Spectral dimension and random walks on the two dimensional uniform spanning tree

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Abstract

We study simple random walk on the uniform spanning tree on $\mathbb{Z}^2$. We obtain estimates for the transition probabilities of the random walk, the distance of the walk from its starting point after n steps, and exit times of both Euclidean balls and balls in the intrinsic graph metric. In particular, we prove that the spectral dimension of the uniform spanning tree on $\mathbb{Z}^2$ is $16/13$ almost surely.

Keywords: Uniform spanning tree, loop erased random walk, random walk on a random graph

Subject Classification: 60G50, 60J10

1 Introduction

Note. After this paper was published Lawler proved that the growth function $G(n)$ of the LERW in $\mathbb{Z}^2$ can be taken (up to constants) to be $G(n) = n^{5/4}$. (See `arXiv.1301.5331`.) This allows simpler statements of the results and shortens some proofs. This revision incorporates some of these simplifications. In particular in Section 4 the functions G, g, F, f, k can be replaced by simple powers. To allow for comparison with the published version, unneeded results have been replaced by placeholders. Other changes have been kept to a minimum, and no attempt has been made to update or improve any other aspects of the paper. The main changes in the text are marked in blue.

A *spanning tree* on a finite graph $G = (V, E)$ is a connected subgraph of G which is a tree and has vertex set V. A *uniform spanning tree* in G is a random spanning tree chosen uniformly from the set of all spanning trees. Let $Q_n = [-n, n]^d \subset \mathbb{Z}^d$, and write $\mathcal{U}_{Q_n}$ for a uniform spanning tree on Q_n. Pemantle [Pem91] showed that the weak limit of $\mathcal{U}_{Q_n}$ exists and is connected if and only if $d \leq 4$. (He also showed that the limit does not depend on the particular sequence of sets Q_n chosen, and that 'free' or 'wired' boundary conditions give

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rise to the same limit.) We will be interested in the case $d = 2$, and will call the limit the *uniform spanning tree* (UST) on $\mathbb{Z}^2$ and denote it by $\mathcal{U}$. For further information on USTs, see for example [BLPS01, BKPS04, Lyo98]. The UST can also be obtained as a limit as $p, q \to 0$ of the random cluster model – see [Häg95].

A loop erased random walk (LERW) on a graph is a process obtained by chronologically erasing the loops of a random walk on the graph. There is a close connection between the UST and the LERW. Pemantle [Pem91] showed that the unique path between any two vertices v and w in a UST on a finite graph G has the same distribution as the loop-erasure of a simple random walk on G from v to w. Wilson [Wil96] then proved that a UST could be generated by a sequence of LERWs by the following algorithm. Pick an arbitrary vertex $v \in G$ and let $T_0 = \{v\}$. Now suppose that we have generated the tree T_k and that T_k does not span. Pick any point $w \in G \setminus T_k$ and let T_{k+1} be the union of T_k and the loop-erasure of a random walk started at w and run until it hits T_k. We continue this process until we generate a spanning tree T_m. Then T_m has the distribution of the UST on G.

We now fix our attention on $\mathbb{Z}^2$. By letting the root v in Wilson's algorithm go to infinity, one sees that one can obtain the UST $\mathcal{U}$ on $\mathbb{Z}^2$ by first running an infinite LERW from a point x_0 (see Section 2 for the precise definition) to create the first path in $\mathcal{U}$, and then using Wilson's algorithm to generate the rest of $\mathcal{U}$. This construction makes it clear that $\mathcal{U}$ is a 1-sided tree: from each point x there is a unique infinite (self-avoiding) path in $\mathcal{U}$.

Both the LERW and the UST on $\mathbb{Z}^2$ have conformally invariant scaling limits. Lawler, Schramm and Werner [LSW04] proved that the LERW in simply connected domains scales to SLE_2 – Schramm-Loewner evolution with parameter 2. Using the relation between LERW and UST, this implies that the UST has a conformally invariant scaling limit in the sense of [Sch00] where the UST is regarded as a measure on the set of triples (a, b, γ) where $a, b \in \mathbb{R}^2 \cup \{\infty\}$ and γ is a path between a and b. In addition [LSW04] proves that the UST Peano curve – the interface between the UST and the dual UST – has a conformally invariant scaling limit, which is SLE_8.

In this paper we will study properties of the UST $\mathcal{U}$ on $\mathbb{Z}^2$. We have two natural metrics on $\mathcal{U}$; the intrinsic metric given by the shortest path in $\mathcal{U}$ between two points, and the Euclidean metric. For $x, y \in \mathbb{Z}^2$ let $\gamma(x, y)$ be the unique path in $\mathcal{U}$ between x and y, and let $d(x, y) = |\gamma(x, y)|$ be its length. If U_0 is a connected subset of $\mathcal{U}$ then we write $\gamma(x, U_0)$ for the unique path from x to U_0. Write $\gamma(x, \infty)$ for the path from x to infinity. We define balls in the intrinsic metric by

$$B_d(x, r) = \{y : d(x, y) \le r\}$$

and let $|B_d(x, r)|$ be the number of points in $B_d(x, r)$ (the *volume* of $B_d(x, r)$). We write

$$B(x, r) = \{y \in \mathbb{Z}^d : |x - y| \le r\},$$

for balls in the Euclidean metric, and let $B_R = B(R) = B(0, R)$, $B_d(R) = B_d(0, R)$.

Our goals in this paper are to study the volume of balls in the d metric, to obtain estimates of the degree of 'metric distortion' between the intrinsic and Euclidean metrics, and to study the behaviour of simple random walk (SRW) on $\mathcal{U}$.

To state our results we need some further notation. Let $\widetilde{G}(n)$ be the expected number of steps of an infinite LERW started at 0 until it leaves $B(0, n)$. Clearly $\widetilde{G}(n)$ is strictly increasing. extend G to a continuous strictly increasing function from $[1,\infty)$ to $[1,\infty)$, with $G(1)=1$. Let $\widetilde{g}(t)$ be the inverse of $\widetilde{G}$. Set $G(r)=r^{5/4}$, $g(r)=r^{4/5}$. By [Law13] we have

$$\widetilde{G}(n) \asymp G(n) = n^{5/4}, \quad \widetilde{g}(n) \asymp g(n) = n^{4/5}. \tag{1.1}$$

Our first result is on the relation between balls in the two metrics.

Theorem 1.1 *(a) There exist constants $c, C > 0$ such that for all $r \ge 1$, $\lambda \ge 1$,*

$$\mathbb{P}\big(B_d(0, \lambda^{-1} r^{5/4}) \not\subset B(0,r)\big) \le C e^{-c\lambda^{2/3}}. \tag{1.2}$$

(b) For all $\varepsilon > 0$, there exist $c(\varepsilon), C(\varepsilon) > 0$ and $\lambda_0(\varepsilon) \ge 1$ such that for all $r \ge 1$ and $\lambda \ge 1$,

$$\mathbb{P}\big(B(0,r) \not\subset B_d(0, \lambda r^{5/4})\big) \le C\lambda^{-4/15+\varepsilon}, \tag{1.3}$$

and for all $r \ge 1$ and all $\lambda \ge \lambda_0(\varepsilon)$,

$$\mathbb{P}\big(B(0,r) \not\subset B_d(0, \lambda r^{5/4})\big) \ge c\lambda^{-4/5-\varepsilon}. \tag{1.4}$$

We do not expect any of these bounds to be optimal. In fact, we could improve the exponent in the bound (1.2), but to simplify our proofs we have not tried to find the best exponent that our arguments yield when we have exponential bounds. However, we will usually attempt to find the best exponent given by our arguments when we have polynomial bounds, as in (1.3) and (1.4).

The reason we have a polynomial lower bound in (1.4) is that if we have a point w such that $|w| = r$, then the probability that $\gamma(0,w)$ leaves the ball $B(0,\lambda r)$ is bounded below by λ^{-1} (see Lemma 2.6). This in turn implies that the probability that $w \notin B_d(0, \lambda r^{5/4})$ is bounded from below by $c\lambda^{-4/5-\varepsilon}$ (Proposition 2.7).

Theorem 1.1 leads immediately to bounds on the tails of $|B_d(0,R)|$. However, while (1.2) gives a good bound on the upper tail, (1.3) only gives polynomial control on the lower tail. By working harder (see Theorem 3.4) we can obtain the following stronger bound.

Theorem 1.2 *Let $R \ge 1$, $\lambda \ge 1$. Then*

$$\mathbb{P}(|B_d(0,R)| \ge \lambda R^{8/5}) \le C e^{-c\lambda^{1/3}}, \tag{1.5}$$

$$\mathbb{P}(|B_d(0,R)| \le \lambda^{-1} R^{8/5}) \le C e^{-c\lambda^{1/9}}. \tag{1.6}$$

So in particular there exists C such that for all $R \ge 1$,

$$C^{-1} R^{8/5} \le \mathbb{E}|B_d(0,R)| \le C R^{8/5}. \tag{1.7}$$

Remark 1.3 (1) The two main ingedients for the proof of these theorems are Wilson's algorithm, plus good control of the LERWs. For the LERW we need bounds on the length of a LERW run from O until it leaves a domain $D \subset \mathbb{Z}^d$ – see Theorem 2.2. We remark that although we do not need the stretched exponential tails given there, we do need quite rapid decay of the tail probabilities. The UST is constructed from a large number of LERW paths, and a single 'bad path' (i.e. one which is much too short or much too long) could mean the inclusions in Theorem 1.1 fail.

In addition, to control the way that Wilson's algorithm 'fills in' in the tree, we use the discrete Beurling estimate, which states that the probability a random walk started at 0 will hit a path γ connecting $B(0,n)$ with $B(0,2n)^c$ is bounded below by a constant.

It is possible that a similar strategy would work on other graphs (such as $\mathbb{Z}^3$), conditional on having an analogue to Theorem 2.2, and also a Beurling estimate giving lower bounds on the probability a SRW hits a LERW path α inside a ball $B(0,n)$. Since however such bounds could not be uniform in α and n, some extensions of the arguments here would certainly be needed.

(2) We do not use directly the fact that the LERW and UST have SLE limits. Indeed, there is only one point in the whole argument leading to the results of this paper where this connection is needed. That is in [Mas09], which it is used to show that the function $Es(n)$ (the probability a LERW and independent SRW do not hit inside $B(0,n)$) satisfies $Es(n) \approx n^{-3/4}$.

We now discuss the simple random walk on the UST $\mathcal{U}$. To help distinguish between the various probability laws, we will use the following notation. For LERW and simple random walk in $\mathbb{Z}^2$ we will write $\mathbb{P}^z$ for the law of the process started at z. The probability law of the UST will be denoted by $\mathbb{P}$, and the UST will be defined on a probability space $(\Omega, \mathbb{P})$; we let ω denote elements of Ω. For the tree $\mathcal{U}(\omega)$ write $x \sim y$ if x and y are connected by an edge in $\mathcal{U}$, and for $x \in \mathbb{Z}^2$ let

$$\mu_x = \mu_x(\omega) = |\{y : x \sim y\}|$$

be the degree of the vertex x.

The random walk on $\mathcal{U}(\omega)$ is defined on a second space $\mathcal{D} = (\mathbb{Z}^2)^{\mathbb{Z}_+}$. Let X_n be the coordinate maps on $\mathcal{D}$, and for each $\omega \in \Omega$ let P^x_ω be the probability on $\mathcal{D}$ which makes $X = (X_n, n \ge 0)$ a simple random walk on $\mathcal{U}(\omega)$ started at x. Thus we have $P^x_\omega(X_0 = x) = 1$, and

$$P^x_\omega(X_{n+1} = y | X_n = x) = \frac{1}{\mu_x(\omega)} \quad \text{if } y \sim x.$$

We remark that since the UST $\mathcal{U}$ is a subgraph of $\mathbb{Z}^2$ the SRW X is recurrent. We define the heat kernel (transition density) with respect to μ by

$$p^\omega_n(x,y) = \mu_y^{-1} P^x_\omega(X_n = y). \tag{1.8}$$

Define the stopping times

$$\tau_R = \min\{n \ge 0 : d(0, X_n) > R\}, \tag{1.9}$$
$$\widetilde{\tau}_r = \min\{n \ge 0 : |X_n| > r\}. \tag{1.10}$$

Given functions f and g we write $f \approx g$ to mean

$$\lim_{n\to\infty} \frac{\log f(n)}{\log g(n)} = 1,$$

and $f \asymp g$ to mean that there exists $C \geq 1$ such that

$$C^{-1} f(n) \leq g(n) \leq C f(n), \quad n \geq 1.$$

The following summarizes our main results on the behaviour of X. Some more precise estimates, including heat kernel estimates, can be found in Theorems 4.3 – 4.7 in Section 4.

Theorem 1.4 *We have for $\mathbb{P}$ -a.a. ω, P^0_ω-a.s.,*

$$p_{2n}(0,0) \approx n^{-8/13}, \tag{1.11}$$

$$\tau_R \approx R^{13/5}, \tag{1.12}$$

$$\widetilde{\tau}_r \approx r^{13/4}, \tag{1.13}$$

$$\max_{0 \leq k \leq n} d(0, X_k) \approx n^{5/13}. \tag{1.14}$$

We now explain why these exponents arise. If G is a connected graph, with graph metric d, we can define the volume growth exponent (called by physicists the fractal dimension of G) by

$$d_f = d_f(G) = \lim_{R \to \infty} \frac{\log |B_d(0,R)|}{\log R},$$

if this limit exists. Using this notation, Theorem 1.2 and (1.1) imply that

$$d_f(\mathcal{U}) = 8/5, \quad \mathbb{P} - \text{a.s.}$$

Following work by mathematical physicists in the early 1980s, random walks on graphs with fractal growth of this kind have been studied in the mathematical literature. (Much of the initial mathematical work was done on diffusions on fractal sets, but many of the same results carry over to the graph case). This work showed that the behaviour of SRW on a (sufficiently regular) graph G can be summarized by two exponents. The first of these is the volume growth exponent d_f, while the second, denoted d_w, and called the walk dimension, can be defined by

$$d_w = d_w(G) = \lim_{R \to \infty} \frac{\log E^0 \tau_R}{\log R} \qquad \text{(if this limit exists).}$$

Here 0 is a base point in the graph, and τ_R is as defined in (1.9); it is easy to see that if G is connected then the limit is independent of the base point. One finds that $d_f \geq 1$, $2 \leq d_w \leq 1 + d_f$, and that all these values can arise – see [Bar04].

Many of the early papers required quite precise knowledge of the structure of the graph in order to calculate d_f and d_w. However, [BCK05] showed that in some cases it is sufficient to know two facts: the volume growth of balls, and the growth of effective resistance between points in the graph. Write $R_{\text{eff}}(x,y)$ for the effective resistance between points x and y in a graph G – see Section 3 for a precise definition. The results of [BCK05] imply that if G has uniformly bounded vertex degree, and there exist $\alpha > 0$, $\zeta > 0$ such that

$$c_1 R^\alpha \leq |B_d(x,R)| \leq c_2 R^\alpha, \quad x \in G,\ R \geq 1, \tag{1.15}$$

$$c_1 d(x,y)^\zeta \leq R_{\text{eff}}(x,y) \leq c_2 d(x,y)^\zeta, \quad x, y \in G, \tag{1.16}$$

then writing $\tau_R^x = \min\{n : d(x, X_n) > R\}$,

$$p_{2n}(x,x) \asymp n^{-\alpha/(\alpha+\zeta)}, \quad x \in G,\, n \ge 1, \tag{1.17}$$

$$E^x \tau_R^x \asymp R^{\alpha+\zeta}, \quad x \in G,\, R \ge 1. \tag{1.18}$$

(They also obtained good estimates on the transition probabilities $P^x(X_n = y)$ – see [BCK05, Theorem 1.3].) From (1.17) and (1.18) one sees that if G satisfies (1.15) and (1.16) then

$$d_f = \alpha, \quad d_w = \alpha + \zeta.$$

The decay n^{-d_f/d_w} for the transition probabilities in (1.17) can be explained as follows. If $R \ge 1$ and $2n = R^{d_w}$ then with high probability X_{2n} will be in the ball $B(x, cR)$. This ball has $cR^{d_f} \approx cn^{d_f/d_w}$ points, and so the average value of $p_{2n}(x,y)$ on this ball will be n^{-d_f/d_w}. Given enough regularity on G, this average value will then be close to the actual value of $p_{2n}(x,x)$.

In the physics literature a third exponent, called the spectral dimension, was introduced; this can be defined by

$$d_s(G) = -2 \lim_{n\to\infty} \frac{\log P^x_\omega(X_{2n} = x)}{\log 2n}, \quad \text{(if this limit exists)}. \tag{1.19}$$

This gives the rate of decay of the transition probabilities; one has $d_s(\mathbb{Z}^d) = d$. The discussion above indicates that the three indices d_f, d_w and d_s are not independent, and that given enough regularity in the graph G one expects that

$$d_s = \frac{2d_f}{d_w}.$$

For graphs satisfying (1.15) and (1.16) one has $d_s = 2\alpha/(\alpha + \zeta)$.

Note that if G is a tree and satisfies (1.15) then $R_{\text{eff}}(x,y) = d(x,y)$ and so (1.16) holds with $\zeta = 1$. Thus

$$d_f = \alpha, \quad d_w = \alpha + 1, \quad d_s = \frac{2\alpha}{\alpha+1}. \tag{1.20}$$

For random graphs arising from models in statistical physics, such as critical percolation clusters or the UST, random fluctuations will mean that one cannot expect (1.15) and (1.16) to hold uniformly. Nevertheless, providing similar estimates hold with high enough probability, it was shown in [BJKS08] and [KM08] that one can obtain enough control on the properties of the random walk X to calculate d_f, d_w and d_s. An additional contribution of [BJKS08] was to show that it is sufficient to estimate the volume and resistance growth for balls from one base point. In section 4, we will use these methods to show that (1.20) holds for the UST, namely that

Theorem 1.5 *We have for $\mathbb{P}$ -a.a. ω*

$$d_f(\mathcal{U}) = \frac{8}{5}, \quad d_w(\mathcal{U}) = \frac{13}{5}, \quad d_s(\mathcal{U}) = \frac{16}{13}. \tag{1.21}$$

The methods of [BJKS08] and [KM08] were also used in [BJKS08] to study the incipient infinite cluster (IIC) for high dimensional oriented percolation, and in [KN09] to show the IIC for standard percolation in high dimensions has spectal dimension 4/3. These critical percolation clusters are close to trees and have $d_f = 2$ in their graph metric. Our results for the UST are the first time these exponents have been calculated for a two-dimensional model arising from the random cluster model. It is natural to ask about critical percolation in two dimensions, but in spite of what is known via SLE, the values of d_w and d_s appear at present to be out of reach.

The rest of this paper is laid out as follows. In Section 2, we define the LERW on $\mathbb{Z}^2$ and recall the results from [Mas09, BM10] which we will need. The paper [BM10] gives bounds on M_D, the length of the loop-erasure of a random walk run up to the first exit of a simply connected domain D. However, in addition to these bounds, we require estimates on $d(0, w)$ which by Wilson's algorithm is the length of the loop-erasure of a random walk started at 0 and run up to the first time it hits w; we obtain these bounds in Proposition 2.7.

In Section 3, we study the geometry of the two dimensional UST $\mathcal{U}$, and prove Theorems 1.1 and 1.2. In addition (see Proposition 3.6) we show that with high probability the electrical resistance in the network $\mathcal{U}$ between 0 and $B_d(0, R)^c$ is greater than R/λ. The proofs of all of these results involve constructing the UST $\mathcal{U}$ in a particular way using Wilson's algorithm and then applying the bounds on the lengths of LERW paths from Section 2.

In Section 4, we use the techniques from [BJKS08, KM08] and our results on the volume and effective resistance of $\mathcal{U}$ from Section 3 to prove Theorems 1.4 and 1.5.

Throughout the paper, we use c, c', C, C' to denote positive constants which may change between each appearance, but do not depend on any variable. If we wish to fix a constant, we will denote it with a subscript, e.g. c_0.

2 Loop erased random walks

In this section, we look at LERW on $\mathbb{Z}^2$. We let S be a simple random walk on $\mathbb{Z}^2$, and given a set $D \subset \mathbb{Z}^2$, let

$$\sigma_D = \min\{j \geq 1 : S_j \in \mathbb{Z}^2 \setminus D\}$$

be the first exit time of the set D, and

$$\xi_D = \min\{j \geq 1 : S_j \in D\}$$

be the first hitting time of the set D. If $w \in \mathbb{Z}^2$, we write ξ_w for $\xi_{\{w\}}$. We also let $\sigma_R = \sigma_{B(R)}$ and use a similar convention for ξ_R. The outer boundary of a set $D \subset \mathbb{Z}^2$ is

$$\partial D = \{x \in \mathbb{Z}^2 \setminus D : \text{ there exists } y \in D \text{ such that } |x - y| = 1\},$$

and its inner boundary is

$$\partial_i D = \{x \in D : \text{ there exists } y \in \mathbb{Z}^2 \setminus D \text{ such that } |x - y| = 1\}.$$

A *path* λ in $\mathbb{Z}^2$ is a sequence (finite or infinite) $\lambda_0, \lambda_1, \ldots$ with $|\lambda_{j+1} - \lambda_j| = 1$. We occasionally will write $\lambda(j)$ rather than λ_j if j is a complicated expression. Given a path

$\lambda = [\lambda_0, \ldots, \lambda_m]$ in $\mathbb{Z}^2$, let $\mathrm{L}(\lambda)$ denote its chronological loop-erasure. More precisely, we let

$$s_0 = \max\{j : \lambda_j = \lambda_0\},$$

and for $i > 0$,

$$s_i = \max\{j : \lambda_j = \lambda(s_{i-1} + 1)\}.$$

Let

$$n = \min\{i : s_i = m\}.$$

Then

$$\mathrm{L}(\lambda) = [\lambda(s_0), \lambda(s_1), \ldots, \lambda(s_n)].$$

We note that by Wilson's algorithm, $\mathrm{L}(S[0, \xi_w])$ has the same distribution as $\gamma(0, w)$ – the unique path from 0 to w in the UST $\mathcal{U}$. We will therefore use $\gamma(0, w)$ to denote $\mathrm{L}(S[0, \xi_w])$ even when we make no mention of the UST $\mathcal{U}$.

For positive integers l, let Ω_l be the set of paths $\omega = [0, \omega_1, \ldots, \omega_k] \subset \mathbb{Z}^2$ such that $\omega_j \in B(0, l)$, $j = 1, \ldots, k-1$ and $\omega_k \in \partial B(0, l)$. For $n \geq l$, define the measure $\mu_{l,n}$ on Ω_l to be the distribution on Ω_l obtained by restricting $\mathrm{L}(S[0, \sigma_n])$ to the part of the path from 0 to the first exit of $B(0, l)$.

For a fixed l and $\omega \in \Omega_l$, it was shown in [Law91] that the sequence $\mu_{l,n}(\omega)$ is Cauchy. Therefore, there exists a limiting measure μ_l such that

$$\lim_{n \to \infty} \mu_{l,n}(\omega) = \mu_l(\omega).$$

The μ_l are consistent and therefore there exists a measure μ on infinite self-avoiding paths. We call the associated process the infinite LERW and denote it by $\widehat{S}$. We denote the exit time of a set D for $\widehat{S}$ by $\widehat{\sigma}_D$. By Wilson's algorithm, $\widehat{S}[0, \infty)$ has the same distribution as $\gamma(0, \infty)$, the unique infinite path in $\mathcal{U}$ starting at 0. Depending on the context, either notation will be used.

For a set D containing 0, we let M_D be the number of steps of $\mathrm{L}(S[0, \sigma_D])$. Notice that if $D = \mathbb{Z}^2 \setminus \{w\}$ and S is a random walk started at x, then $M_D = d(x, w)$. In addition, if $D' \subset D$ then we let $M_{D',D}$ be the number of steps of $\mathrm{L}(S[0, \sigma_D])$ while it is in D', or equivalently the number of points in D' that are on the path $\mathrm{L}(S[0, \sigma_D])$.

We let $\widehat{M}_n$ be the number of steps of $\widehat{S}[0, \widehat{\sigma}_n]$. As in the introduction, we set $\widehat{G}(n) = \mathrm{E}[\widehat{M}_n]$, extend $\widehat{G}$ to a continuous strictly increasing function from $[1, \infty)$ to $[1, \infty)$ with $\widehat{G}(1) = 1$, and let $\widehat{g}$ be the inverse of $\widehat{G}$. It was shown in [Law13] that $\widehat{G}(n) \asymp G(n) = n^{5/4}$.

Lemma 2.1 *We have*

$$G(\lambda r) \asymp \lambda^{5/4} G(r), \tag{2.1}$$

$$g(\lambda r) \asymp \lambda^{4/5} g(r). \tag{2.2}$$

Proof. This is immediate from (1.1). □

The following result from [BM10] gives bounds on the tails of $\widehat{M}_n$ and of $M_{D',D}$ for a broad class of sets D and subsets $D' \subset D$. We call a subset of $\mathbb{Z}^2$ *simply connected* if all connected components of its complement are infinite.

Theorem 2.2 *[BM10, Theorems 5.8 and 6.7] There exist positive global constants C and c, and given $\varepsilon > 0$, there exist positive constants $C(\varepsilon)$ and $c(\varepsilon)$ such that for all $\lambda > 0$ and all n, the following holds.*

1. *Suppose that $D \subset \mathbb{Z}^2$ contains 0, and $D' \subset D$ is such that for all $z \in D'$, there exists a path in D^c connecting $B(z, n+1)$ and $B(z, 2n)^c$ (in particular this will hold if D is simply connected and $\operatorname{dist}(z, D^c) \leq n$ for all $z \in D'$). Then*
$$\mathbb{P}\left(M_{D',D} > \lambda n^{5/4}\right) \leq 2e^{-c\lambda}. \tag{2.3}$$

2. *For all $D \supset B(0,n)$,*
$$\mathbb{P}\left(M_D < \lambda^{-1} n^{5/4}\right) \leq C(\varepsilon) e^{-c(\varepsilon)\lambda^{4/5-\varepsilon}}. \tag{2.4}$$

3. $$\mathbb{P}\left(\widehat{M}_n > \lambda n^{5/4}\right) \leq Ce^{-c\lambda}. \tag{2.5}$$

4. $$\mathbb{P}\left(\widehat{M}_n < \lambda^{-1} n^{5/4}\right) \leq C(\varepsilon) e^{-c(\varepsilon)\lambda^{4/5-\varepsilon}}. \tag{2.6}$$

We would like to use (2.3) in the case where $D = \mathbb{Z}^2 \setminus \{w\}$ and $D' = B(0,n) \setminus \{w\}$. However these choices of D and D' do not satisfy the hypotheses in (2.3), so we cannot use Theorem 2.2 directly. The idea behind the proof of the following proposition is to get the distribution on $\gamma(0,w)$ using Wilson's algorithm by first running an infinite LERW γ (whose complement is simply connected) and then running a LERW from w to γ.

Proposition 2.3 *There exist positive constants C and c such that the following holds. Let $n \geq 1$ and $w \in B(0,n)$. Let $Y_w = w$ if $\gamma(0,w) \subset B(0,n)$; otherwise let Y_w be the first point on the path $\gamma(0,w)$ which lies outside $B(0,n)$. Then,*
$$\mathbb{P}\left(d(0, Y_w) > \lambda n^{5/4}\right) \leq Ce^{-c\lambda}. \tag{2.7}$$

Proof. Let γ be any infinite path starting from 0, and let $\widetilde{D} = \mathbb{Z}^2 \setminus \gamma$. Then $\widetilde{D}$ is the union of disjoint simply connected subsets D_i of $\mathbb{Z}^2$; we can assume $w \in D_1$ and let $D_1 = D$. By (2.3), (taking $D' = B_n \cap D$) there exist $C < \infty$ and $c > 0$ such that
$$\mathbb{P}^w\left(M_{D',D} > \lambda n^{5/4}\right) \leq Ce^{-c\lambda}. \tag{2.8}$$

Now suppose that γ has the distribution of an infinite LERW started at 0. By Wilson's algorithm, if S^w is an independent random walk started at w, then $\gamma(0,w)$ has the same distribution as the path from 0 to w in $\gamma \cup \mathrm{L}(S^w[0, \sigma_D])$. Therefore,
$$d(0, Y_w) = |\gamma(0, Y_w)| \leq \widehat{M}_n + M_{D',D},$$
and so,
$$\mathbb{P}\left(d(0,Y_w) > \lambda n^{5/4}\right) \leq \mathbb{P}\left(\widehat{M}_n > (\lambda/2) n^{5/4}\right) + \max_D \mathbb{P}^w\left(M_{D',D} > (\lambda/2) n^{5/4}\right).$$
The result then follows from (2.5) and (2.8). □

Lemma 2.4 *There exists a positive constant C such that for all $k \geq 2$, $n \geq 1$, and $K \subset \mathbb{Z}^2 \setminus B(0,4kn)$, the following holds. The probability that $\mathrm{L}(S[0,\xi_K])$ reenters $B(0,n)$ after leaving $B(0,kn)$ is less than Ck^{-1}. This also holds for infinite LERWs, namely*

$$\mathbb{P}\big(\widehat{S}[\widehat{\sigma}_{kn},\infty) \cap B(0,n) \neq \emptyset\big) \leq Ck^{-1}. \tag{2.9}$$

Proof. The result for infinite LERWs follows immediately by taking $K = \mathbb{Z}^2 \setminus B(0,m)$ and letting m tend to ∞.

We now prove the result for $\mathrm{L}(S[0,\xi_K])$. Let γ be the part of the path $\mathrm{L}(S[0,\xi_K])$ from 0 up to the first point z where it exits $B(0,kn)$. Then by the domain Markov property for LERW [Law91], conditioned on γ, the rest of $\mathrm{L}(S[0,\xi_K])$ has the same distribution as the loop-erasure of a random walk started at z, conditioned on the event $\{\xi_K < \xi_\gamma\}$. Therefore, it is sufficient to show that for any fixed path α from 0 to $\partial B(0,kn)$ and $z \in \partial B(0,kn)$,

$$\mathbb{P}^z\left(\xi_n < \xi_K \mid \xi_K < \xi_\alpha\right) = \frac{\mathbb{P}^z\left(\xi_n < \xi_K; \xi_K < \xi_\alpha\right)}{\mathbb{P}^z\left(\xi_K < \xi_\alpha\right)} \leq Ck^{-1}. \tag{2.10}$$

On the one hand,

$$\begin{aligned}&\mathbb{P}^z\left(\xi_n < \xi_K; \xi_K < \xi_\alpha\right)\\ &\leq \mathbb{P}^z\left(\xi_{kn/2} < \xi_\alpha\right) \max_{x\in\partial_i B(0,kn/2)} \mathbb{P}^x\left(\xi_n < \xi_\alpha\right) \max_{w\in\partial B(0,n)} \mathbb{P}^w\left(\sigma_{2kn} < \xi_\alpha\right) \max_{y\in\partial B(0,2kn)} \mathbb{P}^y\left(\xi_K < \xi_\alpha\right).\end{aligned}$$

However, by the discrete Beurling estimates (see [LL, Theorem 6.8.1]), for any $x \in \partial_i B(0,kn/2)$ and $w \in \partial B(0,n)$,

$$\mathbb{P}^x\left(\xi_n < \xi_\alpha\right) \leq Ck^{-1/2};$$

$$\mathbb{P}^w\left(\sigma_{2kn} < \xi_\alpha\right) \leq Ck^{-1/2}.$$

Therefore,

$$\mathbb{P}^z\left(\xi_n < \xi_K; \xi_K < \xi_\alpha\right) \leq Ck^{-1}\mathbb{P}^z\left(\xi_{kn/2} < \xi_\alpha\right) \max_{y\in\partial B(0,2kn)} \mathbb{P}^y\left(\xi_K < \xi_\alpha\right).$$

On the other hand,

$$\mathbb{P}^z\left(\xi_K < \xi_\alpha\right) \geq \mathbb{P}^z\left(\sigma_{2kn} < \xi_\alpha\right) \min_{y\in\partial B(0,2kn)} \mathbb{P}^y\left(\xi_K < \xi_\alpha\right).$$

By the discrete Harnack inequality (see [LL, Theorem 6.3.9]),

$$\max_{y\in\partial B(0,2kn)} \mathbb{P}^y\left(\xi_K < \xi_\alpha\right) \leq C \min_{y\in\partial B(0,2kn)} \mathbb{P}^y\left(\xi_K < \xi_\alpha\right).$$

Therefore, in order to prove (2.10), it suffices to show that

$$\mathbb{P}^z\left(\sigma_{2kn} < \xi_\alpha\right) \geq c\mathbb{P}^z\left(\xi_{kn/2} < \xi_\alpha\right).$$

Let $B = B(z; kn/2)$. By [Mas09, Proposition 3.5], there exists $c > 0$ such that

$$\mathbb{P}^z \left(|\arg(S(\sigma_B) - z)| \leq \pi/3 \mid \sigma_B < \xi_\alpha\right) > c.$$

Therefore,

$$\begin{aligned}\mathbb{P}^z \left(\sigma_{2kn} < \xi_\alpha\right) &\geq \sum_{\substack{y \in \partial B \\ |\arg(y-z)| \leq \pi/3}} \mathbb{P}^y \left(\sigma_{2kn} < \xi_\alpha\right) \mathbb{P}^z \left(\sigma_B < \xi_\alpha; S(\sigma_B) = y\right) \\ &\geq c\mathbb{P}^z \left(\sigma_B < \xi_\alpha; |\arg(S(\sigma_B) - z)| \leq \pi/3\right) \\ &\geq c\mathbb{P}^z \left(\sigma_B < \xi_\alpha\right) \\ &\geq c\mathbb{P}^z \left(\xi_{kn/2} < \xi_\alpha\right).\end{aligned}$$

□

Remark 2.5 One can also show that there exists $\delta > 0$ such that

$$\mathbb{P}\left(\widehat{S}[\widehat{\sigma}_{kn}, \infty) \cap B(0, n) \neq \emptyset\right) \geq ck^{-\delta}. \tag{2.11}$$

As we will not need this bound we only give a sketch of the proof. Further, we will not try to find the value of δ that the argument yields, since it would be far from optimal.

First, we have

$$\mathbb{P}\left(\widehat{S}[\widehat{\sigma}_{kn}, \infty) \cap B(0, n) \neq \emptyset\right) \geq \mathbb{P}\left(\widehat{S}[\widehat{\sigma}_{kn}, \widehat{\sigma}_{4kn}) \cap B(0, n) \neq \emptyset\right).$$

However, by [Mas09, Corollary 4.5], the latter probability is comparable to the probability that $\mathrm{L}(S[0, \sigma_{16kn}])$ leaves B_{kn} and then reenters $B(0, n)$ before leaving $B(0, 4kn)$. Call the latter event F.

Partition $\mathbb{Z}^2$ into the three cones $A_1 = \{z \in \mathbb{Z}^2 : 0 \leq \arg(z) < 2\pi/3\}$, $A_2 = \{z \in \mathbb{Z}^2 : 2\pi/3 \leq \arg(z) < 4\pi/3\}$ and $A_3 = \{z \in \mathbb{Z}^2 : 4\pi/3 \leq \arg(z) < 2\pi\}$. Then the event F contains the event that a random walk started at 0

(1) leaves $B(0, 2kn)$ before leaving $A_1 \cup B(0, n/2)$,
(2) then enters A_2 while staying in $B(0, 4kn) \setminus B(0, kn)$,
(3) then enters $B(0, n)$ while staying in $A_2 \cap B(0, 4kn)$,
(4) then enters A_3 while staying in $A_2 \cap B(0, n) \setminus B(0, n/2)$,
(5) then leaves $B(0, 16kn)$ while staying in $A_3 \setminus B(0, n/2)$.

One can bound the probabilities of the events in steps (1), (3) and (5) from below by $ck^{-\beta}$ for some $\beta > 0$. The other steps contribute terms that can be bounded from below by a constant; combining these bounds gives (2.11).

Lemma 2.6 *There exists a positive constant C such that for all $k \geq 1$ and $w \in \mathbb{Z}^2$,*

$$\frac{1}{8}k^{-1} \leq \mathbb{P}\left(\gamma(0, w) \not\subset B(0, k\,|w|)\right) \leq Ck^{-1/3}. \tag{2.12}$$

Proof. We first prove the upper bound. By adjusting the value of C we may assume that $k \geq 4$. In order to obtain $\gamma(0,w)$, we first run an infinite LERW γ started at 0 and then run an independent random walk started at w until it hits γ and then erase its loops. By Wilson's algorithm, the resulting path from 0 to w has the same distribution as $\gamma(0,w)$.

By Lemma 2.4, the probability that γ reenters $B_{k^{2/3}|w|}$ after leaving $B_{k|w|}$ is less than $Ck^{-1/3}$. Furthermore, by the discrete Beurling estimates [LL, Proposition 6.8.1],

$$\mathbb{P}^w\left(\sigma_{k^{2/3}|w|} < \xi_\gamma\right) \leq C(k^{2/3})^{-1/2} = Ck^{-1/3}.$$

Therefore,

$$\mathbb{P}\left(\gamma(0,w) \not\subset B_{k|w|}\right) \leq Ck^{-1/3}.$$

To prove the lower bound, we follow the method of proof of [BLPS01, Theorem 14.3] where it was shown that if v and w are nearest neighbors then

$$\mathbb{P}\left(\text{diam } \gamma(v,w) \geq n\right) \geq \frac{1}{8n}.$$

If $w = (w_1, w_2)$, let $u = (w_1 - w_2, w_1 + w_2)$ and $v = (-w_2, w_1)$ so that $\{0, w, u, v\}$ form four vertices of a square of side length $|w|$. Now consider the sets

$$Q_1 = \{jw : j = 0, \ldots, 2k\}, \quad Q_2 = \{2kw + j(u-w) : j = 0, \ldots, 2k\},$$
$$Q_3 = \{jv : j = 0, \ldots, 2k\}, \quad Q_4 = \{2kv + j(u-v) : j = 0, \ldots, 2k\},$$

and let $Q = \bigcup_{i=1}^4 Q_i$. Then Q consists of $8k$ lattice points on the perimeter of a square of side length $2k\,|w|$. Let $x_1, \ldots, x_{8k}$ be the ordering of these points obtained by letting $x_1 = 0$ and then travelling along the perimeter of the square clockwise. Thus $|x_{i+1} - x_i| = |w|$. Now consider any spanning tree U on $\mathbb{Z}^2$. If for all i, $\gamma(x_i, x_{i+1})$ stayed in the ball $B(x_i, k\,|w|)$ then the concatenation of these paths would be a closed loop, which contradicts the fact that U is a tree. Therefore,

$$1 = \mathbb{P}\left(\exists i : \gamma(x_i, x_{i+1}) \not\subset B(x_i, k\,|w|)\right) \leq \sum_{i=1}^{8k} \mathbb{P}\left(\gamma(x_i, x_{i+1}) \not\subset B(x_i, k\,|w|)\right).$$

Finally, using the fact that $\mathbb{Z}^2$ is transitive and is invariant under rotations by 90 degrees, all the probabilities on the right hand side are equal. This proves the lower bound. $\square$

Proposition 2.7 *For all $\varepsilon > 0$, there exist $c(\varepsilon), C(\varepsilon) > 0$ and $\lambda_0(\varepsilon) \geq 1$ such that for all $w \in \mathbb{Z}^2$ and all $\lambda \geq 1$,*

$$\mathbb{P}\left(d(0,w) > \lambda\,|w|^{5/4}\right) \leq C(\varepsilon)\lambda^{-4/15+\varepsilon}, \tag{2.13}$$

and for all $w \in \mathbb{Z}^2$ and all $\lambda \geq \lambda_0(\varepsilon)$,

$$\mathbb{P}\left(d(0,w) > \lambda\,|w|^{5/4}\right) \geq c(\varepsilon)\lambda^{-4/5-\varepsilon}. \tag{2.14}$$

Proof. (As $G(r) = r^{5/4}$ this argument could be simplified.) To prove the upper bound, let $k = \lambda^{4/5-3\varepsilon}$. Then by Lemma 2.1, there exists $C(\varepsilon) < \infty$ such that

$$G(k\,|w|) \le C(\varepsilon) k^{5/4+\varepsilon} G(|w|) \le C(\varepsilon)\lambda^{1-\varepsilon} G(|w|). \tag{2.15}$$

Then,

$$\mathbb{P}\left(d(0,w) > \lambda G(|w|)\right) \le \mathbb{P}\left(\gamma(0,w) \not\subset B_{k|w|}\right) + \mathbb{P}\left(d(0,w) > \lambda G(|w|); \gamma(0,w) \subset B_{k|w|}\right).$$

However, by Lemma 2.6,

$$\mathbb{P}\left(\gamma(0,w) \not\subset B_{k|w|}\right) \le C k^{-1/3} = C\lambda^{-4/15+\varepsilon}, \tag{2.16}$$

while by (2.15),

$$\begin{aligned}\mathbb{P}\left(d(0,w) > \lambda G(|w|); \gamma(0,w) \subset B_{k|w|}\right) &\le \mathbb{P}\left(d(0,w) > c(\varepsilon)\lambda^{\varepsilon} G(k\,|w|); \gamma(0,w) \subset B_{k|w|}\right) \\ &\le C\exp(-c(\varepsilon)\lambda^{\varepsilon}).\end{aligned}$$

Therefore,

$$\mathbb{P}\left(d(0,w) > \lambda G(|w|)\right) \le C\exp(-c(\varepsilon)\lambda^{\varepsilon}) + C\lambda^{-4/15+\varepsilon} \le C(\varepsilon)\lambda^{-4/15+\varepsilon}. \tag{2.17}$$

To prove the lower bound we fix $k = \lambda^{4/5+\varepsilon}$ and assume $k \ge 2$ and $\varepsilon < 1/4$. Then by Lemma 2.1, there exists $C(\varepsilon) < \infty$ such that

$$G((k-1)\,|w|) \ge C(\varepsilon)^{-1} k^{5/4-\varepsilon} G(|w|) \ge C(\varepsilon)^{-1}\lambda^{1+\varepsilon/3} G(|w|).$$

Hence,

$$\mathbb{P}\left(d(0,w) > \lambda G(|w|)\right) \ge \mathbb{P}\left(d(0,w) > C(\varepsilon)\lambda^{-\varepsilon/3} G((k-1)\,|w|)\right).$$

Now consider the UST on $\mathbb{Z}^2$ and recall that $\gamma(0,\infty)$ and $\gamma(w,\infty)$ denote the infinite paths starting at 0 and w. We write Z_{0w} for the unique point where these meet: thus $\gamma(Z_{0w},\infty) = \gamma(0,\infty)\cap\gamma(w,\infty)$. Then $\gamma(0,w)$ is the concatenation of $\gamma(0,Z_{0w})$ and $\gamma(w,Z_{0w})$. By Lemma 2.6,

$$\mathbb{P}\left(\gamma(0,w) \not\subset B_{k|w|}\right) \ge \frac{1}{8k}.$$

Therefore,

$$\mathbb{P}\left(\gamma(0,Z_{0w}) \not\subset B_{k|w|} \quad \text{or} \quad \gamma(w,Z_{0w}) \not\subset B_{k|w|}\right) \ge \frac{1}{8k}.$$

By the transitivity of $\mathbb{Z}^2$, the paths $\gamma(0,Z_{0,-w})$ and $\gamma(w,Z_{0w})-w$ have the same distribution, and therefore

$$\mathbb{P}\left(\gamma(0,Z_{0w}) \not\subset B_{(k-1)|w|}\right) \ge \frac{1}{16k}.$$

Since Z_{0w} is on the path $\gamma(0,\infty)$, by (2.6),

$$\begin{aligned}&\mathbb{P}\left(d(0,w) > C(\varepsilon)\lambda^{-\varepsilon/3} G((k-1)\,|w|)\right) \\ &\quad\ge \mathbb{P}\left(d(0,Z_{0w}) > C(\varepsilon)\lambda^{-\varepsilon/3} G((k-1)\,|w|)\right) \\ &\quad\ge \mathbb{P}\left(\widehat{M}_{(k-1)|w|} > C(\varepsilon)\lambda^{-\varepsilon/3} G((k-1)\,|w|); \gamma(0,Z_{0w}) \not\subset B_{(k-1)|w|}\right) \\ &\quad\ge \mathbb{P}\left(\gamma(0,Z_{0w}) \not\subset B_{(k-1)|w|}\right) - \mathbb{P}\left(\widehat{M}_{(k-1)|w|} < C(\varepsilon)\lambda^{-\varepsilon/3} G((k-1)\,|w|)\right) \\ &\quad\ge \frac{1}{16k} - C\exp\{-c\lambda^{\varepsilon/4}\}.\end{aligned}$$

Finally, since $k = \lambda^{4/5+\varepsilon}$, the previous quantity can be made greater than $c(\varepsilon)\lambda^{-4/5-\varepsilon}$ for λ sufficiently large. $\square$

3 Uniform spanning trees

We recall that $\mathcal{U}$ denotes the UST in $\mathbb{Z}^2$, and we write $x \sim y$ if x and y are joined by an edge in $\mathcal{U}$.

Let $\mathcal{E}$ be the quadratic form given by

$$\mathcal{E}(f,g) = \tfrac{1}{2}\textstyle\sum_{x\sim y}(f(x)-f(y))(g(x)-g(y)), \tag{3.1}$$

If we regard $\mathcal{U}$ as an electrical network with a unit resistor on each edge, then $\mathcal{E}(f,f)$ is the energy dissipation when the vertices of $\mathbb{Z}^2$ are at a potential f. Set $H^2 = \{f : \mathbb{Z}^2 \to \mathbb{R} : \mathcal{E}(f,f) < \infty\}$. Let A, B be disjoint subsets of $\mathcal{U}$. The effective resistance between A and B is defined by:

$$R_{\text{eff}}(A,B)^{-1} = \inf\{\mathcal{E}(f,f) : f \in H^2, f|_A = 1, f|_B = 0\}. \tag{3.2}$$

Let $R_{\text{eff}}(x,y) = R_{\text{eff}}(\{x\},\{y\})$, and $R_{\text{eff}}(x,x) = 0$. For general facts on effective resistance and its connection with random walks see [AF, DS84, LP09].

In this section, we establish the volume and effective resistance estimates for the UST $\mathcal{U}$ that will be used in the next section to study random walks on $\mathcal{U}$.

Theorem 3.1 *There exist positive constants C and c such that for all $r \geq 1$ and $\lambda > 0$,*
(a)

$$\mathbb{P}\left(B_d(0,\lambda^{-1}r^{5/4}) \not\subset B(0,r)\right) \leq Ce^{-c\lambda^{2/3}}. \tag{3.3}$$

(b)

$$\mathbb{P}\left(R_{\text{eff}}(0,B(0,r)^c) < \lambda^{-3}r^{5/4}\right) \leq Ce^{-c\lambda^{2/3}}. \tag{3.4}$$

Proof. By adjusting the constants c and C we can assume $\lambda \geq 4$. For $k \geq 1$, let $\delta_k = \lambda^{-1}2^{-k}$, and $\eta_k = (2k)^{-1}$. Let k_0 be the smallest integer such that $r\delta_{k_0} < 1$. Set

$$A_k = B(0,r) - B(0,(1-\eta_k)r), \qquad k \geq 1.$$

Let D_k be a finite collection of points in A_k such that $|D_k| \leq C\delta_k^{-2}$ and

$$A_k \subset \bigcup_{z\in D_k} B(z,\delta_k r).$$

Write $\mathcal{U}_1, \mathcal{U}_2, \dots$ for the random trees obtained by running Wilson's algorithm (with root 0) with walks first starting at all points in D_1, then adding those points in D_2, and so on. So $\mathcal{U}_k$ is a finite tree which contains $\bigcup_{i=1}^k D_i \cup \{0\}$, and the sequence $(\mathcal{U}_k)$ is increasing. Since $r\delta_{k_0} < 1$ we have $\partial_i B(0,r) \subset A_{k_0} \subset \mathcal{U}_{k_0}$. We then complete a UST $\mathcal{U}$ on $\mathbb{Z}^2$ by applying Wilson's algorithm to the remaining points in $\mathbb{Z}^2$.

For $z \in D_1$, let N_z be the length of the path $\gamma(0,z)$ until it first exits from $B(0,r/8)$. By first applying [Mas09, Proposition 4.4] and then (2.6),

$$\mathbb{P}(N_z < \lambda^{-1} r^{5/4}) \le C \mathbb{P}(\widehat{M}_{r/8} < \lambda^{-1} r^{5/4}) \le C e^{-c\lambda^{2/3}},$$

so if

$$\widetilde{F}_1 = \{N_z < \lambda^{-1} r^{5/4} \text{ for some } z \in D_1 \} = \bigcup_{z \in D_1} \{N_z < \lambda^{-1} r^{5/4}\},$$

then

$$\mathbb{P}(\widetilde{F}_1) \le |D_1| C e^{-c\lambda^{2/3}} \le C \delta_1^{-2} e^{-c\lambda^{2/3}} \le C\lambda^2 e^{-c\lambda^{2/3}}. \tag{3.5}$$

For $z \in A_{k+1}$, let H_z be the event that the path $\gamma(z,0)$ enters $B(0,(1-\eta_k)r)$ before it hits $\mathcal{U}_k$. For $k \ge 1$, let

$$F_{k+1} = \bigcup_{z \in D_{k+1}} H_z.$$

Let $z \in D_{k+1}$ and S^z be a simple random walk started at z and run until its hits $\mathcal{U}_k$. Then by Wilson's algorithm, for the event H_z to occur, S^z must enter $B(0,(1-\eta_k)r)$ before it hits $\mathcal{U}_k$. To bound $\mathbb{P}(H_z)$, note that z is a distance at least $(\eta_k - \eta_{k+1})r$ from $B(0,(1-\eta_k)r)$, and that each each point in A_k is within a distance $\delta_k r$ of $\mathcal{U}_k$. Let $m = (\eta_k - \eta_{k+1})/\delta_k$. For H_z to occur, the random walk must move through at least $\frac{1}{2}m$ balls of radius $2\delta_k r$ without hitting $\mathcal{U}_k$. Since $\mathcal{U}_k$ is connected, in each of these balls $B(w_i, 2\delta_k r)$ there is a path in $\mathcal{U}_k$ from $B(w_i,\delta_k r)$ to $B(w_i, 2\delta_k r)^c$. So, by the discrete Beurling estimate,

$$\mathbb{P}(H_z) \le \exp(-c\tfrac{1}{2}m) \le \exp(-c'(\eta_k - \eta_{k+1})/\delta_k).$$

Hence

$$\mathbb{P}(F_{k+1}) \le C\delta_{k+1}^{-2} \exp(-c(\eta_k - \eta_{k+1})/\delta_k) \le C\lambda^2 4^k \exp(-c\lambda 2^k k^{-2}). \tag{3.6}$$

Now define G by

$$G^c = \widetilde{F}_1 \cup \bigcup_{k=2}^{k_0} F_k,$$

so that

$$\mathbb{P}(G^c) \le C\lambda e^{-c\lambda^{2/3}} + \sum_{k=2}^{\infty} C\lambda^2 4^k \exp(-c\lambda 2^k k^{-2}) \le C e^{-c\lambda^{2/3}}. \tag{3.7}$$

Now suppose that $\omega \in G$. Then we claim that:

(1) For every $z \in D_1$ the part of the path $\gamma(0,z)$ until its first exit from $B(0,r/2)$ is of length greater than $\lambda^{-1} r^{5/4}$,

(2) If $z \in D_k$ for any $k \ge 2$ then the path $\gamma(z,0)$ hits $\mathcal{U}_1$ before it enters $B(0,r/2)$.

Of these, (1) is immediate since $\omega \notin \widetilde{F}_1$, while (2) follows by induction on k using the fact that $\omega \notin F_k$ for any k.

Hence, if $\omega \in G$, then $|\gamma(0,z)| \ge \lambda^{-1} r^{5/4}$ for every $z \in \partial_i B(0,r)$, which proves (a).

To prove (b) we use the Nash-Williams bound for resistance [NW59]. For $1 \le k \le \lambda^{-1} r^{5/4}$ let Γ_k be the set of z such that $d(0,z) = k$ and z is connected to $B(0,r)^c$ by a path in $\{z\} \cup (\mathcal{U} - \gamma(0,z))$. Assume now that the event G holds. Then the Γ_k are disjoint sets disconnecting 0 and $B(0,r)^c$, and so

$$R_{\text{eff}}(0, B(0,r)^c) \ge \sum_{k=1}^{\lambda^{-1} r^{5/4}} |\Gamma_k|^{-1}.$$

Furthermore, each $z \in \Gamma_k$ is on a path from 0 to a point in D_1, and so $|\Gamma_k| \le |D_1| \le C\delta_1^{-2} \le C\lambda^2$. Hence on G we have $R_{\text{eff}}(0, B(0,r)^c) \ge c\lambda^{-3} r^{5/4}$, which proves (b). $\square$

A similar argument will give a (much weaker) bound in the opposite direction. We begin with a result we will use to control the way the UST fills in a region once we have constructed some initial paths.

Proposition 3.2 *There exist positive constants c and C such that for each $\delta_0 \le 1$ the following holds. Let $r \ge 1$, and U_0 be a fixed tree in $\mathbb{Z}^2$ connecting 0 to $B(0,2r)^c$ with the property that $\operatorname{dist}(x, U_0) \le \delta_0 r$ for each $x \in B(0,r)$ (here dist refers to the Euclidean distance). Let $\mathcal{U}$ be the random spanning tree in $\mathbb{Z}^2$ obtained by running Wilson's algorithm with root U_0. Then there exists an event G such that*

$$\mathbb{P}(G^c) \le C e^{-c\delta_0^{-1/3}}, \tag{3.8}$$

and on G we have that for all $x \in B(0, r/2)$,

$$d(x, U_0) \le G(\delta_0^{1/2} r); \tag{3.9}$$

$$\gamma(x, U_0) \subset B(0,r). \tag{3.10}$$

Proof. We follow a similar strategy to that in Theorem 3.1. Define sequences (δ_k) and (λ_k) by $\delta_k = 2^{-k}\delta_0$, $\lambda_k = 2^{k/2}\lambda_0$, where $\lambda_0 = 5^{-1}\delta_0^{-1/2}$. For $k \ge 0$, let

$$A_k = B(0, \tfrac{1}{2}(1 + (1+k)^{-1})r),$$

and let $D_k \subset A_k$ be such that for $k \ge 1$,

$$|D_k| \le C\delta_k^{-2},$$

$$A_k \subset \bigcup_{z \in D_k} B(z, \delta_k r).$$

Let $\mathcal{U}_0 = U_0$ and as before let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be the random trees obtained by performing Wilson's algorithm with root U_0 and starting first at points in D_1, then in D_2 etc. Set

$$M_z = d(z, \mathcal{U}_{k-1}), \quad z \in D_k,$$

$$F_z = \{\gamma(z, \mathcal{U}_{k-1}) \not\subset A_{k-1}\}, \quad z \in D_k,$$

$$M_k = \max_{z \in D_k} M_z,$$

$$F_k = \bigcup_{z \in D_k} F_z.$$

For $z \in D_k$,

$$\mathbb{P}\left(M_z > \lambda_k G(\delta_{k-1} r)\right) \le \mathbb{P}(F_z) + \mathbb{P}\left(M_z > \lambda_k G(\delta_{k-1} r); F_z^c\right). \tag{3.11}$$

Since z is a distance at least $\frac{1}{2} r (k^{-1} - (k+1)^{-1})$ from A_{k-1}^c, and each point in A_{k-1} is within a distance $\delta_{k-1} r$ of $\mathcal{U}_{k-1}$,

$$\mathbb{P}\left(F_z\right) \le C \exp(-c\delta_{k-1}^{-1}(k^{-1} - (k+1)^{-1})) \le C \exp(-c\delta_{k-1}^{-1} k^{-2}). \tag{3.12}$$

By (2.3), again using the fact that each point in A_{k-1} is within distance $\delta_{k-1} r$ of $\mathcal{U}_{k-1}$,

$$\mathbb{P}\left(M_z > \lambda_k G(\delta_{k-1} r); F_z^c\right) \le C \exp(-c\lambda_k^{2/3}). \tag{3.13}$$

So, combining (3.11)–(3.13), for $k \ge 1$,

$$\mathbb{P}\left(M_k > \lambda_k G(\delta_{k-1} r)\right) + \mathbb{P}\left(F_k\right) \le C \left|D_k\right| \left[\exp(-c\delta_{k-1}^{-1} k^{-2}) + \exp(-c\lambda_k^{2/3})\right]. \tag{3.14}$$

Now let

$$G = \bigcap_{k=1}^{\infty} F_k^c \cap \{M_k \le \lambda_k G(\delta_{k-1} r)\}. \tag{3.15}$$

Summing the series given by (3.14), and using the bound $|D_k| \le c\delta_k^{-2}$, we have

$$\begin{aligned}
\mathbb{P}\left(G^c\right) &\le C\delta_0^{-2} \sum_k 2^{2k} \left[\exp(-c\delta_0^{-1} 2^k k^{-2}) + \exp(-c 2^{k/3} \delta_0^{-1/3})\right] \\
&\le C\delta_0^{-2} e^{-c\delta_0^{-1/3}} \\
&\le C e^{-c'\delta_0^{-1/3}}.
\end{aligned}$$

Using Lemma 2.1 with $\varepsilon = \frac{1}{4}$ gives

$$\lambda_k G(\delta_{k-1} r) \le \lambda_k \delta_0^{1/2} 2^{-(k-1)} G(\delta_0^{1/2} r) = 2\lambda_0 \delta_0^{1/2} 2^{-k/2} G(\delta_0^{1/2} r).$$

So

$$\sum_{k=1}^{\infty} \lambda_k G(\delta_{k-1} r) \le 5\lambda_0 \delta_0^{1/2} G(\delta_0^{1/2} r) = G(\delta_0^{1/2} r).$$

Since $B(0, r/2) \subset \bigcap_k A_k$, we have $B(0, r/2) \subset \bigcup_k \mathcal{U}_k$. Therefore on the event G, for any $x \in B(0, r/2)$, $d(x, U_0) \le G(\delta_0^{1/2} r)$. Further, on G, for each $z \in D_k$, we have $\gamma(z, \mathcal{U}_{k-1}) \subset A_{k-1}$. Therefore if $x \in B(0, r/2)$ the connected component of $\mathcal{U} - U_0$ containing x is contained in $B(0, r)$, which proves (3.10). □

Theorem 3.3 *For all $\varepsilon > 0$, there exist $c(\varepsilon), C(\varepsilon) > 0$ and $\lambda_0(\varepsilon) \ge 1$ such that for all $r \ge 1$ and $\lambda \ge 1$,*

$$\mathbb{P}\big(B(0, r) \not\subset B_d(0, \lambda r^{5/4})\big) \le C\lambda^{-4/15+\varepsilon}, \tag{3.16}$$

and for all $r \ge 1$ and all $\lambda \ge \lambda_0(\varepsilon)$,

$$\mathbb{P}\big(B(0, r) \not\subset B_d(0, \lambda r^{5/4})\big) \ge c\lambda^{-4/5-\varepsilon}.$$

Proof. The lower bound follows immediately from the lower bound in Proposition 2.7.

To prove the upper bound, let $E \subset B(0,4r)$ be such that $|E| \le C\lambda^{\varepsilon/2}$ and

$$B(0,4r) \subset \bigcup_{z\in E} B(z,\lambda^{-\varepsilon/4}r).$$

We now let $\mathcal{U}_0$ be the random tree obtained by applying Wilson's algorithm with points in E and root 0. Therefore, by Proposition 2.7, for any $z \in E$,

$$\mathbb{P}\big(d(0,z) > \lambda r^{5/4}/2\big) \le \mathbb{P}\big(d(0,z) > c\lambda G(|z|)/2\big) \le C(\varepsilon)\lambda^{-4/15+\varepsilon/2}.$$

Let

$$F = \{d(0,z) \le \lambda r^{5/4}/2 \text{ for all } z \in E\};$$

then

$$\mathbb{P}(F^c) \le |E|\, C(\varepsilon)\lambda^{-4/15+\varepsilon/2} \le C(\varepsilon)\lambda^{-4/15+\varepsilon}.$$

We have now constructed a tree $\mathcal{U}_0$ connecting 0 to $B(0,4r)^c$ and by the definition of the set E, for all $z \in B(0,2r)$, $\mathrm{dist}\,(z,\mathcal{U}_0) \le \lambda^{-\varepsilon/4}r$. We now use Wilson's algorithm to produce the UST $\mathcal{U}$ on $\mathbb{Z}^2$ with root $\mathcal{U}_0$. Let G be the event given by applying Proposition 3.2 (with r replaced by $2r$), so that

$$\mathbb{P}\left(G^c\right) \le Ce^{-c\lambda^{\varepsilon/12}}.$$

On the event G we have $d(x,\mathcal{U}_0) \le G(\lambda^{-\varepsilon/2}r) \le \lambda r^{5/4}/2$ for all $x \in B(0,r)$. Therefore, on the event $F \cap G$ we have $d(x,0) \le \lambda r^{5/4}$ for all $x \in B(0,r)$. Thus,

$$\mathbb{P}\Big(\max_{x\in B(0,r)} d(x,0) > \lambda r^{5/4}\Big) \le C(\varepsilon)\lambda^{-4/15+\varepsilon} + Ce^{-c\lambda^{\varepsilon/12}} \le C(\varepsilon)\lambda^{-4/15+\varepsilon}.$$

□

Theorem 1.1 is now immediate from Theorem 3.1 and Theorem 3.3.

While Theorem 3.1 immediately gives the exponential bound (1.5) on the upper tail of $|B_d(0,r)|$ in Theorem 1.2, it only gives a polynomial bound for the lower tail. The following theorem gives an exponential bound on the lower tail of $|B_d(0,r)|$ and consequently proves Theorem 1.2.

Theorem 3.4 *There exist constants c and C such that if $R \ge 1$, $\lambda \ge 1$ then*

$$\mathbb{P}(|B_d(0,R)| \le \lambda^{-1}R^{8/5}) \le Ce^{-c\lambda^{1/9}}. \tag{3.17}$$

Proof. Let $k \ge 1$ and let $r = g(R/k^{1/2})$, so that $R = k^{1/2}r^{5/4}$. Fix a constant $\delta_0 < 1$ such that the right side of (3.8) is less than 1/4. Fix a further constant $\theta < 1$, to be chosen later but which will depend only on δ_0.

We begin the construction of $\mathcal{U}$ with an infinite LERW $\widehat{S}$ started at 0 which gives the path $\gamma_0 = \mathcal{U}_0 = \gamma(0,\infty)$. Let z_j, $j = 1,\dots k$ be points on $\widehat{S}[0.\widehat{\sigma}_r]$ chosen such that $B_j = B(z_j, r/k)$

are disjoint. (We choose these according to some fixed algorithm so that they depend only on the path $\widehat{S}[0,\widehat{\sigma}_r]$.) Let

$$F_1 = \{ \ \widehat{S}[\widehat{\sigma}_{2r},\infty) \text{ hits more than } k/2 \text{ of } B_1,\dots B_k\}, \tag{3.18}$$

$$F_2 = \{|\widehat{S}[0,\widehat{\sigma}_{2r}]| \geq \tfrac{1}{2}k^{1/2}r^{5/4}\}. \tag{3.19}$$

We have

$$\mathbb{P}(F_1) \leq Ce^{-ck^{1/3}}, \tag{3.20}$$

$$\mathbb{P}(F_2) \leq Ce^{-ck^{1/2}}. \tag{3.21}$$

Of these, (3.21) is immediate from (2.5) while (3.20) will be proved in Lemma 3.7 below.

If either F_1 or F_2 occurs, we terminate the algorithm with a 'Type 1' or 'Type 2' failure. Otherwise, we continue as follows to construct $\mathcal{U}$ using Wilson's algorithm. We define

$$B'_j = B(z_j,\theta r/k), \qquad B''_j = B(z_j,\theta^2 r/k).$$

The algorithm is at two 'levels' which we call 'ball steps' and 'point steps'. We begin with a list J_0 of good balls. These are the balls B_j such that $B_j \cap \widehat{S}[\widehat{\sigma}_{2r},\infty) = \emptyset$. The nth ball step starts by selecting a good ball B_j from the list J_{n-1} of remaining good balls. We then run Wilson's algorithm with paths starting in B'_j. The ball step will end either with success, in which case the whole algorithm terminates, or with one of three kinds of failure. In the event of failure the ball B_j, and possibly a number of other balls also, will be labelled 'bad', and J_n is defined to be the remaining set of good balls. If more than $k^{1/2}/4$ balls are labelled bad at any one ball step, we terminate the whole algorithm with a 'Type 3 failure'. Otherwise, we proceed until, if we have tried $k^{1/2}$ balls steps without a success, we terminate the algorithm with a 'Type 4 failure'.

We write $\mathcal{U}_n$ for the tree obtained after n ball steps. After ball step n, any ball B_j in J_n will have the property that $B'_j \cap \mathcal{U}_n = B'_j \cap \mathcal{U}_0$.

We now describe in detail the second level of the algorithm, which works with a fixed (initially good) ball B_j. We assume that this is the nth ball step (where $n \geq 1$), so that we have already built the tree $\mathcal{U}_{n-1}$. Let $D' \subset B(0,\theta^2 r/k)$ satisfy

$$|D'| \leq c\delta_0^{-2}, \qquad B(0,\theta^2 r/k) \subset \bigcup_{x\in D'} B(x,\delta_0\theta^2 r/k).$$

Let $D_j = z_j + D'$, so that $D_j \subset B''_j$.

We now proceed to use Wilson's algorithm to build the paths $\gamma(w,\mathcal{U}_{n-1})$ for $w \in D_j$. For $w \in D_j$ let S^w be a random walk started at w. For each $w \in D_j$ let G_w be the event that $\gamma(w,\mathcal{U}_{n-1}) \subset B'_j$. If F_w is the event that S^w exits from B'_j before it hits $\mathcal{U}_0$, then

$$\mathbb{P}(G_w^c) \leq \mathbb{P}(F_w) \leq c\theta^{1/2}. \tag{3.22}$$

Here the first inequality follows from Wilson's algorithm, while the second is by the discrete Beurling estimates ([LL, Proposition 6.8.1]).

Let $M_w = d(w, \mathcal{U}_{n-1})$, and T_w be the first time S^w hits $\mathcal{U}_{n-1}$. Then by Wilson's algorithm and (2.3),

$$\mathbb{P}(M_w \geq \theta^{-1} G(\theta r/k); G_w) = \mathbb{P}(M_w \geq \theta^{-1} G(\theta r/k); \mathrm{L}(S^w[0, T_w]) \subset B_j') \leq c e^{-c\theta^{-1}}. \quad (3.23)$$

We now define sets corresponding to three possible outcomes to this procedure:

$$H_{1,n} = \bigcup_{w \in D_j} G_w^c,$$

$$H_{2,n} = \left\{ \max_{w \in D_j} M_w \geq \theta^{-1} G(\theta r/k) \right\} \cap \bigcap_{w \in D_j} G_w,$$

$$H_{3,n} = \left\{ \max_{w \in D_j} M_w < \theta^{-1} G(\theta r/k) \right\} \cap \bigcap_{w \in D_j} G_w.$$

By (3.22),

$$\mathbb{P}(H_{1,n}) \leq \sum_{w \in D_j} \mathbb{P}(G_w) \leq c \delta_0^{-2} \theta^{1/2}, \quad (3.24)$$

and by (3.23),

$$\mathbb{P}(H_{2,n}) \leq \sum_{w \in D_j} \mathbb{P}(M_w \geq \theta^{-1} G(\theta r/k); G_w) \leq c \delta_0^{-2} e^{-c\theta^{-1}}. \quad (3.25)$$

We now choose the constant θ small enough so that each of $\mathbb{P}(H_{i,n}) \leq \frac{1}{4}$ for $i = 1, 2$, and therefore

$$\mathbb{P}(H_{3,n}) \geq \tfrac{1}{2}. \quad (3.26)$$

If $H_{3,n}$ occurs then we have constructed a tree $\mathcal{U}_n'$ which contains $\mathcal{U}_{n-1}$ and D_j. Further, we have that for each point $w \in D_j$, the path $\gamma(w, 0)$ hits $\mathcal{U}_0$ before it leaves B_j'. Hence,

$$d(w, 0) \leq M_w + \max_{z \in \mathcal{U}_0 \cap B_j} d(0, z) \leq \tfrac{1}{2} k^{1/2} r^{5/4} + \theta^{-1} G(\theta r/k).$$

We now use Wilson's algorithm to fill in the remainder of B_j'. Let G_n be the event given by applying Proposition 3.2 to the ball B_j'' with $U_0 = \mathcal{U}_n'$. Then

$$\mathbb{P}(G_n^c) \leq c e^{-c \delta_0^{-1/3}} \leq \tfrac{1}{4}$$

by the choice of δ_0, and therefore $\mathbb{P}(H_{3,n} \cap G_n) \geq \frac{1}{4}$. If this event occurs, then all points in $B(z_j, \theta^2 r/2k)$ are within distance $G(\delta_0^{1/2} \theta^2 r/k)$ of $\mathcal{U}_n'$ in the graph metric d; in this case we label ball step n as successful, and we terminate the whole algorithm. Then for all $z \in B(z_j, \theta^2 r/2k)$,

$$\begin{aligned} d(0, z) &\leq d(z, \mathcal{U}_n') + \max_{w \in \mathcal{U}_n'} d(w, 0) \\ &\leq G(\delta_0^{1/2} \theta^2 r/k) + \tfrac{1}{2} k^{1/2} r^{5/4} + \theta^{-1} G(\theta r/k) \\ &\leq k^{1/2} r^{5/4}, \end{aligned}$$

provided that k is large enough. So there exists $k_0 \geq 1$ such that, provided that $k \geq k_0$, if $H_{3,n} \cap G_n$ occurs then $B(z_j, \theta^2 r/2k) \subset B_d(0, k^{1/2} r^{5/4})$. Since $R = k^{1/2} r^{5/4} \leq G(k^{1/2} r)$ we have $g(R) \leq k^{1/2} r$, and therefore

$$|B_d(0,R)| \geq |B(z_j, \theta^2 r/2k)| \geq ck^{-2} r^2 \geq cg(R)^2/k^3. \tag{3.27}$$

If $H_{1,n} \cup H_{2,n} \cup (H_{3,n} \cap G_n^c)$ occurs then as soon as we have a random walk S^w that 'misbehaves' (either by leaving B_j' before hitting $\mathcal{U}_0$, or by having M_w too large), then we terminate the ball step and mark the ball B_j as 'bad'. If $\omega \in H_{2,n}$ only the ball B_j becomes bad, but if $\omega \in H_{1,n} \cup (H_{3,n} \cap G_n^c)$ then S^w may hit several other balls B_i' before it hits $\mathcal{U}_{n-1}$. Let N_w^B denote the number of such balls hit by S^w. By Beurling's estimate, the probability that S^w enters a ball B_i' and then exits B_i without hitting $\mathcal{U}_0$ is less than $c\theta^{1/2}$. Since the balls B_i are disjoint,

$$\mathbb{P}(N_w^B \geq m) \leq (c\theta^{1/2})^m \leq e^{-c'm}. \tag{3.28}$$

A Type 3 failure occurs if $N_w^B \geq k^{1/2}/4$; using (3.28) we see that the probability that a ball step ends with a Type 3 failure is bounded by $\exp(-ck^{1/2})$. If we write F_3 for the event that some ball step ends with a Type 3 failure, then since there are at most $k^{1/2}$ ball steps,

$$\mathbb{P}(F_3) \leq k^{1/2} \exp(-ck^{1/2}) \leq C \exp(-c'k^{1/2}). \tag{3.29}$$

The final possibility is that $k^{1/2}$ ball steps all end in failure; write F_4 for this event. Since each ball step has a probability at least 1/4 of success (conditional on the previous steps of the algorithm), we have

$$\mathbb{P}(F_4) \leq (3/4)^{k^{1/2}} \leq e^{-ck^{1/2}}. \tag{3.30}$$

Thus either the algorithm is successful, or it ends with one of four types of failure, corresponding to the events F_i, $i = 1, \dots 4$. By Lemma 3.7 and (3.21), (3.29), (3.30) we have $\mathbb{P}(F_i) \leq C \exp(-ck^{1/3})$ for each i. Therefore, we have that provided $k \geq k_0$, (3.27) holds except on an event of probability $C \exp(-ck^{1/3})$. Taking $k = c\lambda^{1/3}$ for a suitable constant c, and adjusting the constant C so that (3.17) holds for all λ completes the proof. $\square$

The reason why we can only get a polynomial bound in the Theorem 3.3 is that one cannot get exponential estimates for the probability that $\gamma(0,w)$ leaves $B(0, k\,|w|)$ (see Lemma 2.6). However, if we let U_r be the connected component of 0 in $\mathcal{U} \cap B(0,r)$, then the following proposition enables us to get exponential control on the length of $\gamma(0,w)$ for $w \in U_r$. This will allow us to obtain an exponential bound on the lower tail of $R_{\text{eff}}(0, B_d(0,R)^c)$ in Proposition 3.6.

Proposition 3.5 *There exist positive constants c and C such that for all $\lambda \geq 1$ and $r \geq 1$,*

$$\mathbb{P}\left(U_r \not\subset B_d(0, \lambda r^{5/4})\right) \leq Ce^{-c\lambda}. \tag{3.31}$$

Proof. This proof is similar to that of Theorem 3.3. Let $E \subset B(0,2r)$ be such that $|E| \leq C\lambda^6$ and

$$B(0,2r) \subset \bigcup_{z \in E} B(z, \lambda^{-3} r),$$

and let $\mathcal{U}_0$ be the random tree obtained by applying Wilson's algorithm with points in E and root 0. For each $z \in E$, let Y_z be defined as in Proposition 2.3, so that $Y_z = z$ if $\gamma(0,z) \subset B(0,2r)$, and otherwise Y_z is the first point on $\gamma(0,z)$ which is outside $B(0,2r)$. Let

$$G_1 = \{d(Y_z, 0) \le \tfrac{1}{2}\lambda r^{5/4} \text{ for all } z \in E\}.$$

Then by Proposition 2.3,

$$\mathbb{P}(G_1^c) \le \sum_{z\in E} \mathbb{P}(d(Y_z,0) > \tfrac{1}{2}\lambda G(2r)) \le |E|\, Ce^{-c\lambda} \le C\lambda^6 e^{-c\lambda}. \tag{3.32}$$

We now complete the construction of $\mathcal{U}$ by using Wilson's algorithm. Then Proposition 3.2 with $\delta_0 = \lambda^{-3}$ implies that there exists an event G_2 with

$$\mathbb{P}\left(G_2^c\right) \le e^{-c\delta_0^{-1/3}} = e^{-c\lambda}, \tag{3.33}$$

and on G_2,

$$\max_{x\in B(0,r)} d(x, \mathcal{U}_0) \le G(\lambda^{-3/2} r).$$

Suppose $G_1 \cap G_2$ occurs, and let $x \in U_r$. Write Z_x for the point where $\gamma(x,0)$ meets $\mathcal{U}_0$. Since $x \in U_r$, we must have $Z_x \in B(0,r)$, and $\gamma(Z_x, 0) \subset B(0,r)$. As $Z_x \in \mathcal{U}_0$, there exists $z \in E$ such that $Z_x \in \gamma(0,z)$. Since G_1 occurs, $d(0, Z_x) \le d(0, Y_z) \le \frac{1}{2}\lambda r^{5/4}$, while since G_2 occurs $d(x, Z_x) \le G(\lambda^{-3/2} r)$. So, provided λ is large enough,

$$d(0,x) \le d(0,Z_x) + d(Z_x, x) \le \tfrac{1}{2}\lambda r^{5/4} + G(\lambda^{-3/2} r) \le \lambda r^{5/4}.$$

Using (3.32) and (3.33), and adjusting the constant C to handle the case of small λ completes the proof. □

Proposition 3.6 *There exist positive constants c and C such that for all $R \ge 1$ and $\lambda \ge 1$,*
(a)

$$\mathbb{P}(R_{\text{eff}}(0, B_d(0,R)^c) < \lambda^{-1} R) \le Ce^{-c\lambda^{2/11}}; \tag{3.34}$$

(b)

$$\mathbb{E}(R_{\text{eff}}(0, B_d(0,R)^c) |B_d(0,R)|) \le CR^{13/5}. \tag{3.35}$$

Proof. (a) Recall the definition of U_r given before Proposition 3.5, and note that for all $r \ge 1$, $R_{\text{eff}}(0, B(0,r)^c) = R_{\text{eff}}(0, U_r^c)$. Given R and λ, let r be such that $R = \lambda^{2/11} r^{5/4}$. By monotonicity of resistance we have that if $U_r \subset B_d(0,R)$, then

$$R_{\text{eff}}(0, B_d(0,R)^c) \ge R_{\text{eff}}(0, U_r^c).$$

So, writing $B_d = B_d(0,R)$,

$$\begin{aligned}\mathbb{P}(R_{\text{eff}}(0,B_d^c) < \lambda^{-1}R) &= \mathbb{P}(R_{\text{eff}}(0,B_d^c) < \lambda^{-1}R; U_r \not\subset B_d) + \mathbb{P}(R_{\text{eff}}(0,B_d^c) < \lambda^{-1}R; U_r \subset B_d) \\ &\le \mathbb{P}(U_r \not\subset B_d(0,\lambda^{2/11} r^{5/4})) + \mathbb{P}(R_{\text{eff}}(0,U_r^c) < \lambda^{-9/11} r^{5/4}).\end{aligned}$$

By Proposition 3.5,

$$\mathbb{P}(U_r \not\subset B_d(0, \lambda^{2/11} r^{5/4})) \leq Ce^{-c\lambda^{2/11}},$$

while by (3.4),

$$\mathbb{P}(R_{\text{eff}}(0, U_r^c) < \lambda^{-9/11} r^{5/4}) \leq Ce^{-c\lambda^{2/11}}.$$

This proves (a).

(b) Since $R_{\text{eff}}(0, B_d(0,R)^c) \leq R$, this is immediate from Theorem 1.2. □

We conclude this section by proving the following technical lemma that was used in the proof of Theorem 3.4.

Lemma 3.7 *Let F_1 be the event defined by* (3.18). *Then*

$$\mathbb{P}(F_1) \leq Ce^{-ck^{1/3}}. \tag{3.36}$$

Proof. Let $b = e^{k^{1/3}}$. Then by Lemma 2.4

$$\mathbb{P}\left(\widehat{S}[\widehat{\sigma}_{br}, \infty) \cap B_r \neq \emptyset\right) \leq Cb^{-1} \leq Ce^{-k^{1/3}}. \tag{3.37}$$

If $\widehat{S}[\widehat{\sigma}_{2r}, \infty)$ hits more than $k/2$ balls (from the family $B_1, \ldots, B_k$) then either $\widehat{S}$ hits B_r after time $\widehat{\sigma}_{br}$, or $\widehat{S}[\widehat{\sigma}_{2r}, \widehat{\sigma}_{br}]$ hits more than $k/2$ balls. Given (3.37), it is therefore sufficient to prove that

$$\mathbb{P}(\widehat{S}[\widehat{\sigma}_{2r}, \widehat{\sigma}_{br}] \text{ hits more than } k/2 \text{ balls}) \leq Ce^{-ck^{1/3}}. \tag{3.38}$$

Let S be a simple random walk started at 0, and let $L' = \mathrm{L}(S[0, \sigma_{4br}])$. Then by [Mas09, Corollary 4.5], in order to prove (3.38), it is sufficient to prove that

$$\mathbb{P}(L' \text{ hits more than } k/2 \text{ balls}) \leq Ce^{-ck^{1/3}}. \tag{3.39}$$

Define stopping times for S by letting $T_0 = \sigma_{2r}$ and for $j \geq 1$,

$$R_j = \min\{n \geq T_{j-1} : S_n \in B(0,r)\},$$
$$T_j = \min\{n \geq R_j : S_n \notin B(0,2r)\}.$$

Note that the balls B_j can only be hit by S in the intervals $[R_j, T_j]$ for $j \geq 1$. Let $M = \min\{j : R_j \geq \sigma_{4br}\}$. Then

$$\mathbb{P}(M = j+1 | M > j) = \frac{\log(2r) - \log(r)}{\log(4br) - \log r} = \frac{\log 2}{\log(4b)} \geq ck^{-1/3}.$$

Hence

$$\mathbb{P}(M \geq k^{2/3}) \leq C \exp(-ck^{1/3}).$$

For each $j \geq 1$ let $L_j = \mathrm{L}(S[0, T_j])$, let α_j be the first exit by L_j from $B(0,2r)$, and β_j be the number of steps of L_j.

If L' hits more than $k/2$ balls then there must exist some $j \leq M$ such that $L_j[\alpha_j, \beta_j]$ hits more than $k/2$ balls B_i. (We remark that since the balls B_i are defined in terms of the loop erased walk path, they will depend on $L_j[0, \alpha_j]$. However, they will be fixed in each of the intervals $[R_j, T_j]$.) Hence, if $M \leq k^{2/3}$ and L' hits more than $k/2$ balls then S must hit more than $ck^{1/3}$ balls in one of the intervals $[R_j, T_j]$, without hitting the path $L_j[0, \alpha_j]$. However, by Beurling's estimate the probability of this event is less than $C \exp(-ck^{1/3})$. Combining these estimates concludes the proof. □

4 Random walk estimates

We recall the notation of random walks on the UST given in the introduction. We write $\overline{\omega}$ for elements of $\mathcal{D}$. Finally, we recall the definitions of the stopping times τ_R and $\widetilde{\tau}_r$ from (1.9) and (1.10) and the transition densities $p_n^\omega(x,y)$ from (1.8). To avoid difficulties due to $\mathcal{U}$ being bipartite, we also define

$$\widetilde{p}_n^\omega(x,y) = p_n^\omega(x,y) + p_{n+1}^\omega(x,y). \tag{4.1}$$

Throughout this section, we will write $C(\lambda)$ to denote expressions of the form $C\lambda^p$ and $c(\lambda)$ to denote expressions of the form $c\lambda^{-p}$, where c, C and p are positive constants. To handle the indices it will be helpful to set $\kappa = 5/4$, and

$$d_f = \frac{2}{\kappa} = \frac{8}{5}, \quad d_w = 1 + d_f = \frac{2+\kappa}{\kappa} = \frac{13}{5}.$$

As in [BJKS08, KM08] we define a (random) set $J(\lambda)$:

Definition 4.1 Let $\mathcal{U}$ be the UST. For $\lambda \geq 1$ and $x \in \mathbb{Z}^2$, let $J(x,\lambda)$ be the set of those $R \in [1, \infty]$ such that the following all hold:
(1) $|B_d(x,R)| \leq \lambda R^{8/5}$,
(2) $\lambda^{-1} R^{8/5} \leq |B_d(x,R)|$,
(3) $R_{\text{eff}}(x, B_d(x,R)^c) \geq \lambda^{-1} R$.

Proposition 4.2 *For $R \geq 1$, $\lambda \geq 1$ and $x \in \mathbb{Z}^2$,*
(a)

$$\mathbb{P}(R \in J(x,\lambda)) \geq 1 - Ce^{-c\lambda^{1/9}}; \tag{4.2}$$

(b)

$$\mathbb{E}(R_{\text{eff}}(0, B_d(0,R)^c)|B_d(0,R)|) \leq CR^{d_w}.$$

Therefore conditions (1), (2) and (4) of [KM08, Assumption 1.2] hold with $v(R) = R^{d_f}$ and $r(R) = R$.

Proof. (a) is immediate from Theorem 1.2 and Proposition 3.6(a), while (b) is exactly Proposition 3.6(b). We note that since $r(R) = R$, the condition $R_{\text{eff}}(x,y) \leq \lambda r(d(x,y))$ in

[KM08, Definition 1.1] always holds for $\lambda \geq 1$, so that our definition of $J(\lambda)$ agrees with that in [KM08]. $\square$

We will see that the time taken by the random walk X to move a distance R is of order $Rg(R)^2 = R^{1/d_w}$. We therefore define

$$F(R) = Rg(R)^2 = R^{d_w}. \tag{4.3}$$

and let $f(r) = r^{1/d_w}$ be the inverse of F. We will prove that the heat kernel $\widetilde{p}_T(x,y)$ is of order $g(f(T))^{-2}$ and so we let

$$k(t) = g(f(t))^2 = t^{d_f/d_w}, \quad t \geq 1. \tag{4.4}$$

We have

$$G(R) = R^{\kappa}, \quad g(R) = R^{1/\kappa}, \quad F(R) = R^{d_w}, \tag{4.5}$$

$$f(R) = R^{1/d_w}, \quad k(R) = R^{d_f/d_w}, \quad R^2 G(R) = R^{\kappa d_w}. \tag{4.6}$$

We now state our results for the SRW X on $\mathcal{U}$, giving the asymptotic behaviour of $d(0, X_n)$, the transition densities $\widetilde{p}^{\omega}_n(x,y)$, and the exit times τ_R and $\widetilde{\tau}_r$. We begin with three theorems which follow directly from Proposition 4.2 and [KM08]. The first theorem gives tightness for some of these quantities, the second theorem gives expectations with respect to $\mathbb{P}$, and the third theorem gives 'quenched' limits which hold $\mathbb{P}$-a.s. In various ways these results make precise the intuition that the time taken by X to escape from a ball of radius R is of order R^{d_w}, that X moves a distance of order n^{1/d_w} in time n, and that the probability of X returning to its initial point after $2n$ steps is the same order as $1/|B(0, n^{1/d_w})|$, that is n^{-d_f/d_w}.

We define the averaged measure P^* on $\Omega \times \mathcal{D}$ by setting $P^*(A \times B) = \mathbb{E}[\mathbf{1}_A P^0_\omega(B)]$ and extending this to a probability measure.

Theorem 4.3 *Uniformly with respect to $n \geq 1$, $R \geq 1$ and $r \geq 1$,*

$$\mathbb{P}\Big(\theta^{-1} \leq \frac{E^0_\omega \tau_R}{R^{d_w}} \leq \theta\Big) \to 1 \quad \text{as } \theta \to \infty, \tag{4.7}$$

$$\mathbb{P}\Big(\theta^{-1} \leq \frac{E^0_\omega \widetilde{\tau}_r}{r^{\kappa d_w}} \leq \theta\Big) \to 1 \quad \text{as } \theta \to \infty, \tag{4.8}$$

$$\mathbb{P}(\theta^{-1} \leq n^{-d_f/d_w} p^{\omega}_{2n}(0,0) \leq \theta) \to 1 \quad \text{as } \theta \to \infty, \tag{4.9}$$

$$P^*\Big(\theta^{-1} < \frac{1 + d(0, X_n)}{n^{1/d_w}} < \theta\Big) \to 1 \quad \text{as } \theta \to \infty. \tag{4.10}$$

Theorem 4.4 *There exist positive constants c and C such that for all $n \geq 1$, $R \geq 1$, $r \geq 1$,*

$$cR^{d_w} \leq \mathbb{E}(E^0_\omega \tau_R) \leq CR^{d_w}, \tag{4.11}$$

$$cr^{\kappa d_w} \leq \mathbb{E}(E^0_\omega \widetilde{\tau}_r) \leq Cr^{\kappa d_w}, \tag{4.12}$$

$$cn^{-d_f/d_w} \leq \mathbb{E}(p^{\omega}_{2n}(0,0)) \leq Cn^{-d_f/d_w}, \tag{4.13}$$

$$cn^{1/d_w} \leq \mathbb{E}(E^0_\omega d(0, X_n)). \tag{4.14}$$

Theorem 4.5 *There exist $\alpha_i < \infty$, and a subset Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that the following statements hold.*
(a) For each $\omega \in \Omega_0$ and $x \in \mathbb{Z}^2$ there exists $N_x(\omega) < \infty$ such that

$$(\log\log n)^{-\alpha_1} n^{-d_f/d_w} \le p^\omega_{2n}(x,x) \le (\log\log n)^{\alpha_1} n^{-d_f/d_w}, \quad n \ge N_x(\omega). \tag{4.15}$$

In particular, $d_s(\mathcal{U}) = 16/13$, $\mathbb{P}$-a.s.
(b) For each $\omega \in \Omega_0$ and $x \in \mathbb{Z}^2$ there exists $R_x(\omega) < \infty$ such that

$$(\log\log R)^{-\alpha_2} R^{d_w} \le E^x_\omega \tau_R \le (\log\log R)^{\alpha_2} R^{d_w}, \quad R \ge R_x(\omega), \tag{4.16}$$

$$(\log\log r)^{-\alpha_3} r^{\kappa d_w} \le E^x_\omega \widetilde{\tau}_r \le (\log\log r)^{\alpha_3} r^{\kappa d_w}, \quad r \ge R_x(\omega). \tag{4.17}$$

Hence

$$d_w(\mathcal{U}) = \lim_{R\to\infty} \frac{\log E^x_\omega \tau_R}{\log R} = \frac{13}{5}, \qquad \lim_{r\to\infty} \frac{\log E^x_\omega \widetilde{\tau}_r}{\log r} = \frac{13}{4}. \tag{4.18}$$

(c) Let $Y_n = \max_{0\le k\le n} d(0, X_k)$. For each $\omega \in \Omega_0$ and $x \in \mathbb{Z}^2$ there exist $\overline{N}_x(\overline{\omega})$, $\overline{R}_x(\overline{\omega})$ such that $P^x_\omega(\overline{N}_x < \infty) = P^x_\omega(\overline{R}_x < \infty) = 1$, and such that

$$(\log\log n)^{-\alpha_4} n^{1/d_w} \le Y_n(\overline{\omega}) \le (\log\log n)^{\alpha_4} n^{1/d_w}, \quad n \ge \overline{N}_x(\overline{\omega}), \tag{4.19}$$

$$(\log\log R)^{-\alpha_4} R^{d_w} \le \tau_R(\overline{\omega}) \le (\log\log R)^{\alpha_4} R^{d_w}, \qquad R \ge \overline{R}_x(\overline{\omega}), \tag{4.20}$$

$$(\log\log r)^{-\alpha_4} r^{\kappa d_w} \le \widetilde{\tau}_r(\overline{\omega}) \le (\log\log r)^{\alpha_4} r^{\kappa d_w}, \quad r \ge R_x(\overline{\omega}). \tag{4.21}$$

(d) Let $W_n = \{X_0, X_1, \dots, X_n\}$ and let $|W_n|$ denote its cardinality. For each $\omega \in \Omega_0$ and $x \in \mathbb{Z}^2$,

$$\lim_{n\to\infty} \frac{\log |W_n|}{\log n} = \frac{8}{13}, \quad P^x_\omega\text{-a.s.}. \tag{4.22}$$

The papers [BJKS08, KM08] studied random graphs for which information on ball volumes and resistances were only available from one point. These conditions were not strong enough to bound $E^0_\omega d(0, X_n)$ or $\widetilde{p}^\omega_T(x,y)$ – see [BJKS08, Example 2.6]. Since the UST is stationary, we have the same estimates available from every point x, and this means that stronger conclusions are possible.

Theorem 4.6 *There exist $N_0(\omega)$ with $\mathbb{P}(N_0 < \infty) = 1$, $\alpha > 0$ and for all $q > 0$, C_q such that*

$$E^0_\omega d(0, X_n)^q \le C_q n^{q/d_w} (\log n)^{\alpha q} \quad \text{for } n \ge N_0(\omega). \tag{4.23}$$

Further, for all $n \ge 1$,

$$\mathbb{E}(E^0_\omega d(0, X_n)^q) \le C_q n^{q/d_w} (\log n)^{\alpha q}. \tag{4.24}$$

Write $\Phi(T, x, x) = 0$, and for $x \ne y$ let

$$\Phi(T, x, y) = \frac{d(x,y)}{G((T/d(x,y))^{1/2})} = \Big(\frac{d(x,y)^{d_w}}{T}\Big)^{1/(d_w-1)}. \tag{4.25}$$

Theorem 4.7 *There exists a constant* $\alpha > 0$ *and r.v.* $N_x(\omega)$ *with*

$$\mathbb{P}(N_x \geq n) \leq Ce^{-c(\log n)^2} \tag{4.26}$$

such that provided $T^{d_w} \vee |x-y| \geq N_x(\omega)$ *and* $T \geq d(x,y)$*, then writing* $A = A(x,y,T) = C(\log(|x-y| \vee T^{d_w}))^\alpha$*,*

$$A^{-1}T^{-d_f/d_w} \exp\Big(- A\Phi(T,x,y)\Big) \leq \widetilde{p}_T(x,y) \leq AT^{-d_f/d_w} \exp\Big(- A^{-1}\Phi(T,x,y)\Big). \tag{4.27}$$

Remark 4.8 Except for the logarithmic term A, the bounds in (4.27) are of the same form as those obtained in the diffusions on fractals literature.

Before we prove Theorems 4.3 – 4.7, we summarize some properties of the exit times τ_R.

Proposition 4.9 *Let* $\lambda \geq 1$ *and* $x \in \mathbb{Z}^2$*.*
(a) If $R, R/(4\lambda) \in J(x,\lambda)$ *then*

$$c_1(\lambda)R^{d_w} \leq E^x_\omega \tau(x,R) \leq C_2(\lambda)R^{d_w}. \tag{4.28}$$

(b) Let $0 < \varepsilon \leq c_3(\lambda)$*. Suppose that* $R, \varepsilon R, c_4(\lambda)\varepsilon R \in J(x,\lambda)$*. Then*

$$P^x_\omega(\tau(x,R) < c_5\lambda\varepsilon^{d_w}R^{d_w}) \leq C_6(\lambda)\varepsilon. \tag{4.29}$$

Proof. This follows directly from [BJKS08, Proposition 2.1] and [KM08, Proposition 3.2, 3.5]. □

Proof of Theorems 4.3, 4.4, and 4.5. All these statements, except those relating to $\widetilde{\tau}_r$, follow immediately from Proposition 4.2 and Propositions 1.3 and 1.4 and Theorem 1.5 of [KM08]. Thus it remains to prove (4.8), (4.12), (4.17) and (4.21). By the stationarity of $\mathcal{U}$ it is enough to consider the case $x = 0$.

Recall that U_r denotes the connected component of 0 in $\mathcal{U} \cap B(0,r)$, and therefore

$$\widetilde{\tau}_r = \min\{n \geq 0 : X_n \notin U_r\}.$$

Let

$$H_1(r,\lambda) = \{B_d(0,\lambda^{-1}r^\kappa) \subset U_r \subset B_d(0,\lambda r^\kappa\}.$$

On $H_1(r,\lambda)$ we have

$$\tau_{\lambda^{-1}r^\kappa} \leq \widetilde{\tau}_r \leq \tau_{\lambda r^\kappa}, \tag{4.30}$$

while by Theorem 3.1 and Proposition 3.5 we have for $r \geq 1$, $\lambda \geq 1$,

$$\mathbb{P}(H_1(r,\lambda)^c) \leq e^{-c\lambda^{2/3}}.$$

The upper bound in (4.8) will follow from (4.12). For the lower bound, on $H_1(r,\lambda)$ we have, writing $R = \lambda^{-1}r^\kappa$,

$$\frac{E^0_\omega \widetilde{\tau}_r}{r^{\kappa d_w}} \geq \frac{E^0_\omega \tau_R}{(\lambda R)^{d_w}} \geq \lambda^{-3}\frac{E^0_\omega \tau_R}{R^{d_w}}. \tag{4.31}$$

So

$$\mathbb{P}\Big(\frac{E^0_\omega \widetilde{\tau}_r}{r^{\kappa d_w}} < \lambda^{-4}\Big) \le \mathbb{P}(H_1(r,\lambda)^c) + \mathbb{P}\Big(\frac{E^0_\omega \tau_R}{R^{d_w}} < \lambda^{-1}\Big), \tag{4.32}$$

and the bound on the lower tail in (4.8) follows from (4.7).

We now prove the remaining statements in Theorem 4.5. Let $r_k = e^k$, and $\lambda_k = a(\log k)^{3/2}$, and choose a large enough so that

$$\sum_k \exp(-c\lambda_k^{2/3}) < \infty.$$

Hence by Borel-Cantelli there exists a r.v. $K(\omega)$ with $\mathbb{P}(K < \infty) = 1$ such that $H_1(r_k, \lambda_k)$ holds for all $k \ge K$. So if k is sufficiently large, and α_2 is as in (4.16),

$$E^0_\omega \widetilde{\tau}_{r_k} \le E^0_\omega \tau_{\lambda_k r_k^\kappa} \le [\log\log(\lambda_k r_k^\kappa)^{\alpha_2} (\lambda_k r_k^\kappa)^{d_w} \le C(\log k)^{\alpha_3} r_k^{\kappa d_w}.$$

Since $\widetilde{\tau}_r$ is monotone in r, the upper bound in (4.17) follows. A very similar argument gives the lower bound, and also (4.21).

It remains to prove (4.12). A general result on random walks (see e.g. [BJKS08], (2.21)) implies that

$$E^0_\omega \widetilde{\tau}_r \le R_{\text{eff}}(0, U_r^c) \sum_{x \in U_r} \mu_x \le C r^2 R_{\text{eff}}(0, U_r^c).$$

Let z be the first point on the path $\gamma(0,\infty)$ outside $B(0,r)$. Then $R_{\text{eff}}(0, U_r^c) \le d(0,z)$, and since $\gamma(0,\infty)$ has the law of an infinite LERW, $\mathbb{E}d(0,z) \le \mathbb{E}\widehat{M}_{r+1} \le Cr^\kappa$. Hence

$$\mathbb{E}(E^0_\omega \widetilde{\tau}_r) \le C r^{2+\kappa} = C r^{\kappa d_w}.$$

For the lower bound, let

$$H_2(r,\lambda) = \{\lambda^{-1} r^\kappa, (2\lambda)^{-2} r^\kappa \in J(\lambda)\}.$$

Choose λ_0 large enough so that $\mathbb{P}(H_1(\lambda_0, r) \cap H_2(\lambda_0, r)) \ge \frac{1}{2}$. If $H_2(r, \lambda_0)$ holds then by Proposition 4.9, writing $R = \lambda_0^{-1} r^\kappa$,

$$E^0_\omega \tau_R \ge c(\lambda_0) R^{d_w}.$$

So, since $R^{d_w} = c(\lambda_0) r^{\kappa d_w}$,

$$\begin{aligned}
\mathbb{E}E^0_\omega \widetilde{\tau}_r &\ge \mathbb{E}(E^0_\omega \widetilde{\tau}_r; H_1(\lambda_0, r) \cap H_2(\lambda_0, r)) \\
&\ge \mathbb{E}(E^0_\omega \tau_R; H_1(\lambda_0, r) \cap H_2(\lambda_0, r)) \ge \tfrac{1}{2} c(\lambda_0) R^{d_w} \ge c(\lambda_0) r^{\kappa d_w}.
\end{aligned}$$

□

We now turn to the proofs of Theorems 4.6 and 4.7, and begin with a slight simplification of Lemma 1.1 of [BB89].

Lemma 4.10 *There exists $c_0 > 0$ such that the following holds. Suppose we have nonnegative r.v. ξ_i which satisfy, for some $t_0 > 0$,*

$$\mathbb{P}(\xi_i \le t_0 | \xi_1, \dots, \xi_{i-1}) \le \tfrac{1}{2}.$$

Then

$$\mathbb{P}(\sum_{i=1}^n \xi_i < T) \le \exp(-c_0 n + T/t_0). \tag{4.33}$$

Proof. Write $\mathcal{F}_i = \sigma(\xi_1, \dots \xi_i)$. Let $\theta = 1/t_0$, and let $e^{-c_0} = \frac{1}{2}(1 + e^{-1})$. Then

$$\begin{aligned}\mathbb{E}(e^{-\theta \xi_i} | \mathcal{F}_{i-1}) &\le \mathbb{P}(\xi_i < t_0 | \mathcal{F}_{i-1}) + e^{-\theta t_0} \mathbb{P}(\xi_i \ge t_0 | \mathcal{F}_{i-1}) \\ &= \mathbb{P}(\xi_i < t_0 | \mathcal{F}_{i-1})(1 - e^{-\theta t_0}) + e^{-\theta t_0} \\ &\le \tfrac{1}{2}(1 + e^{-\theta t_0}) = e^{-c_0}.\end{aligned}$$

Then

$$\mathbb{P}(\sum_{i=1}^n \xi_i < T) = \mathbb{P}(e^{-\theta \sum_{i=1}^n \xi_i} > e^{-\theta T}) \le e^{\theta T} \mathbb{E}(e^{-\theta \sum_{i=1}^n \xi_i}) \le e^{\theta T} e^{-n c_0}.$$

□

We also require the following lemma which is an immediate consequence of the definitions of the functions F and G.

Lemma 4.11 *Let $R \ge 1$, $T \ge 1$, and*

$$b_0 = \frac{R}{G((T/R)^{1/2})} = \Big(\frac{R^{d_w}}{T}\Big)^{1/(d_w - 1)}. \tag{4.34}$$

Then,

$$R/b_0 = G((T/R)^{1/2}) = f(T/b_0), \tag{4.35}$$
$$b \le b_0 \Leftrightarrow T/b \le F(R/b) \Leftrightarrow f(T/b) \le R/b \tag{4.36}$$
$$F(\theta R) = \theta^{d_w} F(R), \tag{4.37}$$
$$f(\theta R) = \theta^{1/d_w} f(R). \tag{4.38}$$

For $x \in \mathbb{Z}^2$, let

$$A_x(\lambda, n) = \{\omega : R' \in J(y, \lambda) \text{ for all } y \in B(x, n^2), 1 \le R' \le n^2\}.$$

and let $A(\lambda, n) = A_0(\lambda, n)$.

Proposition 4.12 *Let $\lambda \geq 1$ and suppose that $1 \leq R \leq n$,*

$$T \geq C_9(\lambda)R, \tag{4.39}$$

and $A(\lambda, n)$ occurs. Then,

$$P_\omega^0(\tau_R < T) \leq C_{10}(\lambda)\exp\left(-c_{11}(\lambda)\Big(\frac{R^{d_w}}{T}\Big)^{1/(d_w-1)}\right). \tag{4.40}$$

Proof. In this proof, the constants $c_i(\lambda)$, $C_i(\lambda)$ for $1 \leq i \leq 8$ will be as in Proposition 4.9 and Lemma 4.11, and c_0 will be as in Lemma 4.10. We work with the probability P_ω^0, so that $X_0 = 0$.

Let $b_0 = R/G((T/R)^{1/2})$ be as in (4.34), and define the quantities

$$\varepsilon = (2C_6(\lambda))^{-1}, \qquad \theta = \tfrac{1}{4}C_8^{-1}c_0c_5(\lambda)\varepsilon^2, \qquad C^*(\lambda) = 2\theta^{-1},$$
$$m = \lfloor \theta b_0 \rfloor, \qquad R' = R/m, \qquad t_0 = c_5(\lambda)(\varepsilon R')^{d_w}$$

We now establish the key facts that we will need about the quantities defined above. We can assume that $b_0 \geq C^*(\lambda)$ for if $b_0 \leq C^*(\lambda)$, then by adjusting the constants $C_{10}(\lambda)$ and $c_{11}(\lambda)$ we will still obtain (4.40). Therefore,

$$1 \leq \tfrac{1}{2}\theta b_0 \leq m \leq \theta b_0. \tag{4.41}$$

Furthermore, since $m/\theta \leq b_0$, $\theta R/m = G((T/R)^{1/2}) \geq 1$ and $\theta/\varepsilon < 1$, we have by Lemma 4.11 that

$$T/m \leq \theta^{-1}F(\theta R/m) \leq C_8\theta\varepsilon^{-2}F(\varepsilon R/m) \leq \tfrac{1}{4}c_0c_5(\lambda)F(\varepsilon R/m) = \tfrac{1}{4}c_0t_0.$$

Therefore,

$$T/t_0 < \tfrac{1}{2}c_0m. \tag{4.42}$$

Finally, we choose

$$C_9(\lambda) \geq g(c_4(\lambda)^{-1}\varepsilon^{-1}\theta)^2,$$

so that if $T/R \geq C_9(\lambda)$, then

$$G((T/R)^{1/2}) \geq c_4(\lambda)^{-1}\varepsilon^{-1}\theta,$$

and therefore

$$c_4(\lambda)\varepsilon R' \geq c_4(\lambda)\varepsilon R\theta^{-1}b_0^{-1} \geq 1. \tag{4.43}$$

Having established (4.41), (4.42) and (4.43), the proof of the Proposition is straightforward. Let $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$. Define stopping times for X by

$$T_0 = 0,$$
$$T_k = \min\{j \geq T_{k-1} : X_j \notin B_d(X_{T_{k-1}}, R'-1)\},$$

and let $\xi_k = T_k - T_{k-1}$. Note that $T_m \le \tau_R$, and that if $k \le m$, then

$$X_{T_k} \in B_d(0, kR') \subset B_d(0, n) \subset B(0, n).$$

Therefore, since (4.43) holds and $A(\lambda, n)$ occurs, we can apply Proposition 4.9 to obtain that

$$P^0_\omega(\xi_k < c_5(\lambda) F(\varepsilon R') | \mathcal{F}_{k-1}) \le C_6(\lambda)\varepsilon = \tfrac{1}{2}.$$

Hence by Lemma 4.10 and (4.42),

$$\begin{aligned} P^0_\omega(\tau_R < T) &\le P^0_\omega(\sum_1^m \xi_i < T) \\ &\le \exp(-c_0 m + T/t_0) \\ &\le \exp(-c_0/2m) \\ &\le \exp\left(-c_{11}(\lambda) \frac{R}{G((T/R)^{1/2})}\right). \end{aligned}$$

□

Proof of Theorem 4.6 We will prove Theorem 4.6 with T replacing n. Let $R = f(T)$; we can assume that T is large enough so that $R \ge 2$. We also let $C_9(\lambda)$, $C_{10}(\lambda)$ and $c_{11}(\lambda)$ be as in Proposition 4.12, and let $p > 0$ be such that $C_i(\lambda) \le C\lambda^p$, $i = 9, 10$ and $c_{11}(\lambda) \ge c\lambda^{-p}$.

We have

$$\begin{aligned} E^0_\omega d(0, X_T)^q &\le R^q + E^0_\omega\Big(\sum_{k=1}^\infty 1_{(e^{k-1}R \le d(0,X_T) < e^k R)} d(0, X_T)^q\Big) \\ &\le R^q + R^q \sum_{k=1}^\infty e^{kq} P^0_\omega(e^{k-1}R \le d(0, X_T) \le e^k R). \end{aligned} \tag{4.44}$$

By (4.2) we have

$$\mathbb{P}(A(\lambda, n)^c) \le 4n^3 e^{-c\lambda^{1/9}} \le \exp(-c\lambda^{1/9} + C\log n). \tag{4.45}$$

Let $\lambda_k = k^{10}$. Then $\sum_k \mathbb{P}(A(\lambda_k, e^k)^c) < \infty$, and so by Borel-Cantelli there exists $K_0(\omega)$ such that $A(\lambda_k, e^k)$ holds for all $k \ge K_0$. Furthermore, we have

$$\mathbb{P}(K_0 \ge n) \le Ce^{-cn^{10/9}}. \tag{4.46}$$

Suppose now that $k \ge K_0$. To bound the sum (4.44), we consider two ranges of k. If $C_9(\lambda_k)e^{k-1}R > T$, then we let $A_k = B_d(0, e^k R) - B_d(0, e^{k-1}R)$, and by the Carne-Varopoulos bound (see [Car85]),

$$\begin{aligned} e^{kq} P^0_\omega(e^{k-1}R \le d(0, X_T) \le e^k R) &\le e^{kq} \sum_{y \in A_k} P^0_\omega(X_T = y) \\ &\le e^{kq} \sum_{y \in A_k} C \exp(-d(0, y)^2/2T) \\ &\le Ce^{kq}(e^k R)^2 \exp(-(e^{k-1}R)^2/2T) \\ &\le C \exp(-C_9(\lambda_k)^{-1} e^k R + 2\log(e^k R) + kq) \\ &\le C \exp(-ck^{-10p} e^k + C_q k). \end{aligned} \tag{4.47}$$

On the other hand, if $C_9(\lambda_k)e^{k-1}R \le T$, then we let $m = \lceil k + \log R \rceil$, so that $e^k R \le e^m < e^{k+1}R$. Then by Proposition 4.12,

$$\begin{aligned}
e^{kq}P^0_\omega(e^{k-1}R \le d(0,X_T) \le e^k R) &\le e^{kq}P^0_\omega(\tau_{e^{k-1}R} < T) \\
&\le e^{kq}C_{10}(\lambda_m)\exp\left(-c_{11}(\lambda_m)\frac{e^{k-1}R}{G((e^{-k+1}T/R)^{1/2})}\right) \\
&\le e^{kq}Cm^{10p}\exp\left(-cm^{-10p}e^k\frac{R}{G((T/R)^{1/2})}\right) \\
&\le C(k+\log R)^{10p}\exp(-c(k+\log R)^{-10p}e^k + kq). \quad (4.48)
\end{aligned}$$

Let $k_1 = 20p \log\log R$. Then if $k \ge k_1$,

$$(k+\log R)^{10p} \le (k + e^{k/(20p)})^{10p} \le Ce^{k/2}.$$

Hence for $k \ge k_1$,

$$e^{kq}P^0_\omega(e^{k-1}R \le d(0,X_T) \le e^k R) \le C\exp(-ce^{k/2} + C_q k). \quad (4.49)$$

Let $K' = K_0 \vee k_1$. Then since the series given by (4.47) and (4.49) both converge,

$$\begin{aligned}
\sum_{k=1}^{\infty} e^{kq}P^0_\omega(e^{k-1}R \le d(0,X_T) \le e^k R) &\le \sum_{k=1}^{K'-1} e^{kq} + C_q \\
&\le e^{K'q} + C_q \\
&\le e^{K_0 q} + (\log R)^{20pq} + C_q.
\end{aligned}$$

Hence since $R \le T$, we have that for all $T \ge N_0 = e^{e^{K_0}}$

$$E^0_\omega d(0,X_T)^q \le C_q R^q((\log T)^q + (\log T)^{20pq}), \quad (4.50)$$

so that (4.23) holds. Taking expectations in (4.50) and using (4.46) gives (4.24). □

Remark 4.13 It is natural to ask if (4.24) holds without the term in $\log T$, as with the averaged estimates in Theorem 4.4. It seems likely that this is the case; such an averaged estimate was proved for the incipient infinite cluster on regular trees in [BK06, Theorem 1.4(a)]. The key to obtaining such a bound is to control the exit times $\tau_{e^k R}$; this was done above using the events $A(\lambda, n)$, but this approach is far from optimal. The argument of Proposition 4.12 goes through if only a positive proportion of the points X_{T_k} are at places where the estimate (4.29) can be applied. This idea was used in [BK06] – see the definition of the event $G_2(N,R)$ on page 48.

Suppose we say that $B_d(x,R)$ is λ-bad if $R \notin J(x,\lambda)$. Then it is natural to conjecture that there exists λ_c such that for $\lambda > \lambda_c$ the bad balls fail to percolate on $\mathcal{U}$. Given such a result (and suitable control on the size of the clusters of bad balls) it seems plausible that the methods of this paper and [BK06] would then lead to a bound of the form $\mathbb{E}(E^0_\omega d(0,X_T)^q) \le C_q f(T)^q$.

We now use the arguments in [BCK05] to obtain full heat kernel bounds for $p_T(x,y)$ and thereby prove Theorem 4.7. Since the techniques are fairly standard, we only give full details for the less familiar steps.

Lemma 4.14 *Suppose $A(\lambda,n)$ holds. Let $x,y\in B(0,n)$. Then*
(a)

$$p_T(x,y)\le C_{12}(\lambda)T^{-d_f/d_w},\qquad \text{if } 1\le T\le n^{d_w}. \tag{4.51}$$

(b)

$$\widetilde{p}_T(x,y)\ge c_{13}(\lambda)T^{-d_f/d_w},\qquad \text{if } 1\le T\le n^{d_w} \text{ and } d(x,y)\le c_{14}\lambda T^{1/d_w}. \tag{4.52}$$

Proof. If $x=y$ then (a) is immediate from [KM08, Proposition 3.1]. Since $p_T(x,y)^2\le \widetilde{p}_T(x,x)\widetilde{p}_T(y,y)$, the general case then follows.
(b) The bound when $x=y$ is given by [KM08, Proposition 3.3(2)]. We also have, by [KM08, Proposition 3.1],

$$|\widetilde{p}_T(x,y)-\widetilde{p}_T(x,z)|^2\le \frac{c}{T}d(y,z)p_{2\lfloor T/2\rfloor}(x,x).$$

Therefore using (a),

$$\begin{aligned}
\widetilde{p}_T(x,y)&\ge \widetilde{p}_T(x,x)-|\widetilde{p}_T(x,x)-\widetilde{p}_T(x,y)|\\
&\ge c(\lambda)T^{-d_f/d_w}-\Big(C(\lambda)d(x,y)T^{-1-d_f/d_w}\Big)^{1/2}\\
&= c(\lambda)T^{-d_f/d_w}\Big(1-\big(C(\lambda)d(x,y)T^{-1+d_f/d_w}\big)^{1/2}\Big).
\end{aligned}$$

Since $1+d_f=d_w$, (4.52) follows. □

Recall from (4.25) the definition of $\Phi(T,x,y)$.

Proposition 4.15 *Suppose that $A(\lambda,n)$ holds. Let $x,y\in B(0,n)$. If $d(x,y)\le T\le n^{d_w}$, then*

$$c(\lambda)T^{-d_f/d_w}\exp\Big(-C(\lambda)\Phi(T,x,y)\Big)\le \widetilde{p}_T(x,y)\le C(\lambda)T^{-d_f/d_w}\exp\Big(-c(\lambda)\Phi(T,x,y)\Big). \tag{4.53}$$

Proof. Let $R=d(x,y)$. In this proof we take $c_{13}(\lambda)$ and $c_{14}(\lambda)$ to be as in (4.52).

We will choose a constant $C^*(\lambda)\ge 2$ later. Suppose first that $R\le T\le C^*(\lambda)R$. Then the upper bound in (4.53) is immediate from the Carne-Varopoulos bound. If $R+T$ is even and then we have $p_T(x,y)\ge 4^{-T}$, and this gives the lower bound.

We can therefore assume that $T\ge C^*(\lambda)R$. The upper bound follows from the bounds (4.51) and (4.40) by the same argument as in [BCK05, Proposition 3.8].

It remains to prove the lower bound in the case when $T\ge C^*(\lambda)R$, and for this we use a standard chaining technique which derives (4.53) from the 'near diagonal lower bound' (4.52). For its use in a discrete setting see for example [BCK05, Section 3.3]. As in Lemma 4.11, we set

$$b_0=\frac{R}{G((T/R)^{1/2})}. \tag{4.54}$$

If $b_0 < 1$ then we have from Lemma 4.11 that $R \le C_8 b_0^{2/3} f(T)$. If $C_8 b_0^{2/3} \le c_{14}(\lambda)$ then $R \le c_{14}(\lambda) f(T)$ and the lower bound in (4.53) follows from (4.52). We can therefore assume that $C_8 b_0^{2/3} > c_{14}(\lambda)$. We will choose $\theta > 2(c_{14}/C_8)^{-3/2}$ later; this then implies that $\theta b_0 \ge 2$. Let $m = \lfloor \theta b_0 \rfloor$; we have $\frac{1}{2}\theta b_0 \le m \le \theta b_0$. Let $r = R/m$, $t = T/m$; we will require that both r and t are greater than 4. Choose integers $t_1, \dots, t_m$ so that $|t_i - t| \le 2$ and $\sum t_i = T$. Choose a chain $x = z_0, z_1, \dots, z_m = y$ of points so that $d(z_{i-1}, z_i) \le 2r$, and let $B_i = B(z_i, r)$. If $x_i \in B_i$ for $1 \le i \le m$ then $d(x_{i-1}, x_i) \le 4r$. We choose θ so that we have

$$\widetilde{p}_{t_i}(x_{i-1}, x_i) \ge c_{13}(\lambda) k(t)^{-1} \text{ whenever } x_{i-1} \in B_{i-1},\ x_i \in B_i. \tag{4.55}$$

By (4.52) it is sufficient for this that

$$4R/m = 4r \le c_{14}(\lambda) f(t/2) = c_{14}(\lambda) f(T/2m). \tag{4.56}$$

Since $2m/\theta \ge b_0$, Lemma 4.11 implies that $f(\theta T/(2m)) \ge \theta R/(2m)$, and therefore

$$4R/m \le 8\theta^{-1} f(\theta T/(2m)) \le C\theta^{-1/3} f(T/2m), \tag{4.57}$$

and so taking $\theta = \max(2(c_{14}/C_8)^{-3/2}, (C/c_3(\lambda))^3)$ gives (4.56). The condition $T \ge C^*(\lambda) R$ implies that $f(T/b_0) = R/b_0 \ge G(C^*(\lambda))$, so taking C^* large enough ensures that both r and t are greater than 4.

The Chapman-Kolmogorov equations give

$$\widetilde{p}_T(x,y) \ge \sum_{x_1 \in B_1} \cdots \sum_{x_{m-1} \in B_{m-1}} p_{t_1}(x_0, x_1)\mu_{x_1} p_{t_2}(x_1, x_2)\mu_{x_2} \dots$$
$$p_{t_{m-1}}(x_{m-2}, x_{m-1})\mu_{x_{m-1}} \widetilde{p}_{t_m}(x_{m-1}, y). \tag{4.58}$$

Since $x_{m-1} \in B_{m-1}$ we have $\widetilde{p}_{t_m}(x_{m-1}, y) \ge c_{13}(\lambda) k(t)^{-1} \ge c_{13}(\lambda) k(T)^{-1}$. Note that exactly one of $p_t(x,y)$ and $p_{t+1}(x,y)$ can be non-zero. Using this, and (4.55) we deduce that for $1 \le i \le m-1$,

$$\sum_{x_i \in B_i} p_{t_i}(x_{i-1}, x_i)\mu_{x_i} \ge c(\lambda) k(t)^{-1} g(r)^2. \tag{4.59}$$

The choice of m implies that $c'(\lambda) f(t) \le r \le c(\lambda) f(t)$, and therefore

$$k(t)^{-1} g(r)^2 = g(r)^2 / g(f(t))^2 \ge c(\lambda).$$

So we obtain

$$\widetilde{p}_T(x,y) \ge k(T) c(\lambda)^m \ge k(T) \exp(-c(\lambda) R / G((T/R)^{1/2})). \tag{4.60}$$

□

Proof of Theorem 4.7 As in the proof of Theorem 4.6, we have that by by (4.2)

$$\mathbb{P}(A(\lambda, n)^c) \le 4n^3 e^{-c\lambda^{1/9}} \le \exp(-c\lambda^{1/9} + C \log n).$$

Therefore if we let $\lambda_n = (\log n)^{18}$, then by Borel Cantelli, for each $x \in \mathbb{Z}^2$ there exists N_x such that $A_x(\lambda_n, n)$ holds for all $n \geq N_x$. Further we have that

$$\mathbb{P}(N_x \geq n) \leq Ce^{-c(\log n)^2}.$$

Let $x, y \in \mathbb{Z}^2$ and $T \geq 1$. To apply the bound in Proposition 4.15 we need to find n such that $T \leq F(n)$, $y \in B(x, n)$ and $n \geq N_x$. Hence if $F(T) \vee |x-y| \geq N_x$ we can take $n = F(T) \vee |x-y|$, to obtain (4.53) with constants $c(\lambda_n) = c(\log n)^p$. Choosing α suitably then gives (4.27). □

Remark 4.16 Not now needed.

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