+1}^{p_{k+1}/S} &= \Big(\kappa^{-N} \int_{Q_{k+1}} v^{p_{k+1}} \gamma_{k+1}\Big)^{1/S} \\ &\leq c_8 l_{\mathcal{F}}^{2kd_w} \Big(\kappa^{-N} \int_{Q_k} v^{p_k} \gamma_k\Big) = c_8 l_{\mathcal{F}}^{2kd_w} \Psi_k^{p_k},\end{aligned}$$

or

$$\Psi_{k+1} \leq \Big(c_8 l_{\mathcal{F}}^{2kd_w}\Big)^{1/p_k} \Psi_k.$$

Hence for every m

$$\log \Psi_m \leq \log \Psi_0 + \sum_{k=1}^{m} p_k^{-1} \log(c_8 l_{\mathcal{F}}^{kd_w}). \tag{4.4}$$

As the sum in (4.4) converges, and $\sup_{Q_\infty} v \leq \limsup_{m\to\infty} \Psi_m$, we have

$$\sup_{Q_\infty} v \leq c_9 \Big(\kappa^{-N} \int_{Q_0} v^{2q_0} \gamma_0\Big)^{1/(2q_0)}. \tag{4.5}$$

Now let $q \in (0,2)$. We can take $q_0 = q'S^{-i} < q$. Then by Hölder's inequality, and Lemma 3.1

$$\int_{Q_0} v^{2q_0} \gamma_0 \leq \Big(\int_{Q_0} v^{2q} \gamma_0\Big)^{q_0/q} \Big(\int_{Q_0} \gamma_0\Big)^{1-q_0/q} \leq c_{10} \Big(\int_{Q_0} v^{2q} \gamma_0\Big)^{q_0/q}.$$

Thus

$$\sup_{Q_\infty} v^{2q} \leq c_{11} \int_{Q_0} v^{2q} \gamma_0.$$

By Corollary 3.5 this implies

$$\sup_{Q_\infty} v^{2q} \leq c_{12} \kappa^{-N} \int_{Q_0} (\zeta^{-N} |\nabla v^q|^2 + v^{2q}).$$

□

In the argument above we were tied to the cubes Q_k since we needed to use Theorem 3.9. However, in the remainder of this section it will be more convenient to use the balls $B(x,r)$. An easy covering argument gives us

Corollary 4.3. *Let $u > 0$ be $\mathcal{L}$-harmonic in $B(x_0, 3)$. There exists c_1, independent of u and x_0, such that for $0 < q < 2$ and $v = u$ or $v = u^{-1}$*

$$\sup_{B(x_0,1)} v^{2q} \le c_1 \kappa^{-N} \int_{B(x_0,2)} (\zeta^{-N} |\nabla v^q|^2 + v^{2q}). \tag{4.6}$$

We now follow the ideas of Moser [M2] to link the L^∞ norms of u and u^{-1}. Fix $x_0 \in \mathcal{P}_N$, and write $B(r) = B(x_0, r)$. Let $u > 0$ be $\mathcal{L}$-harmonic in $B(x_0, 4)$. v is either u or u^{-1}.

Corollary 4.4. *Let $1/2 \le s < t \le 4$ and let $r_{st} = U(\cdot, B(s), B(t))$, $\gamma_{st} = 1 + \zeta^{-N} |\nabla r_{st}^2|$. Then if $0 < q < \frac{1}{3}$,*

$$\sup_{B(s)} v^{2q} \le c_1 (t-s)^{-d_w - d_f} \kappa^{-N} \int_{B(t)} v^{2q} \gamma_{st}.$$

Proof. Let $\theta = \frac{1}{4}(t-s)$, $s' = s + 2\theta$. By Corollary 4.3 and scaling, if $B(x, 3\theta) \subset B(4)$ then

$$\sup_{B(x,\theta)} v^{2q} \le c_2 \kappa^{-N} \zeta^{-N} \theta^{d_w - d_f} \int_{B(x,2\theta)} |\nabla v^q|^2 + c_2 \theta^{-d_f} \kappa^{-N} \int_{B(x,2\theta)} v^{2q}. \tag{4.7}$$

We can cover $B(s)$ by a collection of balls $B(x_i, 2\theta) \subset B(s')$ such that no point in $B(s')$ is contained in more than c_3 of these balls. So by (4.7)

$$\sup_{B(s)} v^{2q} \le c_4 \kappa^{-N} \zeta^{-N} \theta^{d_w - d_f} \int_{B(s')} |\nabla v^q|^2 + c_2 \theta^{-d_f} \kappa^{-N} \int_{B(s')} v^{2q}. \tag{4.8}$$

By Lemma 2.2(d) $r_{st} \ge c_4 \theta^{d_w}$ on $B(s')$. Since r_{st} is supported on $B(t)^o$, we have by Proposition 4.1,

$$\begin{aligned}
\zeta^{-N} \int_{B(s')} |\nabla v^q|^2 &\le c_5 \zeta^{-N} \theta^{-2d_w} \int_{B(s')} |\nabla v^q|^2 r_{st}^2 \\
&\le c_5 \zeta^{-N} \theta^{-2d_w} \int_{B(t)} |\nabla v^q|^2 r_{st}^2 \\
&\le c_6 \zeta^{-N} \theta^{-2d_w} \int_{B(t)} |\nabla r_{st}|^2 v^{2q} \\
&\le c_7 \theta^{-2d_w} \int_{B(t)} v^{2q} \gamma_{st}.
\end{aligned}$$

Combining this with (4.8) and noting that $\theta \le 1$ completes the proof. □

Now let $w = \log u$. We will need the following estimate.

Proposition 4.5. *Suppose $h \in [1/2, 2]$ and $w = \log u$. There exists c_1 such that*

$$\zeta^{-N} \int_{B(h)} |\nabla w|^2 \le c_1 |B(h)|.$$

Proof. Again, this is essentially Moser's proof. Let $\phi = U(\cdot, B(h), B(2h))$, and note that by Lemma 2.2 $\phi \ge c_2$ on $B(h)$. So

$$\int_{B(h)} |\nabla w|^2 \le c_3 \int \phi^2 |\nabla w|^2.$$

We write

$$\begin{aligned} 0 = \int \frac{\phi^2}{u} \mathcal{L} u &= - \int \nabla(\phi^2/u) \cdot a \nabla u \\ &= - \int \Big(2\frac{\phi}{u} \nabla\phi \cdot a\nabla u - \frac{\phi^2}{u^2} \nabla u \cdot a \nabla u \Big) \\ &= -2 \int \phi \nabla \phi \cdot a \nabla w + \int \phi^2 \nabla w \cdot a \nabla w. \end{aligned}$$

So

$$\int \phi^2 |\nabla w|^2 \le c_4 \Big| \int \phi \nabla\phi \cdot a \nabla w \Big| \le c_5 \Big(\int |\nabla \phi|^2 \Big)^{1/2} \Big(\int \phi^2 |\nabla w|^2 \Big)^{1/2}.$$

Dividing and squaring,

$$\int \phi^2 |\nabla w|^2 \le c_6 \int |\nabla\phi|^2,$$

and by Lemma 3.1, $\zeta^{-N} \int |\nabla \phi|^2 \le c_7 |B(h)|$. □

For $\frac{1}{2} \le h \le 4$, let $\alpha(h) = \frac{1}{|B(h)|} \int_{B(h)} w$.

Corollary 4.6. *Let $\frac{1}{2} \le s < t \le 1$. Then*

$$\int_{|\{w - \alpha(2)| > A\} \cap B(s)} \gamma_{st} \le \frac{c_1 \kappa^{-N}}{A^2}.$$

Proof. Note first that

$$\int_{\{|w-\alpha(2)|>A\} \cap B(s)} \gamma_{st} \le \int_{\{|w-\alpha(2)|>A\} \cap B(1)} \gamma_{st}\,.$$

By Chebyshev's inequality,

$$\begin{aligned} \int_{\{|w-\alpha(2)|>A\} \cap B(1)} \gamma_{st} &\le A^{-2} \int_{\{|w-\alpha(2)|>A\} \cap B(1)} |w - \alpha(2)|^2 \gamma_{st} \\ &\le A^{-2} \int_{B(1)} |w - \alpha(2)|^2 \gamma_{st}. \end{aligned}$$

Now apply Corollary 3.4(a) with $Q(h) = B(s)$, $Q(k) = B(t)$, $I = B(1)$ and $I^* = B(2)$; we have

$$\int_{B(1)} (w - \alpha(2))^2 \gamma_{st} \leq c_2 \zeta^{-N} \int_{B(2)} |\nabla w|^2,$$

and by Proposition 4.5 this is bounded by $c_3 \kappa^{-N}$. □

Without loss of generality, let us multiply u by a constant so that $\int_{B(2)} \log v = \alpha(2) = 0$. Recall that v is either u or u^{-1} and define

$$\varphi(h) = \sup_{B(h)} \log v.$$

Lemma 4.7. *If* $\frac{1}{2} \leq s < t \leq 1$, *then*

$$\varphi(s) \leq \tfrac{3}{4}\varphi(t) + c_1 (t-s)^{-d_f - d_w}. \tag{4.9}$$

Proof. Fix t and write φ for $\varphi(t)$. Let $c_2 > e$ satisfy $c_2 = 6 \log c_2$. If $\varphi(t) \leq c_2$ then as $\varphi(\cdot)$ is increasing

$$\varphi(s) \leq \varphi(t) \leq \tfrac{3}{4}\varphi(t) + \tfrac{1}{4} c_2,$$

so that (4.9) holds provided $c_1 \geq c_2/4$.

Now suppose $\varphi > c_2$. By Corollary 4.6 and the fact that $v^p \leq e^{p\varphi}$ on $B(t)$,

$$\begin{aligned}
\int_{B(t)} v^{2p} \gamma_{st} &= \int_{B(t) \cap \{\log v \geq \varphi/2\}} v^{2p} \gamma_{st} + \int_{B(t) \cap \{\log v < \varphi/2\}} v^{2p} \gamma_{st} \\
&\leq e^{2p\varphi} \int_{B(t) \cap \{\log v \geq \varphi/2\}} \gamma_{st} + e^{p\varphi} \int_{B(t) \cap \{\log v < \varphi/2\}} \gamma_{st} \\
&\leq \frac{4 c_2 c_3 e^{2p\varphi}}{\varphi^2} \kappa^{-N} + e^{p\varphi} \int_{B(t)} \gamma_{st} \\
&\leq c_4 \Big(\frac{e^{2p\varphi}}{\varphi^2} + e^{p\varphi} \Big) \kappa^N.
\end{aligned}$$

Let $p = \frac{2}{\varphi} \log \varphi$, so that $e^{p\varphi} = \varphi^2$. As $\varphi > c_2$ we have $p < (2/c_2) \log c_2 = \frac{1}{3}$. So

$$\int_{B(t)} v^{2p} \gamma_{st} \leq 2 c_4 e^{p\varphi} \kappa^N.$$

Therefore by Corollary 4.4,

$$\begin{aligned}
\varphi(s) &= \frac{1}{2p} \log[\sup_{B(s)} v^{2p}] \\
&\leq \frac{1}{2p} \log \Big[c_5 (t-s)^{-d_f - d_w} \kappa^{-N} \int_{B(t)} v^{2p} \gamma_{st} \Big] \\
&\leq \frac{1}{2p} \log \Big[c_6 (t-s)^{-d_f - d_w} e^{p\varphi} \Big].
\end{aligned}$$

So

$$\varphi(s) \le \tfrac{1}{2}\varphi(t)\Big[1 + \frac{1}{2}\frac{\log(c_6(t-s)^{-d_f-d_w})}{\log\varphi}\Big]. \tag{4.10}$$

Without loss of generality we may take c_6 larger than c_2. If $\varphi(t) \ge c_6(t-s)^{-d_f-d_w}$, then by (4.10) $\varphi(s) \le \frac{3}{4}\varphi(t)$, and (4.9) is satisfied. If, on the other hand, $\varphi(t) \le c_6(t-s)^{-d_f-d_w}$, then since $\varphi(s) \le \varphi(t)$, we have (4.9) satisfied with $c_1 = c_6$. □

We can now prove the Harnack inequality.

Theorem 4.8. *There exists c_1 such that if u is nonnegative and $\mathcal{L}$-harmonic in $B(3)$, then*

$$\frac{\sup_{B(1/2)} u}{\inf_{B(1/2)} u} \le c_1.$$

Proof. We know that u is continuous and bounded in $B(2)$; we need to show we can bound the ratio of the supremum of u to the infimum of u in $B(1/2)$ by a constant not depending on u. Multiplying u by a constant we can assume $\int_{B(2)} \log u = \alpha(2) = 0$. First let $v = u$.

Choose $t_j = 1 - (1/(j+2))$, so that $t_0 = 1/2$ and $t_i \uparrow 1$. Then by Lemma 4.7, writing $\theta = d_f + d_w$,

$$\begin{aligned}\varphi(t_0) &\le \tfrac{3}{4}\varphi(t_1) + c_2(t_1-t_0)^{-\theta}\\ &\le (\tfrac{3}{4})^2\varphi(t_2) + c_2(t_1-t_0)^{-\theta} + \tfrac{3}{4}c_2(t_2-t_1)^{-\theta}\\ &\le \cdots\\ &\le (\tfrac{3}{4})^n\varphi(t_n) + \tfrac{4}{3}\textstyle\sum_{i=1}^n (\tfrac{3}{4})^i c_2(t_i-t_{i-1})^{-\theta},\end{aligned}$$

for any $n \ge 0$. Since $\varphi(t_n) \le \varphi(1) < \infty$, and

$$\sum_{i=1}^{\infty} (\tfrac{3}{4})^i c_2(t_i - t_{i-1})^{-\theta} = c_3 < \infty,$$

we obtain

$$\sup_{B(1/2)} \log u \le c_3.$$

Now let $v = u^{-1}$; $\log v = -\log u$ so we still have $\int_{B(2)} \log v = 0$. The same argument as above now implies $\sup_{B(1/2)} \log v \le c_3$, or

$$\inf_{B(1/2)} \log u \ge -c_3.$$

Combining we deduce

$$e^{-c_3} \le \inf_{B(1/2)} u \le \sup_{B(1/2)} u \le e^{c_3},$$

which is what we wanted to prove. □

Theorem 1.1 follows from Theorem 4.8 by a scaling and covering argument.

Standard arguments now yield

Corollary 4.9. *Suppose $D \subset E$ are open subsets of $\mathcal{P}$. There exists c_1 depending only on the ratio of* $\operatorname{dist}(\partial_a D, \partial_a E)$ *to the diameter of D such that if u is nonnegative and harmonic in E, then*

$$u(x) \le c_1 u(y), \qquad x, y \in D.$$

5. Heat kernel estimates.

In this section we study the fundamental solutions $q(t,x,y)$ of the heat equation in $\mathcal{P}$:

$$\frac{\partial u}{\partial t}(x,t) = \mathcal{L}u(x,t),$$

where u has conormal reflection on $\partial\mathcal{P}$. In probabilistic terms $q(t,x,y)$ is the transition density of the diffusion X on $\mathcal{P}$ with generator $\mathcal{L}$. We continue to assume the a_{ij} are smooth, although our bounds will not depend on the smoothness.

Let D be a domain in $\mathcal{P}$. Let $u_{\mathcal{L},D}(x,y)$ be the Green function for $\mathcal{L}$ for the process killed on exiting D and let $u_{\Delta,D}$ be the corresponding Green function for reflecting Brownian motion in D killed on hitting $\partial_a D$. Let $C_{\mathcal{L},D}(A)$ and $C_{\Delta,D}(A)$ be the capacities of a set $A \subset D$ with respect to $\mathcal{L}$ and Δ respectively. We have

$$C_{\mathcal{L},D}(A) = \inf\Big\{ \int_D \nabla f \cdot a \nabla f : f = 0 \text{ on } \partial\mathcal{P} - D, f = 1 \text{ on } A \Big\}$$
$$C_{\Delta,D}(A) = \inf\Big\{ \int_D |\nabla f|^2 : f = 0 \text{ on } \partial\mathcal{P} - D, f = 1 \text{ on } A \Big\}.$$

It is immediate from these definitions that

$$\lambda_1 C_{\Delta,D}(A) \le C_{\mathcal{L},D}(A) \le \lambda_2 C_{\Delta,D}(A), \tag{5.1}$$

where λ_1, λ_2 depend only on the bounds on the matrix a in (1.1).

Let Y be either of the processes W or X. We write u_D, C_D for the Green's functions and capacities for Y, and τ_D for the exit time of Y from D. We define $U_D\mu(x) = \int u_D(x,y)\mu(dy)$.

Lemma 5.1. *Let $x \in \mathcal{P}$, $R > 0$, $D = B(x,R)$, and $B_n = B(x, 2^{-n}R)$ for $n \ge 0$. Then*

$$c_1 \sum_{n=1}^{\infty} |B_n| C_D(B_n)^{-1} \le \mathbb{E}^x \tau_D \le c_2 \sum_{n=1}^{\infty} |B_n| C_D(B_n)^{-1}. \tag{5.2}$$

Proof. Note first that $u_D(x,\cdot)$ is continuous and harmonic on $D - \{x\}$, and is zero on $\{y : d(x,y) = R\} = \partial_a D$. So by the maximum principle it follows that $u_D(x,\cdot)$ attains its maximum on $B_{n-1} - B_n$ at a point z_n with $d(x,z_n) = 2^{-n}R$.

Let μ_n be the capacitary measure for $\overline{B}_n$. Thus $C_D(B_n) = \mu_n(\overline{B}_n)$ and $U_D\mu_n \le 1$ on D, and equals 1 on B_n. We know that μ_n is concentrated on $\partial\overline{B}_n$. We have

$$1 = U_D\mu_n(x) = \int_{\overline{B}_n} u_D(x,z)\mu_n(dz) \le u_D(x,z_n)C_D(B_n). \tag{5.3}$$

Now $u_D(z_n,.)$ is harmonic in B_{n+1}, so by the Harnack inequality Corollary 4.9 we have $u_D(z_n,y) \asymp u_D(z_n,x)$ on B_{n+1}. Therefore

$$1 \ge U_D\mu_{n+1}(z_n) = \int_{\overline{B}_n} u_D(y,z_n)\mu_{n+1}(dy) \ge c_3 u_D(z_n,x)C_D(B_{n+1}). \tag{5.4}$$

Now write $A_n = B(z_n, 2^{-(n+1)}R)$. Note that $A_n \subset B_n - B_{n+1}$, so that the sets A_n are disjoint. The estimate (2.2) implies that A_n, B_n and B_{n-1} all have comparable volume. By Corollary 4.9 we have $u_D(x,y) \asymp u_D(x,z_n)$ on A_n.

Using these estimates we have

$$\begin{aligned}
\mathbb{E}^x\tau_D &= \int_D u_D(x,y) \\
&\ge \sum_{n=1}^\infty \int_{A_n} u_D(x,y)dy \\
&\ge c_4 \sum_{n=1}^\infty \int_{A_n} u_D(x,z_n)dy \ge c_5 \sum_{n=1}^\infty |B_n|C_D(B_n)^{-1}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbb{E}^x\tau_D &= \sum_{n=1}^\infty \int_{B_{n-1}-B_n} u_D(x,y)dy \\
&\le c_6 \sum_{n=1}^\infty \int_{B_{n-1}-B_n} u_D(x,z_n)dy \\
&\le c_6 \sum_{n=1}^\infty |B_{n-1}|u_D(x,z_n) \\
&\le c_7 \sum_{n=1}^\infty |B_{n+1}|C_D(B_{n+1})^{-1} \le c_7 \sum_{m=1}^\infty |B_m|C_D(B_m)^{-1}.
\end{aligned}$$

□

Let $\tau_D^{\mathcal{L}}$ be the time for the process X_t associated to $\mathcal{L}$ to exit D, and let τ_D^{Δ} be the analogous time for the Brownian motion W_t on $\mathcal{P}$ to exit D. Recall from Section 2 the definition $\psi(r) = r^{d_w} \vee \zeta^N r^2$, and from Lemma 2.2 that if $D = B(x_0,R)$ then

$$\mathbb{E}^x\tau_{D^*}^{\Delta} \le c_1\psi(R), \quad x \in D^*, \qquad \mathbb{E}^x\tau_{D^*}^{\Delta} \ge c_2\psi(R), \quad x \in D.$$

We have the same bounds for the exit times of X.

Corollary 5.2. *There exist c_1, c_2 such that*

$$\mathbb{E}^x \tau^{\mathcal{L}}_{D^*} \le c_1 \psi(R), \quad x \in D^*,$$
$$\mathbb{E}^x \tau^{\mathcal{L}}_{D^*} \ge c_2 \psi(R), \quad x \in D.$$

Proof. This is immediate from Lemma 5.1 and the comparison for capacities given by (5.1). $\square$

Theorem 5.3. *Let $q(t,x,y)$ denote the transition densities for X_t. There exist $c_1, \dots, c_8 \in (0,\infty)$ such that if $x, y \in \mathcal{P}$ and*
(a) $t \ge \max(1, |x-y|)$, then

$$c_1 t^{-d_s/2} \exp\Big(-c_2 \Big(\frac{|x-y|^{d_w}}{t}\Big)^{1/(d_w-1)}\Big) \le q(t,x,y) \le c_3 t^{-d_s/2} \exp\Big(-c_4 \Big(\frac{|x-y|^{d_w}}{t}\Big)^{1/(d_w-1)}\Big);$$

(b) if $t \le 1$, then

$$c_5 t^{-d/2} \exp(-c_6 |x-y|^2/t) \le q(t,x,y) \le c_7 t^{-d/2} \exp(-c_8 |x-y|^2/t).$$

(c) if $t \ge 1$, $|x-y| > t$, then

$$c_5 t^{-d_s/2} \exp(-c_6 |x-y|^2/t) \le q(t,x,y) \le c_7 t^{-d_s/2} \exp(-c_8 |x-y|^2/t).$$

Proof. The upper bound follows the proof in [BB3], Section 6, exactly, using Corollary 5.2. The lower bound is done as follows:

Just as in [BB3], (6.23), we have $q(t,x,x) \ge c_9 t^{-d_s/2}$ for $t \ge 1$. Let D be a $\mathcal{P}$-cube of side length t^{d_w} and let $q_D(t,x,y)$ denote the transition densities for X_t killed on exiting D. X is a symmetric process, so there is an eigenvalue expansion for q_D:

$$q_D(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y).$$

As in [BB2], Section 7, we deduce (6.21) and (6.22) of [BB3]. Since $q(t,x,y) \ge q_D(t,x,y)$, we argue as in [BB3], Theorem 6.9, and derive the lower bound. $\square$

We also have a parabolic Harnack inequality for $\mathcal{P}$. The statement and proof are the same as in [BB3], Theorem 7.12.

Remark. Using the results of this section we can construct a diffusion process on $\mathcal{F}$ corresponding to $\mathcal{L}$. As in [BB1], Section 5, we use Corollary 5.2 to obtain a tightness estimate. The Harnack inequality, following [BB1], Section 5, implies that λ-resolvents are Hölder continuous. We first take a subsequence as in [BB1], Section 6 to construct a process corresponding to $\mathcal{L}$ on $\mathcal{F}$ when the a_{ij} are smooth. In the case when the a_{ij} are not smooth, we take a^n_{ij} smooth satisfying the same bound and uniform ellipticity as the a_{ij} and take a limit. It is then straightforward, as in [BB3] Section 6, to derive heat kernel bounds and a parabolic Harnack inequality for this process.

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