Which values of the volume growth and escape time exponent are possible for a graph?

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Abstract Let $\Gamma = (G, E)$ be an infinite weighted graph which is Ahlfors α-regular, so that there exists a constant c such that $c^{-1} r^\alpha \le V(x,r) \le c r^\alpha$, where $V(x,r)$ is the volume of the ball centre x and radius r. Define the escape time $T(x,r)$ to be the mean exit time of a simple random walk on Γ starting at x from the ball centre x and radius r. We say Γ has escape time exponent $\beta > 0$ if there exists a constant c such that $c^{-1} r^\beta \le T(x,r) \le c r^\beta$ for $r \ge 1$. Well known estimates for random walks on graphs imply that $\alpha \ge 1$ and $2 \le \beta \le 1 + \alpha$. We show that these are the only constraints, by constructing for each α_0, β_0 satisfying the inequalities above a graph $\widetilde{\Gamma}$ which is Ahlfors α_0-regular and has escape time exponent β_0. In addition we can make $\widetilde{\Gamma}$ sufficiently uniform so that it satisfies an elliptic Harnack inequality.

Keywords Graph, random walk, volume growth, anomalous diffusion

Mathematics Subject classifications 2000 Primary 60J05, Secondary 58J65, 82C41

0. Introduction.

Let $\Gamma = (G, E)$ be an infinite connected locally finite graph. We call $a = (a_{xy})$, $x, y \in G$ a *conductance matrix* if $a_{xy} \ge 0$ and $a_{xy} = a_{yx}$ for all $x, y \in G$ and in addition a is linked to the graph structure by the requirement that there exists $C_1 > 0$ such that

$$\begin{aligned} &a_{xy} = 0 \quad \text{ if } \{x,y\} \text{ is not an edge in } \Gamma, \\ &a_{xy} \ge C_1 > 0 \text{ if } \{x,y\} \in E. \end{aligned} \tag{0.1}$$

We call the pair (Γ, a) a *weighted graph*. We call the *natural weight* on Γ the weights given by taking the conductance matrix a to be the adjacency matrix of Γ; that is

$$a_{xy} = \begin{cases} 1 & \text{if } \{x,y\} \in E, \\ 0 & \text{if } \{x,y\} \notin E. \end{cases}$$

Whenever we discuss below a graph without any weights specified, we will assume we are using the natural weights. We set $\mu_x = \sum_y a_{xy}$, and extend μ to a measure on G. Let $d(x,y)$ be the usual graph distance on G, and let for $x \in G$, $r \in (0, \infty)$,

$$B(x,r) = \{y : d(x,y) < r\}, \quad V(x,r) = \mu(B(x,r)).$$

[1] Research partially supported by grants from NSERC (Canada) and CNRS (France).

We say that Γ is *Ahlfors α-regular* (here $\alpha \in (0,\infty)$) if there exists a constant $c \geq 1$ such that the volume growth function V satisfies

$$c^{-1}r^\alpha \leq V(x,r) \leq cr^\alpha, \qquad r \in [1,\infty),\, x \in G. \tag{V_α}$$

Note that, with (0.1), (V_α) implies that the vertex degree is uniformly bounded.

A random walk $X = (X_n, n \geq 0, \mathbb{P}^x, x \in G)$ on (Γ, a) is a μ-symmetric G-valued Markov chain with transition probabilities given by

$$p_{xy} = \mathbb{P}^{\cdot}(X_{n+1} = y | X_n = x) = \frac{a_{xy}}{\mu_x}, \quad x, y \in G,\ n \geq 0.$$

The heat kernel on (Γ, a) is the density of X_n with respect to the measure μ:

$$p_n(x,y) = \mathbb{P}^x(X_n = y)/\mu_y,$$

and is easily seen to be symmetric: $p_n(x,y) = p_n(y,x)$. For $A \subset G$ write

$$T_A = \min\{n \geq 0 : X_n \in A\}, \qquad T_x = T_{\{x\}},$$

and set

$$\tau_{x,r} = T_{B(x,r)^c} = \min\{n \geq 0 : d(x, X_n) \geq r\}.$$

We say that Γ satisfies (E_β) if for some constant $c \geq 1$,

$$c^{-1}r^\beta \leq \mathbb{E}^x \tau_{x,r} \leq cr^\beta, \qquad r \in [1,\infty),\, x \in G. \tag{E_β}$$

There has recently been much activity in the general area of geometry and heat kernels. While many of the questions in this field arose in the context of manifolds, they can also be posed for graphs, where the initial technical difficulties are fewer, but the same basic principles apply. The overall object is to relate geometric properties of these spaces (such as (V_α) or the weaker volume doubling property), and analytic ones, such as the space satisfying various kinds of Sobolev, Poincaré or Harnack inequalities, with the global properties of the random walk X and its transition density $p_n(x,y)$.

In particular, it has been discovered that spaces satisfying (E_β) with $\beta > 2$ provide natural families of examples of spaces which satisfy the elliptic Harnack inequality (see below), but fail to satisfy the stronger parabolic Harnack inequality. See [BB1], and [HSC] for a recent discussion.

The weighted graph (Γ, a) satisfies the *volume doubling* condition (VD) if there exists $c > 1$ such that

$$V(x, 2R) \leq cV(x,R) \text{ for all } x \in G,\ R \geq 1. \tag{VD}$$

Volume doubling, together with a Poincaré inequality, is a necessary and sufficient condition for the parabolic Harnack inequality to hold – see [G], [SC] (for manifolds) and [D1] for graphs. The condition (V_α) immediately implies (VD), but gives much more regularity in the spatial structure of Γ.

Probabilistic conditions like (E_β) have only been introduced more recently. In [GT1], [GT2] it is shown that, combined with (V_α) or (VD) and an elliptic Harnack inequality, (E_β) yields very good upper and lower bounds on $p_n(x,y)$.

In this paper we answer the following question:

If (Γ, a) satisfies (V_α) and (E_β) what values of (α, β) are possible?

The theorem below is well known to experts, and follows easily from known estimates on random walks due to Varopoulos, Carne, Kesten, Kusuoka and Telcs; for completeness we give a quick proof in Section 1.

Theorem 1. *If (Γ, a) is an infinite connected weighted graph satisfying (0.1), (V_α) and (E_β) then $\alpha \geq 1$, and*

$$2 \leq \beta \leq 1 + \alpha. \tag{0.2}$$

We now recall the definition of the elliptic Harnack inequality. (See [D1] for the parabolic Harnack inequality, which has a more complicated definition, and is not used in this paper.)

Definitions. 1. Let $A \subset G$. We write $\partial A = \{y \in A^c : d(x,y) = 1 \text{ for some } x \in A\}$ for the exterior boundary of A, and set $\overline{A} = A \cup \partial A$.
2. A function $h : \overline{A} \to \mathbb{R}$ is *harmonic on* $A \subset G$ if

$$\Delta h(x) = \frac{1}{\mu_x} \sum_y a_{xy}(h(y) - h(x)) = 0, \quad x \in A.$$

This is equivalent to the assertion that $(h(X_{n \wedge T_{A^c}}), n \geq 0)$ is a martingale.
3. (Γ, a) satisfies *an elliptic Harnack inequality* (EHI) if there exists $c_1 > 0$ such that, for any $x \in G$, $R \geq 1$, and non-negative $h : G \to \mathbb{R}$ harmonic in $B(x, 2R)$,

$$\sup_{B(x,R)} h \leq c_1 \inf_{B(x,R)} h. \tag{0.3}$$

We have taken balls $B(x,R) \subset B(x,2R)$ just for simplicity: if $K > 1$ and (0.3) holds whenever $h \geq 0$ is harmonic in $B(x, KR)$, then an easy chaining argument gives (EHI) (for a different constant c_1).

The main result of this paper is

Theorem 2. *Let $\alpha \geq 1$ and $2 \leq \beta \leq 1 + \alpha$. There there exists an infinite connected locally finite graph $\widetilde{\Gamma}$ which satisfies (V_α), (E_β) and (EHI).*

Examples.
1. The Euclidean space $\mathbb{Z}^d$, $d \geq 1$, (with its natural graph structure and conductances) satisfies (V_d) and (E_2), as well as (EHI). If $d \geq 2$ then the graph Γ consisting of two copies of $\mathbb{Z}^d$ with their origins identified satisfies (V_d) and (E_2), but fails to satisfy (EHI).
2. The binary tree satisfies (E_1), but since $V(x,r) \approx 2^r$ it fails to satisfy (V_α) for any α. ((EHI) also fails.)
3. Examples of graphs with $\beta > 2$ are provided by 'pre-fractal' graphs – see for example [J],

[BB2], [GT1]. The condition (E_β) implies that the mean square displacement $\mathbb{E}^x d(x, X_n)^2$ grows as $n^{2/\beta}$: if $\beta > 2$ then this growth is sublinear and is referred to by physicists as 'anomalous diffusion'. We call β the *'anomalous diffusion exponent'*, or the *'escape time exponent'*. (In the physics literature, or that on diffusions on fractals, one would write $\alpha = d_f$, the 'fractal dimension' and $\beta = d_w$, the 'walk dimension'.)
4. Let Γ_i, $i = 1, 2$ satisfy (V_{α_i}), (E_{β_i}). The product graph $\widehat{\Gamma} = \Gamma_1 \times \Gamma_2$ can be defined by taking the edges of $\widehat{\Gamma}$ to be of the form $\{(x, x_2), (x, y_2)\}$, where $\{x_2, y_2\} \in E_2$, and $\{(x_1, x_2), (y_1, x_2)\}$, where $\{x_1, y_1\} \in E_1$. Then it is easy to see that $\widehat{\Gamma}$ satisfies $(V_{\alpha_1+\alpha_2})$ and $(E_{\beta_1 \wedge \beta_2})$.

The case $\beta = 2$, $\alpha \in [1, \infty)$, α not an integer, has been treated (in the metric space context) in several recent papers. Following a question in [HS], Bourdon and Pajot [BoP] proved that the boundary of certain hyperbolic buildings satisfy (V_α), as well as an analytic condition (a weak (1,1) Poincaré inequality) which is strong enough to imply both (E_2) and (EHI). Here the possible values of α form a countable dense subset in $[1, \infty)$. More recently, Laakso [L] has given another construction of metric spaces satisfying (V_α) and a weak (1,1) Poincaré inequality, which permits any $\alpha > 1$. This is done by taking the product of $[0, 1]$ with a Cantor set and then identifying a dense set of points. The proof of Theorem 2 uses a similar construction, adapted to the graph case.

The examples in [BoP] and [L] disprove a conjecture made in [B1, Section 3].

We now outline the main steps in the proof of Theorem 2.

Definition. We say a weighted graph (Γ, a) is *very strongly recurrent* (VSR) if there exists $p_0 > 0$ such that, for all $R \geq 1$, x, $y \in G$ with $d(x, y) < R$,

$$\mathbb{P}^x(T_y < \tau_{x,2R}) \geq p_0. \qquad (VSR)$$

Remarks. 1. The literature contains (at least) two distinct definitions of strong recurrence for graphs, in [T2] and [D2]. (VSR) is equivalent to the definition in [D2], and stronger than that in [T2].
2. It is easily seen that (VSR) implies recurrence; in Lemma 1.6 below we prove it implies (EHI).
3. $\mathbb{Z}^1$ is very strongly recurrent, while $\mathbb{Z}^2$ is recurrent, but not very strongly recurrent.

The following Proposition, which is proved in Section 1, implies that many of the graphs studied in the fractals literature (such as the pre-Sierpinski gasket) satisfy (VSR).

Proposition 3. *Let (Γ, a) satisfy (V_α), and (E_β).*
(a) If $\beta \geq \alpha$ then Γ is recurrent.
(b) If $\beta > \alpha$ and Γ satisfies (EHI) then Γ satisfies (VSR).
(c) If $\beta = \alpha$ then Γ is recurrent but does not satisfy (VSR).
(d) If $\beta < \alpha$ and Γ satisfies (EHI) then Γ is transient.

Remark. I do not know if the conditions (V_α) and (E_β), with $\alpha > \beta$, are enough to imply that Γ is transient.

Proposition 4. *Let $\alpha \geq 1$. Then there exists a connected locally finite infinite graph Γ_α satisfying (V_α), $(E_{1+\alpha})$, (EHI), and (VSR).*

This is proved in Section 4, by adapting work of the author and Hambly (in [BH]) on mixtures of different types of Sierpinski gaskets to the case of graphs which are trees.

Proposition 5. *Let (Γ, a) satisfy (V_α), (E_β), (VSR), and so (EHI). Let $\lambda > 0$. Then there exists a weighted graph $(\widetilde{\Gamma}, \widetilde{a})$ satisfying $(V_{\alpha+\lambda})$, (E_β) and (EHI).*

This is proved in Sections 2 and 3. In section 2 we construct $\widetilde{\Gamma}$ by taking the product of G with an ultrametric space U, and fitting in edges in such a way that $\widetilde{\Gamma}$ consists of a countable number of copies of Γ, connected at link points. Section 2 deals with the geometry of $\widetilde{\Gamma}$, and proves that it satisfies $(V_{\alpha+\lambda})$. In section 3 we study random walks on $\widetilde{\Gamma}$. It is easy to prove that (E_β) holds, but the elliptic Harnack inequality takes a little more work.

In Section 5 we conclude the paper with some additional examples, motivated by those in [D2], concerning the property (EHI). In particular we have:

Theorem 6. *The elliptic Harnack inequality is not stable under products. That is, there exists a graph Γ which satisfies (EHI) such that the product graph $\Gamma \times \Gamma$ does not satisfy (EHI).*

Proof of Theorem 2. Let $\alpha \ge 1$ and β satisfy (0.2). By Proposition 4 there exists a graph Γ satisfying $(V_{\beta-1})$, (E_β), (EHI), and (VSR). By Proposition 5, taking $\lambda = \alpha + 1 - \beta$, there exists a graph $\widetilde{\Gamma}$ satisfying (V_α), (E_β), and (EHI). $\square$

Throughout this paper we use c, c', c'' for positive constants which may change from line to line, and c_i for positive constants that are fixed for each section. C_i denote positive constants which are fixed for the whole paper.

Acknowledgment. I am grateful to A. Grigor'yan for asking me this question. Much of this paper was written while the author was visiting the Erwin Schrödinger Institute in Vienna.

1. Weighted graphs and random walks

Throughout this section we take (G, a) to be an infinite connected locally finite weighted graph.

Lemma 1.1. *(See [Ku], [Ke].) If Γ satisfies (V_α) then there exists a constant c such that*

$$\mathbb{E}^x \tau_{x,2R} \ge c \frac{R^2}{(\log R)^{1/2}}, \quad x \in G,\ R \ge 1. \tag{1.1}$$

If Γ also satisfies (E_β) then $\beta \ge 2$.

Remark. See [BaP] for an example which shows that it is not possible to remove the $\log R$ term in (1.1).

Proof. First, note that the final assertion is immediate from (1.1).

By [C], [V], the transition probabilities $p_n(x, y)$ for X satisfy

$$p_n(x, y) \le c e^{-d(x,y)^2/2n}, \quad x, y \in G, \quad n \ge 1.$$

Let $R \ge 1$, $1 \le n \le R^2$ and set $\lambda = R/n^{1/2} \ge 1$. Set $A_k = \{y : 2^k R < d(x, y) \le 2^{k+1} R\}$; by (V_α) we have $\mu(A_k) \le c 2^{\alpha k} R^\alpha$. We have (see [B1, Lemma 3.7] for a similar calculation with more details)

$$\begin{aligned}\mathbb{P}^x(d(x,X_n)\geq R) &= \sum_{k=0}^\infty \sum_{y\in A_k} p_n(x,y)\mu_y \\ &\leq c\sum_{k=0}^\infty 2^{\alpha k}R^\alpha e^{-4^k R^2/2n} \\ &= cR^\alpha \sum_{k=0}^\infty 2^{\alpha k} e^{-4^k\lambda^2/2} \leq cR^\alpha e^{-\lambda^2/2}. \end{aligned} \tag{1.2}$$

For $x\in G$,

$$\mathbb{P}^x(\tau_{x,2R}\leq n) = \mathbb{P}^x(\tau_{x,2R}\leq n, X_n\notin B(x,R)) + \mathbb{P}^x(\tau_{x,2R}\leq n, X_n \in B(x,R)). \tag{1.3}$$

The second term in (1.3) equals, writing $\tau=\tau_{x,2R}$,

$$\begin{aligned}\mathbb{E}^x 1_{(\tau\leq n)}\mathbb{P}^{X_\tau}(X_{n-\tau}\in B(x,R)) &\leq \mathbb{E}^x 1_{(\tau\leq n)}\mathbb{P}^{X_\tau}(d(X_0,X_{n-\tau})\geq R) \\ &\leq \sup_{y\in G}\sup_{m\leq n}\mathbb{P}^y(d(y,X_m)\geq R) \\ &\leq cR^\alpha e^{-\lambda^2/2}, \end{aligned}$$

by (1.2). Since the first term in (1.3) also satisfies this bound, we deduce

$$\mathbb{P}^x(\tau_{x,2R}\leq R^2/\lambda)\leq c'R^\alpha e^{-\lambda^2/2} = c'e^{-\lambda^2/2+\alpha\log R}.$$

So if we set $\lambda=\theta(\log R)^{1/2}$ where θ is a constant with $\theta^2/2>\alpha$ then

$$\mathbb{P}^x\left(\tau_{x,2R}\leq \theta^{-1}R^2(\log R)^{-1/2}\right)\leq c'e^{-(\theta^2/2-\alpha)\log R},$$

which implies (1.1). □

The proof that $\beta\leq 1+\alpha$ uses the connection between random walks and electrical networks – see [DS]. If (Γ',a) is any finite weighted graph then (see [Tet])

$$\mathbb{E}^x T_y + \mathbb{E}^y T_x = \mu(G')R_e(x,y). \tag{1.4}$$

Here $R_e(x,y)$ is the effective resistance between x and y in the network where the edge $\{x',y'\}$ has conductivity $a_{x'y'}$. If we collapse the vertices in $B(x,R)^c$ to a single vertex y, and discard the second term in (1.4) then we obtain an inequality proved in [T1]:

$$\mathbb{E}^x\tau_{x,R}\leq V(x,R)R_e(x,\partial B(x,R)). \tag{1.5}$$

Lemma 1.2. *If Γ satisfies (V_α) and (E_β) then $\beta\leq 1+\alpha$.*

Proof. We can take $R\in\mathbb{N}$. As x and $\partial B(x,R)$ are connected by a chain of exactly R wires, each of conductance at least C_1, we have $R_e(x,\partial B(x,R))\leq C_1^{-1}R$. So using (V_α) and (E_β), (1.5) implies that $cR^\beta\leq cR^{\alpha+1}$, giving $\beta\leq 1+\alpha$. □

Proof of Theorem 1. Since G is infinite and Γ is connected, there exists a path to infinity from any point x. Hence $V(x,R) \geq C_1 R$, which implies that $\alpha \geq 1$. The remaining assertions are immediate from Lemmas 1.1 and 1.2. □

From [GT1], [GT2] we have the following estimates.

Lemma 1.3. *Let Γ satisfy (VD), (E_β) and (EHI), and write $\gamma = 1/(\beta-1)$. Suppose in addition that there exists a constant c such that $\mu_x \leq c$ for all $x \in G$.*
(a) There exist constants $c_1 - c_4$ such that for $n \geq 1$, $x, y \in G$,

$$p_n(x,y) \leq c_1 V(x, n^{1/\beta})^{-1} \exp\big(-c_2 (d(x,y)^\beta/n)^\gamma\big),$$
$$p_n(x,y) + p_{n+1}(x,y) \geq c_3 V(x, n^{1/\beta})^{-1} \exp\big(-c_4 (d(x,y)^\beta/n)^\gamma\big).$$

(b) For any $x \in G$, $n \geq 1$, $R \geq 1$,

$$\mathbb{P}^x(\tau_{x,R} \leq n) \leq c_5 \exp\big(-c_6 (R^\beta/n)^\gamma\big).$$

(c) There exists c_7 such that

$$\mathbb{P}^y(\tau_{x,R} > c_7 R^\beta) \leq \tfrac{1}{2}, \quad R \geq 1,\, y \in B(x,R).$$

Let $x_0 \in G$, $R \geq 1$, and let $B = B(x_0, 2R)$, $B' = B(x_0, R)$. Write $\tau = \tau_{x_0,2R}$. Set

$$\overline{p}_n(x,y) = \mathbb{P}^x(X_n = y, \tau > n)\mu_y^{-1}$$

for the density of the process X killed on exiting B, and let

$$\widehat{p}_n(x,y) = p_n(x,y) - \overline{p}_n(x,y) = \mathbb{P}^x(X_n = y, \tau \leq n)\mu_y^{-1}.$$

Lemma 1.4. *Let Γ satisfy (V_α), (E_β) and (EHI). For $x, y \in B'$,*

$$\overline{p}_n(x,y) + \overline{p}_{n+1}(x,y) \geq c_8 n^{-\alpha/\beta} \exp\big(-c_9 (d(x,y)^\beta/n)^\gamma\big), \quad n \leq R^\beta. \tag{1.6}$$

Proof. We begin by using the 'there and back' argument of [BB2] to bound $\widehat{p}$. We have, for $x, y \in B'$,

$$\begin{aligned} &\mathbb{P}^x(X_n = y, \tau \leq n) \\ &\quad= \mathbb{P}^x(X_n = y, \tau \leq n/2) + \mathbb{P}^x(X_n = y, n/2 < \tau < n) \\ &\quad\leq \mathbb{P}^x(X_n = y, \tau \leq n/2) + \mathbb{P}^x(X_n = y, X_m \in B^c \text{ for some } n/2 < m < n). \end{aligned} \tag{1.7}$$

By time reversibility the second term equals

$$\mu_y \mu_x^{-1} \mathbb{P}^y(X_n = x, X_m \in B^c \text{ for some } 0 < m < n/2) \leq c\, \mathbb{P}^y(X_n = x, \tau \leq n/2). \tag{1.8}$$

To bound the first term in (1.7) we have

$$\begin{aligned}\mathbb{P}^x(X_n = y, \tau \le n/2) &\le \mathbb{E}^x 1_{(\tau \le n/2)} \mathbb{P}^{X_\tau}(X_{n-\tau} = y) \\ &\le \mathbb{P}^x(\tau \le n/2) \sup_{z \in G} \sup_{m \le n/2} \mathbb{P}^z(X_{n-m} = y) \\ &\le c n^{-\alpha/\beta} \exp\big(-c'(R^\beta/n)^\gamma\big);\end{aligned}$$

here we used Lemma 1.3(a) and (b) in the last line. The same bound controls the right hand side of (1.8). Therefore, for $x, y \in B'$,

$$\overline{p}_n(x,y) + \overline{p}_{n+1}(x,y) \ge c n^{-\alpha/\beta}\Big(\exp(-c'(d(x,y)^\beta/n)^\gamma) - c'' \exp(-c'''(R^\beta/n)^\gamma)\Big).$$

It follows that there exist constants $c_{10} - c_{12}$, depending only on the constants above, such that

$$\overline{p}_n(x,y) + \overline{p}_{n+1}(x,y) \ge c_{10} n^{-\alpha/\beta} \exp\big(-c_{11}(d(x,y)^\beta/n)^\gamma\big), \quad n \le c_{12} R^\beta,\ d(x,y) \le c_{12} R.$$

This gives the lower bound (1.6) when x, y are sufficiently close together; to extend this to the case $x, y \in B'$ we can use a standard chaining argument – see for example [B1, section 3]. □

We write $\overline{g}_B(x,y) = \sum_n \overline{p}_n(x,y)$ for the Green's function for X killed on exiting $B = B(x_0, 2R)$.

Proposition 1.5. *Let* Γ *satisfy* (V_α), (E_β) *(EHI).*
(a) If $\beta > \alpha$ *then*

$$c_{13} R^{\beta-\alpha} \le \overline{g}_B(x,y) \le c_{14} R^{\beta-\alpha}, \quad x, y \in B'.$$

(b) If $\beta = \alpha$ *then*

$$c_{13} \log R \le \overline{g}_B(x,x) \le c_{14} \log R, \quad x, y \in B'.$$

Proof. The lower bounds in (a) and (b) are immediate on summing the bounds in Lemma 1.4.

For the upper bounds, it is enough to take $y = x$, since $\overline{g}_B(x,y) = \overline{g}_B(y,x) \le \overline{g}_B(x,x)$. Let $m = c_7(2R)^\beta$. Then by Lemma 1.3(c),

$$\begin{aligned}\mu_x \overline{g}_B(x,x) &= \mathbb{E}^x \sum_{n=0}^{\infty} 1_{(X_n = y, \tau > n)} \\ &= \mathbb{E}^x \sum_{n=0}^{m} 1_{(X_n = y, \tau > n)} + \mathbb{E}^x 1_{(\tau > m)} \mathbb{E}^{X_m} \sum_{n=0}^{\infty} 1_{(X_n = y, \tau > n)} \\ &\le \mathbb{E}^x \sum_{n=0}^{m} 1_{(X_n = y)} + \tfrac{1}{2} \mu_x \overline{g}_B(x,x).\end{aligned}$$

So

$$\overline{g}_B(x,x) \le c \sum_{n=0}^{m} p_n(x,x) \le c' \sum_{n=0}^{c'' R^\beta} n^{-\alpha/\beta},$$

which gives the upper bounds in both (a) and (b). □

Proof of Proposition 3. (a) The case when $\beta > \alpha$ is easy; we have (let $A = B(x,R)$)

$$cR^\beta \le \mathbb{E}^x \tau_{x,R} = \sum_{y \in B(x,R)} g_A(x,y)\mu_y \le c' R^\alpha g_A(x,x).$$

So $g_A(x,x) \ge cR^{\beta-\alpha}$, and thus $g(x,x) = \infty$.

If $\alpha = \beta$ then we consider the resistance from $x_0 \in G$ to infinity. Let $S_n = \{y : d(x_0,y) = 2^n\}$, and write $R_e(S_n, S_{n+1})$ for the effective resistance between S_n and S_{n+1}. Consider the finite graph Γ' where all vertices in $B(x_0, 1+2^n)$ are collapsed to a single vertex a and all vertices in $B(x_0, 2^{n+1}-1)^c$ are collapsed to a vertex b. Then, using R'_e to denote effective resistance in Γ', $R_e(S_n, S_{n+1}) = R'_e(a,b)$. By (1.5)

$$E^a T_b \le \mu(G') R'_e(a,b).$$

We have $\mu(G') \le c(2^n)^\alpha$, while $E^a T_b \ge c'(2^{n-1})^\beta$. So $R_e(S_n, S_{n+1}) \ge c'' > 0$, with c'' independent of n. Since $R_e(x_0, S_n) \ge \sum_{k=0}^{n-1} R_e(S_k, S_{k+1}) \ge c'' n$, we deduce from [DS] that Γ is recurrent.

(b) Let $R \ge 1$ and $x, y \in G$ with $d(x,y) < R$. As above write $B = B(x, 2R)$, $B' = B(x,R)$. For $x, y \in B'$, by Proposition 1.5,

$$\mathbb{P}^y(T_x < \tau_{x,2R}) = \frac{\overline{g}_B(y,x)}{\overline{g}_B(x,x)} \ge c,$$

which proves that Γ satisfies (VSR).

(c) Γ is recurrent by (a). Suppose Γ does satisfy (VSR) and (V_α). Let $B = B(x,2R)$, $B' = B(x,R)$. If $y \in B'$ then by (VSR) $\mathbb{P}^x(T_y < \tau_{x,2R}) = g_B(x,y)/g_B(y,y) \ge p_1$. So

$$g_B(x,y) \ge p_1 g_B(y,y) \ge c \log R.$$

Hence

$$\mathbb{E}^x \tau_{x,2R} \ge \sum_{y \in B'} \mu_y g_B(x,y) \ge cV(x,R) \log R \ge cR^\alpha \log R.$$

Thus Γ does not satisfy (E_α).

(d) From Lemma 1.3(a) we have that $g(x,x) = \sum_{n=0}^\infty p_n(x,x) < \infty$. □

We conclude this section with the following easy Lemma.

Lemma 1.6. *Suppose (VSR) holds for* Γ. *Then* Γ *satisfies (EHI).*

Proof. Let $h \ge 0$ be harmonic on $B(x,2R)$, and let x_0 and x_1 be the points in $B = B(x,R)$ where h attains its minimum and maximum respectively. Then using (VSR) repeatedly we obtain $\mathbb{P}^{x_0}(T_{x_1} < \tau_{x,2R}) \ge c_{15}$. So

$$h(x_0) \ge \mathbb{E}^{x_0} 1_{(T_{x_1} < \tau_{x,2R})} h(x_1) \ge c_{15} h(x_1).$$ □

2. Construction of Product Graph

Let $\Gamma = (G,E)$ be an infinite locally finite connected graph. Further hypotheses will be added later in this section, but this is all that is needed at this point. We write $\#(A)$ for the number of elements in the set A.

Proposition 2.1. *Let $b \geq 3$. There exists a partition*

$$G = \cup_{n=0}^{\infty} A_n$$

of G such that
(a) $\{B(x, b^n)\}$, $x \in \cup_{k=n}^{\infty} A_k$ are disjoint.
(b) For each n, $G \subset \cup_{k=n}^{\infty} \cup_{x \in A_k} B(x, 3b^n)$.
(c) For each n, $G = \cup_{x \in A_n} B(x, 9b^n)$.

Proof. Let $D_0 = G$. We construct a decreasing chain of sets D_k, $k \geq 0$, by choosing, for each k, D_{k+1} to be a maximal subset of D_k such that $B(x, b^{k+1})$, $x \in D_{k+1}$ are disjoint. We also need to ensure that $\cap_{k=0}^{\infty} D_k = \emptyset$. To do this, write $G = \{z_1, z_2, \ldots\}$. If then $D_n = \{z_{i_1}, z_{i_2}, \ldots\}$, with $i_1 < i_2 < \ldots$, let z' be a closest point in $D_n - \{z_{i_1}\}$ to z_{i_1}, and include z' in D_{n+1}. We will see below this implies that $z_{i_1} \notin D_{n+1}$. Let $b_0 = 1/3$, and $b_n = \sum_{i=1}^{n} b^i$ for $n \geq 1$. Then as $b \geq 3$,

$$2b_n + 1 = \frac{2b^{n+1} - b - 1}{b - 1} < \frac{2b}{b-1} b^n \leq 3b^n \leq b^{n+1}, \quad n \geq 0.$$

The construction of the sets D_k gives:

$$B(x, b^k), \quad x \in D_k \text{ are disjoint }, \quad k \geq 0, \tag{2.1}$$
$$D_k \subset \cup_{x \in D_{k+1}} B(x, 2b^{k+1}), \quad k \geq 0. \tag{2.2}$$

(If (2.2) fails then D_{k+1} would not be maximal.) Note that iterating (2.2) we obtain:

$$\text{for any } x_0 \in G \text{ there exists } x_n \in D_n \text{ with } d(x_0, x_n) < 2b_n < 3b^n,\ n \geq 0. \tag{2.3}$$

Let $x \in D_n$; we bound the distance from x to its closest neighbour x' in D_n. Let m be the integer with $2b_n \leq m < 2b_n + 1$, and choose $y \in G$ with $d(x, y) = m$. By (2.3) there exists $x'' \in D_n$ with $d(y, x'') < 2b_n$ – note this implies that $x \neq x''$. Thus for any $x \in D_n$ there exists $x' \in D_n$, with $d(x, x') < m + 2b_n$. Choosing y' on a shortest path between x and x' with $d(x, y') = m$ we have $y' \in B(x, b^{n+1}) \cap B(x', b^{n+1})$, so that x and x' cannot both be in D_{n+1}.

This calculation shows that $z_{i_1} \notin D_{n+1}$, and thus that $\cap D_k = \emptyset$. Set $A_k = D_k - D_{k+1}$: we have $\cup_n A_n = G$. (a) is now immediate from (2.1), and (b) from (2.3).

To prove (c), let $x_0 \in G$. By (2.3) there exists $x_n \in D_n$ with $d(x_0, x_n) < 2b_n$, and if x_n' is a closest neighbour to x_n in D_n then $d(x_n, x_n') < 4b_n + 1$, so that $d(x_0, x_n') < 6b_n + 1 < 9b^n$. Since at least one of x_n and x_n' is in A_n, this proves (c). □

Lemma 2.2. *Let Γ, (A_n) be as above, and let $x \in G$, $n \geq 1$. Suppose that $b \geq 9$.*
(a) $B(x, b^n)$ contains at least one point in A_k for each $k \leq n - 1$.
(b) $B(x, 3b^n)$ contains at least one point in $\cup_{k=n}^{\infty} A_k$.
(c) $B(x, b^n)$ contains at most one point in $\cup_{k=n}^{\infty} A_k$.

Proof. (b) and (c) are immediate from Proposition 2.1(b) and (a). Let $x \in G$ and $k \leq n - 1$. By Proposition 2.1(c) there exists $y \in A_k$ such that $d(x, y) < 9b^k \leq b^n$. □

Let $M \geq 2$ be an integer, and U be a discrete ultrametric space with 'family size' M. We set

$$U = \{u = (u_1, u_2 \ldots), u_i \in \{0, 1, \ldots, M-1\}, \sum u_i < \infty\}.$$

We can regard the sequence (u_i) as the address of $u \in U$: u_1 denotes the district, u_2 the town, u_3 the county etc. – all points $u \in U$ have an address with components which are 0 from some point on. Set $\delta(u, u) = 0$, and for $u \neq v$,

$$\delta(u, v) = \max\{i : u_i \neq v_i\};$$

δ is a metric on U, and the ultrametric property $\delta(u, v) \leq \delta(u, w) \vee \delta(w, v)$ for $u, v, w \in U$ is easy to verify. Write

$$C_n(u) = \{v \in U : \delta(u, v) \leq n\}$$

for the closed ball radius n in U, and note that $\#(C_n(u)) = M^n$.

Now let $\Gamma = (G, E)$ be as above, and choose $b \geq 9$. Let (A_n) be a partition of G satisfying Proposition 2.1: we call the points in A_n *links of order* n. (We could manage with smaller b, but there is no advantage in doing so, while a large b will allow us to use Lemma 2.2 to reduce the number of possible types of 'high level' link points in a ball.) We now construct a new graph $\widetilde{\Gamma}$. Set $\widetilde{G} = G \times U$. We define the edge set $\widetilde{E}$ for the graph $\widetilde{\Gamma} = (\widetilde{G}, \widetilde{E})$ as follows:

$$\{(x, u), (y, u)\} \in \widetilde{E} \text{ if } \{x, y\} \in E,$$
$$\{(x, u), (x, v)\} \in \widetilde{E} \text{ if } u_i = v_i, i \neq n, u_n \neq v_n, x \in A_n, n \geq 1.$$

These are the only edges of $\widetilde{\Gamma}$. Thus we construct links from (x, u) to points (x, v) at $M-1$ other levels of $\widetilde{G}$ if $x \in G - A_0$. If x has degree r, then $(x, u) \in \widetilde{G}$ has degree $M - 1 + r$ if $x \notin A_0$, and degree r if $x \in A_0$. The graph $\widetilde{\Gamma}$ thus consists of a countable number of copies of Γ, glued together at the link points.

If $\Gamma = (G, E, a)$ is a weighted graph then we define the weights $\widetilde{a}_{(x,u),(y,v)}$ by

$$\widetilde{a}_{(x,u),(y,v)} = \begin{cases} a_{xy}, & \text{if } u = v, \\ 1, & \text{if } x = y \text{ and } \{(x, u), (y, v)\} \text{ is an edge in } \widetilde{E}, \\ 0, & \text{otherwise.} \end{cases}$$

Notation. We denote points in $\widetilde{G}$ by $\widetilde{x} = (x, u)$. We write $\widetilde{\mu}$ for the measure associated with $\widetilde{a}$, $\widetilde{B}(x, u, r)$ for the ball in $\widetilde{\Gamma}$ centre (x, u) and radius r, and $\widetilde{V}(x, u, r) = \widetilde{\mu}(\widetilde{B}(x, u, r))$. For $Q \subset G$ and $\widetilde{x} = (x, u) \in Q \times U$ let $W_Q(\widetilde{x})$ be the unique subset of U such that the connected component of $Q \times U$ containing $\widetilde{x}$ is $Q \times W_Q(\widetilde{x})$. Write

$$W_n(x, u) = W_{B(x, b^n)}(x, u).$$

Thus $W_n(x, u)$ is the set of $v \in U$ such that (x, v) is connected to (x, u) by a path $(x_i, u^{(i)})$ with $x_i \in B(x, b^n)$ for all i.

By Lemma 2.2 we obtain

Lemma 2.3. *Exactly one of the following holds:*

$$W_n(x,u) = C_{n-1}(u), \tag{2.4}$$
$$W_n(x,u) = C_n(u), \tag{2.5}$$
$$W_n(x,u) = C_{n-1}(u) \cup C_{n-1}(u'), \text{ for some } u' \text{ with } \delta(u,u') > n. \tag{2.6}$$

Lemma 2.4. *For $n \geq 1$, $(x,u) \in \widetilde{G}$,*

$$\widetilde{B}(x,u,b^n) \subset B(x,b^n) \times W_n(x,u) \subset \widetilde{B}(x,u,6b^n).$$

Proof. The first inclusion is clear from the definition of $W_n(x,u)$. To prove the second, let $(y,v) \in B(x,b^n) \times W_n(x,u)$. We begin by constructing a path from x to y which contains at least one point in $A_k \cap B(x,b^n)$ for each $k \geq 1$ for which this set is non-empty.

If $1 \leq k \leq n-1$ then by Proposition 2.1(c) there exists $x_k \in A_k$ with $d(x,x_k) < 9b^k \leq b^n$. So if π is the path which successively visits $x, x_1, x, x_2, \dots, x_{n-1}, x$ the length of π is at most $18\sum_1^{n-1} b^k \leq (9/4)b^n$. If (2.4) holds the path contains all the necessary link points; if not then there is an extra high level link point $x' \in B(x,b^n) \cap A_m$ for some $m \geq n$, and another section of length at most $2b^n$ is needed. Finally, we add a section from x to y of length again at most b^n.

To construct a path from (x,u) to (y,v) we simply add (if necessary) an extra 'vertical' edge of the form $\{(x_k,u^{(k)}),(x_k,v^{(k)})\}$ at each of the link points, to switch the kth component of u to that of v. The overall length of the path is therefore at most $(21/4)b^n + n \leq 6b^n$. □

Proposition 2.5. *(a) Let $\alpha \geq 1$ and $\lambda = \log M/\log b$. Then Γ satisfies (V_α) if and only if $\widetilde{\Gamma}$ satisfies $(V_{\alpha+\lambda})$.*
(b) $\widetilde{\Gamma}$ satisfies (VD) if and only if Γ satisfies (VD).

Proof. Since Γ is connected, $\mu_x \geq C_1$ for all $x \in G$. Thus $V(x,R) \geq C_1 \# B(x,R)$. For $(x,u) \in \widetilde{G}$ we have $\mu_x \leq \widetilde{\mu}_{(x,u)} \leq \mu_x + (M-1)$, and so

$$\begin{aligned} M^n V(x,b^n) \leq \widetilde{\mu}(B(x,b^n) \times W_n(x,u)) &\leq M^n V(x,b^n) + (M-1)M^n \#(B(x,b^n)) \\ &\leq cM^n V(x,b^n). \end{aligned}$$

Using Lemma 2.4 we therefore have, since $M = b^\lambda$,

$$\widetilde{V}(x,u,b^n) \leq c(b^n)^\lambda V(x,b^n), \qquad \widetilde{V}(x,u,6b^n) \geq c'(b^n)^\lambda V(x,b^n),$$

and (a) and (b) now follow easily. □

3. Random walk on $\widetilde{\Gamma}$.

It will be technically easier to work with continuous time random walks in this section. Let $(\Gamma,a) = (G,E,a)$ be a weighted graph. The continuous time random walk (CTRW) on (Γ,a) is the Markov process $X = (X_t, t \in [0,\infty), \mathbb{P}^x, x \in G)$ with generator

$$\mathcal{L}f(x) = \sum_{y \in G} a_{xy}(f(y) - f(x)).$$

The process X waits at a vertex x for an exponential time with mean μ_x^{-1}, and then jumps to one of the neighbours of x, moving to y with probability $p_{xy} = a_{xy}/\mu_x$. So, if we write S_i for the jump times, the process $Z_n = X_{S_n}$ is exactly the discrete time random walk on Γ defined in the introduction.

Assume that

$$C_2 \leq \mu_x \leq C_3, \qquad x \in G; \tag{3.1}$$

of course this condition follows from (0.1) and (V_α). Then we have $\mathbb{E}(S_{n+1} - S_n | X_s, s \leq S_n) \in [C_3^{-1}, C_2^{-1}]$, and it follows that (E_β) holds for the process X if and only if it holds for Z.

Now fix $(\Gamma, a) = (G, E, a)$ satisfying (3.1), and let $(\widetilde{\Gamma}, \widetilde{a})$ be the weighted graph constructed in the previous section. Let X be the CTRW on (Γ, a). Then the CTRW $\widetilde{X}$ on $\widetilde{\Gamma}$ can be constructed from X and a process Y which is defined as follows.

Let η_k, $k \geq 1$ be independent Poisson point processes with rate M on $\mathbb{R}_+$, and $\theta_{k,n}$, $k \geq 1$, $n \geq 1$ be i.i.d.r.v. uniform on $\{0, 1, \ldots, M-1\}$. Let N_t^k, $t \geq 0$ be the counting processes associated with η_k. Write $\sigma_{k,n}$, $n \geq 1$ for the points in η_k: $\sigma_{k,n+1} - \sigma_{k,n}$ is an exponential r.v. with mean $1/M$. Given $(X_t,\ t \geq 0)$ we construct $Y_t = (Y_t^1, Y_t^2, \ldots) \in U$, with $Y_0 = u = (u_1, \ldots)$, by taking Y_t^k to be constant on each interval $[\sigma_{k,n}, \sigma_{k,n+1})$, and setting

$$Y^k_{\sigma_{k,n+1}} = \begin{cases} Y^k_{\sigma_{k,n}} & \text{if } X_{\sigma_{k,n+1}} \notin A_k, \\ \theta_{k,n+1} & \text{if } X_{\sigma_{k,n+1}} \in A_k. \end{cases}$$

Note that the times of the point process η_k are not the jump times of Y^k: Y^k only makes a jump at one of these times if, first X is in A_k, and second, the r.v. $\theta_{k,\cdot}$ gives a new state. The process $\widetilde{X}_t = (X_t, Y_t)$ is then a CTRW on $\widetilde{\Gamma}$.

Define the filtration

$$\mathcal{G}_t = \sigma(X_s, N_s^k, s \leq t, k \geq 1), \quad t \geq 0.$$

Set also

$$\Lambda_k = \{\sigma_{k,n} : X_{\sigma_{k,n}} \in A_k, n \geq 1\} \subset [0, \infty).$$

Note that $\Lambda_k \cap [0, t]$ is a $\mathcal{G}_t$-measurable random set. Let

$$\begin{aligned} \tau_{x,r} &= \inf\{t \geq 0 : X_t \notin B(x, r)\}, \\ \widetilde{\tau}_{x,u,r} &= \inf\{t \geq 0 : \widetilde{X}_t \notin \widetilde{B}(x, u, r)\}, \\ \tau_n(x, u) &= \inf\{t \geq 0 : \widetilde{X}_t \notin B(x, r) \times W_n(x, u)\}. \end{aligned}$$

Lemma 3.1. *$(\widetilde{\Gamma}, \widetilde{a})$ satisfies (E_β) if and only if (Γ, a) satisfies (E_β).*

Proof. Note that $\widetilde{X}$ can only exit from $B(x, b^n) \times W_n(x, u)$ when X exits from $B(x, b^n)$, so that $\tau_n(x, u) = \tau_{x,b^n}$. Hence using Lemma 2.4,

$$\mathbb{E}^{(x,u)} \widetilde{\tau}_{x,u,6b^n} \geq \mathbb{E}^{(x,u)} \tau_n(x, u) = \mathbb{E}^x \tau_{x,b^n} \geq \mathbb{E}^{(x,u)} \widetilde{\tau}_{x,u,b^n},$$

and (E_β) for $\widetilde{\Gamma}$ follows immediately. □

We now turn to the proof of the elliptic Harnack inequality on $(\widetilde{\Gamma}, \widetilde{a})$. For the remainder of this section we will assume that (VSR) holds for (Γ, a).

Lemma 3.2. *There exists $c_1 > 0$ such that for $k \ge 1$, $x \in G$,*

$$\mathbb{P}^x(\Lambda_k \cap [0, \tau_{x,18b^k}) \neq \emptyset) \ge c_1.$$

Proof. By Proposition 2.1(c) there exists $x' \in A_k \cap B(\xi_r, 9b^k)$. So, using (VSR), and writing $\tau = \tau_{x,18b^k}$, $\mathbb{P}^x(T_{x'} < \tau) \ge p_0 > 0$. If $X_t = x'$ then the time to the next jump of X is exponential with rate μ_x, while points in η_k occur at rate M. So the probability that a point in η_k will occur before X leaves x' is $M/(M+\mu_x) \ge M/(M+C_3)$. Combining these estimates completes the proof, with $c_1 = p_0 M/(M + C_3)$. □

Fix $(x_0, u^0) \in \widetilde{G}$, let $R \ge 1$, let n be such that $b^{n-1} \le R < b^n$, write

$$B = B(x_0, R), \quad B' = B(x_0, \tfrac{3}{2}R), \quad B^* = B(x_0, 2R),$$

and let τ', τ^* be the first exit times of X from B' and B^*. We begin by considering harmonic functions h on $B^* \times W_{B^*}(u^0)$ of the form

$$h(x,u) = h^{z_0,v}(x,u) = \mathbb{P}^{(x,u)}(X_{\tau^*} = z_0, Y_{\tau^*} = v), \tag{3.2}$$

where $z_0 \in \partial B^*$ and $v \in W_{B^*}(u^0)$. Let

$$H_k = \{\text{there exists } n \ge 0 \text{ with } \sigma_{k,n} < \tau' \text{ and } X_{\sigma_{k,n}} \in A_k\} = \{\Lambda_k \cap [0, \tau') \neq \emptyset\}.$$

Lemma 3.3. *For $x \in B$*

$$\mathbb{P}^{(x,u)}(H_k^c) \le c_2 e^{-c_3 b^{n-k}}, \quad 1 \le k \le n.$$

Proof. Define a sequence of stopping times, and points in $\widetilde{G}$ as follows.

$$T_0 = 0, \quad \xi_0 = x, \quad T_{r+1} = \tau_{\xi_r, 18b^k}, \quad \xi_{r+1} = X_{T_{r+1}}, \quad r \ge 0.$$

Let $m \in \mathbb{Z}_+$ satisfy $m \in [R/(36b^k) - 1, R/(36b^k))$; then $d(x, \xi_m) \le 18b^k m < \frac{1}{2}R$, so that $T_m < \tau'$ for $1 \le r \le m$.

By Lemma 3.2 we obtain

$$\mathbb{P}^x(\Lambda_k \cap [T_r, T_{r+1}) \neq \emptyset | \mathcal{G}_{T_r}) \ge c_3. \tag{3.3}$$

Therefore

$$\mathbb{P}^x(H_k^c) \le (1 - c_3)^m \le \exp(-cm) \le c' \exp(-c'' b^{n-k}).$$ □

Set

$$\overline{h}(x) = \mathbb{P}^x(X_{\tau^*} = z_0).$$

As $\overline{h}$ is harmonic on Γ, it satisfies an elliptic Harnack inequality by Lemma 1.6. Write $\overline{h}_{\max}$ for the maximum value of $\overline{h}$ on B'. The next result bounds h above by $\overline{h}_{\max}$. Naturally we expect h to be maximised if $v = u^0$, but we do not need this.

Proposition 3.4. *Let h, B, B', B^* be as above. Then for $(x,u) \in B \times W_B(u^0)$,*

$$h(x,u) \le c_4 M^{-n} \overline{h}_{\max}.$$

Proof. Define an integer valued random variable J by taking $\{J = 0\} = H_1^c$, $\{J = k\} = H_1 \cap \dots H_k \cap H_{k+1}^c$ for $1 \le k \le n-1$, and $\{J = n\} = H_1 \cap \dots H_n$. Note that J is $\mathcal{G}_{\tau'}$ measurable.

Let $k \ge 1$, and define

$$\kappa = \begin{cases} \max\{n : \sigma_{k,n} \in \Lambda_k \cap [0,\tau^*)\} & \text{on } H_k, \\ 0 & \text{on } H_k^c. \end{cases}$$

Then $Y_{\tau^*}^k = \theta_{k,\kappa}$ on H_k, so that (on H_k) $Y_{\tau^*}^k$ is equal to a random variable which is independent of $\mathcal{G}_\infty$, and is uniformly distributed on $\{0,\dots,M-1\}$. So, if $J = k$, then the first k components of Y_{τ^*} will have been randomised and so we have, for any $v \in U$,

$$\mathbb{P}^{(x,u)}(X_{\tau^*} = z_0, Y_{\tau^*} = v | J = k, X_{\tau'} = y) \le M^{-k}\overline{h}(y) \le M^{-k}\overline{h}_{\max}. \tag{3.4}$$

Now choose j_0 to be the smallest strictly positive integer so that $Me^{-c_3 b^{j_0}} \le \frac{1}{2}$. Then

$$\begin{aligned} \sum_{k=0}^{n-j_0} M^{-k}\mathbb{P}^{(x,u)}(J=k) &\le \sum_{k=0}^{n-j_0} M^{-k}\mathbb{P}^{(x,u)}(H_{k+1}^c) \\ &\le \sum_{k=0}^{n-j_0} M^{-k} c_2 \exp(-c_3 b^{n-k-1}) \\ &= c_2 M^{-n+j_0} \sum_{i=0}^{n-j_0} M^i \exp(-c_3 b^{j_0-1} b^i) \\ &\le c_2 M^{-n+j_0} \sum_{i=0}^{\infty} M^i \exp(-c_3 b^{j_0-1} i b) \\ &= c_2 M^{-n+j_0}(1 - Me^{-c_3 b^{j_0}})^{-1} \le 2c_2 M^{-n+j_0}. \end{aligned} \tag{3.5}$$

So, using (3.4) and (3.5) we have

$$\begin{aligned} h(x,u) &= \sum_{k=0}^{n} \sum_{y \in \partial B'} \mathbb{P}^{(x,u)}(X_{\tau^*} = z_0, Y_{\tau^*} = v | J = k, X_{\tau'} = y)\mathbb{P}^{(x,u)}(J = k, X_{\tau'} = y) \\ &\le \sum_{k=0}^{n} M^{-k}\overline{h}_{\max} \sum_{y \in \partial B'} \mathbb{P}^{(x,u)}(J = k, X_{\tau'} = y)) \\ &= \overline{h}_{\max} \sum_{k=0}^{n} M^{-k}\mathbb{P}^{(x,u)}(J = k) \\ &\le \overline{h}_{\max}\Big(2c_2 M^{-n+j_0} + \sum_{k=n-j_0+1}^{n} M^{-k}\Big) \le c_4 \overline{h}_{\max} M^{-n}. \end{aligned}$$

□

Proposition 3.5. *Let $v \in W_{B'}$, and $h = h^{z_0,v}$ where $z_0 \in \partial B^*$. Then*

$$h(x,u) \geq c_5 M^{-n} \inf_{y \in B'} \overline{h}(y), \quad (x,u) \in B \times W_B(u^0).$$

Proof. Since $b^{n-1} \leq R \leq b^n$, by Lemma 2.2 the ball B contains link points in $A_1, \dots A_{n-2}$. In addition B' may contain some additional 'higher level' link points. We will deal with the worst case, when both $A_{n-1} \cap B' \neq \emptyset$ and $D_n \cap B' \neq \emptyset$: if either of these sets is empty, then the relevant part of the construction below can be omitted. Note that $D_n \cap B'$ can contain at most one link point, in A_m say. Let $n_0 \geq 1$ be the smallest integer such that $c_2 e^{-c_3 b^{n_0}}/(1 - e^{-c_3 b^{n_0}}) < \frac{1}{2}$. We begin by assuming that $n \geq n_0$.

We can use the symmetry of U to take $u^0 = (0, \dots)$. Then since $v \in W_{B'}$, we have $v_i = 0$ for $i \geq n$, $i \neq m$. We now estimate from below the probability that X and η_k satisfy the following:

(1) Each of the events H_k, $1 \leq k \leq n - j_0$ occur before time τ'.
(2) X then hits a link point in each of $A_k \cap B$, for $n - j_0 + 1 \leq k \leq n - 2$ without leaving B^*.
(3) X then hits the link points in $A_{n-1} \cap B'$ and $A_m \cap B'$ before τ^*.

We will write F_i, $1 \leq i \leq 3$ for the events described above, and define stopping times T_i for the time this event is completed. More precisely, we set

$$\begin{aligned} T_1 &= \inf\{t \geq 0 : \Lambda_k \cap [0,t] \neq \emptyset, \text{ for each } k \text{ with } 1 \leq k \leq n - n_0\}, \\ T_2 &= \inf\{t \geq T_1 : \Lambda_k \cap [T_1,t] \neq \emptyset, \text{ for each } k \text{ with } n - n_0 < k \leq n-2\}, \\ T_3 &= \inf\{t \geq T_2 : \Lambda_k \cap [T_2,t] \neq \emptyset, \text{ for each } k \text{ with } k = n-1, m\}. \end{aligned}$$

We have, writing $\mathbb{P} = \mathbb{P}^{(x,u)}$,

$$\begin{aligned} \mathbb{P}(F_1^c) \leq \sum_{k=0}^{n-n_0} P(H_k^c) &\leq \sum_{k=0}^{n-n_0} c_2 e^{-c_3 b^{n-k}} \\ &= c_2 \sum_{i=0}^{n-n_0} e^{-c_3 b^{n_0+i}} \\ &\leq c_2 \sum_{i=0}^{\infty} e^{-c_3 b^{n_0}(i+1)} \leq \tfrac{1}{2}, \end{aligned}$$

by the choice of n_0. So $\mathbb{P}(F_1) \geq \frac{1}{2}$. We have $T_1 \leq \tau'$ on F_1. So, using (VSR) repeatedly to 'move' X around in B and B' without leaving B^*, we deduce that

$$\mathbb{P}(F_2 \cap F_3 | \mathcal{G}_{T_1}) \geq c_6^{n_0} \quad \text{on } F_1.$$

Thus if $F = \cap_{i=1}^3 F_i$ we have $\mathbb{P}(F) \geq c_7 > 0$, and $X_{T_3} \in B'$ on F.

Since on the event F all the components of Y which can change while X remains in B^* have had an opportunity to do so, we have

$$\begin{aligned} h(x,u) &\geq \mathbb{E}^x 1_F \mathbb{E}^x(X_{\tau^*}=z_0, Y_{\tau^*}=v|\mathcal{G}_{T_3}) \\ &= \mathbb{E}^x 1_F \mathbb{E}^x(X_{\tau^*}=z_0|\mathcal{G}_{T_3}) M^{-n} \\ &= M^{-n}\mathbb{E}^x 1_F \overline{h}(X_{T_3}) \geq c_7 M^{-n} \inf_{y\in B'} \overline{h}(y). \end{aligned}$$

If $n < n_0$ then we can omit step (1) above, and in step (2) require that X hits each of $A_k \cap B$, $1 \leq k \leq n-2$, before τ'. This event has probability bounded below by $c_6^n \leq c_6^{n_0-2}$. A similar argument to that above then completes the proof. □

Lemma 3.6. *Let $(x_0, u^0) \in \widetilde{G}$, $R \geq 1$, and let $h \geq 0$ be harmonic on $Q_1 = B(x_0, 2R) \times W_{B(x_0,2R)}(u^0)$. Suppose that*

$$W_{B(x_0,R)}(u^0) = W_{B(x_0,2R)}(u^0). \tag{3.6}$$

Then there exists c_9 such that, writing $Q_0 = B(x_0,R) \times W_{B(x_0,R)}(u^0)$,

$$\sup_{Q_0} h \leq c_9 \inf_{Q_0} h. \tag{3.7}$$

Proof. If $h = h^{z_0,v}$, with $v \in W_{B(x_0,R)}(u^0)$ then (3.7) is immediate from the estimates in Propositions 3.4 and 3.5, and the elliptic Harnack inequality for (Γ, a). Now if $h \geq 0$ is harmonic in $B(x_0,2R) \times W_{B(x_0,2R)}(u^0)$, then h can be written

$$h(x,u) = \sum_{z_0\in\partial B(X_0,2R)} \sum_{v\in W_{B(x_0,2R)}(u^0)} h(z_0,v) h^{z_0,v}(x,u),$$

so that (3.7) follows. □

Lemma 3.7. *Let $(x_0,u^0) \in \widetilde{G}$, $n \geq 1$, and let $h \geq 0$ be harmonic on $Q_1 = B(x_0, b^{n+1}) \times W_{n+1}(x_0,u^0)$. Then writing $Q_0 = B(x_0,b^n) \times W_n(u^0)$,*

$$\sup_{Q_0} h \leq c_9 \inf_{Q_0} h.$$

Proof. Set $B_j = B(x_0, 2^j b^n)$ for $j = 0,1,2,3$. By Lemma 2.2 B_0 contains link points in $A_1, \ldots, A_{n-1}$, while B_3 contains at most two additional kinds of link points – in A_n and, possibly, in A_m for some $m \geq n+1$. So we must have $W_{B_j} = W_{B_{j+1}}$ for at least one $j \in \{0,1,2\}$. So we can apply Lemma 3.6 for this B_j and B_{j+1}, and obtain, writing $Q' = B_j \times W_{B_j}$, $\sup_{Q_0} h \leq \sup_{Q'} h \leq c_9 \inf_{Q'} h \leq c_9 \inf_{Q_0} h$. □

We now obtain the elliptic Harnack inequality for $(\widetilde{\Gamma}, \widetilde{a})$.

Theorem 3.8. *Suppose that* (Γ, a) *satisfies (VSR), and* $(\widetilde{\Gamma}, \widetilde{a})$ *is constructed by the procedure of Section 2. Then* $(\widetilde{\Gamma}, \widetilde{a})$ *satisfies (EHI).*

Proof. Let $\widetilde{x} = (x, u) \in \widetilde{\Gamma}$, $R \geq 1$ and $h \geq 0$ be harmonic in $\widetilde{B}(\widetilde{x}, 2R)$. Choose n so that $6b^{n+1} \leq 2R < 6b^{n+2}$. Then by Lemma 2.4, h is harmonic in $B(x, u, b^{n+1}) \times W_{n+1}(x, u)$. Since, also by Lemma 2.4, $\widetilde{B}(x, u, R/(3b^2) \subset B(x, u, b^n) \times W_n(x, u)$, we obtain from Lemma 3.8

$$\sup_{\widetilde{B}(x,u,R/(3b^2))} h \leq c_9 \inf_{\widetilde{B}(x,u,R/(3b^2))} h.$$

This is the elliptic Harnack inequality, but with a tighter condition on the ratio of the sizes of the two balls. A routine chaining argument now gives the (EHI) in its standard form. □

Proof of Proposition 5. This is immediate from Proposition 2.5, Lemma 3.1, and Theorem 3.8.

4. Construction of trees satisfying (V_α) and $(E_{1+\alpha})$.

In this section we prove Proposition 4, by constructing a family of graphs which we will call Vicsek trees. These are most easily defined via their embedding in $\mathbb{R}^N$. Let $N \geq 2$, and let $\mathcal{C}_N$ be the collection of unit cubes in $\mathbb{R}^N$ with corners in $\mathbb{Z}^N$ and edges parallel to the axes. We write $\lambda\mathcal{C}_N = \{\lambda Q : Q \in \mathcal{C}_N\}$. We call any connected set $A \subset \mathbb{R}^N$ which is a union (finite or infinite) of cubes in $\mathcal{C}_N$ a *cubical set*. As we will be working with cubes, we obtain some slight simplification if we use the L_∞ metric on $\mathbb{R}^N$, in which balls are cubes. For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ we set $|x|_\infty = \max\{x_1, \ldots, x_N\}$. We also write $|A|$ for the Lebesgue measure of $A \subset \mathbb{R}^N$.

Definition 4.1. Given a cubical set A we define a graph $\Gamma = \Gamma(A)$, which we call the *graph generated by* A, as follows. The vertex set of Γ is the set of corners and centers of the cubes in $\mathcal{C}_N$. The edges of Γ connect the center of any cube $Q \in \mathcal{C}_N$ with $Q \subset A$ to each corner of that cube.

Note that since A is connected, Γ is also connected, and that each cube in A contains 2^N edges in $\Gamma(A)$. We also remark that each edge of Γ has length $\frac{1}{2}$ (in the L_∞ metric).

Given any suitably regular fractal F one can construct an infinite 'pre-fractal' graph Γ_F such that the large scale structure of Γ_F mimics the small scale structure of F. We begin by constructing a family of regular fractal subsets of $[0,1]^N$.

Let $L \geq 1$, $N \geq 2$, $F_0 = [0,1]^N$, and $x_0 = (\frac{1}{2}, \ldots, \frac{1}{2})$ be the center of F_0. Let J be the union of the 2^N line segments connecting x_0 with the corners of F_0, and let F_1 be the union of the $2^N L + 1$ cubes in $(2L+1)^{-1}\mathcal{C}_d$ with centers in J. (See Figure 1). Label these cubes Q_i, $1 \leq i \leq 2^N L + 1$, and for each i let $\psi_i^{(N,L)}$ be the orientation preserving linear map which maps F_0 onto Q_i. For compact sets $K \subset \mathbb{R}^N$, set

$$\Psi^{(N,L)}(K) = \bigcup_{i=1}^{2^N L+1} \psi_i^{(N,L)}(K).$$

Thus we have $F_1 = \Psi^{(N,L)}(F_0)$. Now let $F_{n+1} = \Psi^{(N,L)}(F_n)$, and note that (F_n) is a decreasing sequence of compact sets. The intersection F is a fractal tree which contains J and has Hausdorff dimension

$$\alpha_{N,L} = \frac{\log(2^N L + 1)}{\log(2L+1)}. \qquad (4.1)$$

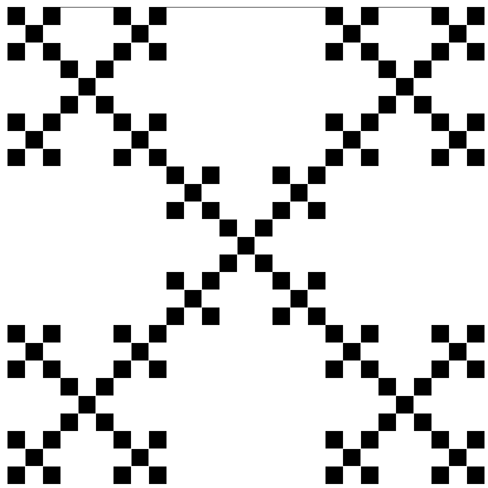

Figure 1. The set F_3 in the case $N = 2$, $L = 1$.

We next construct a graph $\Gamma^{(N,L)}$ which has at the large scale the same structure as F at the small scale. (See [BCG] for some more details and pictures). Let

$$H_n = (2L+1)^n F_n.$$

Thus H_n is a cubical set and is contained in $[0, (2L+1)^n]^N$. It is easy to check that (H_n) is an increasing sequence of sets, and that if $m > n$ then $H_m \cap [0, (2L+1)^n]^N = H_n$. Set

$$H = \cup_{n=0}^{\infty} H_n.$$

Then H is a cubical set, and we define $\Gamma^{(N,L)}$ to be the graph induced by H. (It is easy to see that $\Gamma^{(N,L)}$ is a tree.)

Lemma 4.2. *$\Gamma^{(N,L)}$ is an infinite connected graph which satisfies $(V_{\alpha_{N,L}})$, $(E_{1+\alpha_{N,L}})$, (EHI) and (VSR).*

This Lemma gives Proposition 4 for a countable dense set of α in $[1, \infty)$. We will not prove it at this point, since it will follow from the more general construction below which is needed to prove the full version of Proposition 4.

We now consider fractals and graphs obtained by mixtures of the iterations $\Psi^{(N,L)}$. This was done for Sierpinski gaskets in the fractal case in [BH]. The construction here is quite similar, except for one point: here we work 'outwards', starting with the small scale structure, while [BH] worked 'inwards'. We fix N in what follows, and let $1 \le L_1 < L_2$. (We could allow more than two values of L, as in [BH], but this complicates the notation without giving us anything more.)

Let $\Xi = \{1,2\}^{\mathbb{N}}$ and let $\xi = (\xi_1, \dots) \in \Xi$: we call ξ an *environment sequence.* Let $F_0^\xi = [0,1]^N$, and define

$$\begin{aligned} a_n &= \prod_{i=1}^{n}(2L_{\xi_i}+1), \\ b_n &= \prod_{i=1}^{n}(2^N L_{\xi_i}+1), \\ F_n^\xi &= \Psi^{(N,L_{\xi_n})}(F_{n-1}^\xi), \\ H_n^\xi &= a_n F_n^\xi, \\ H^\xi &= \cup_{i=1}^\infty H_n^\xi. \end{aligned}$$

If ξ is constant then we obtain one of the sets $H^{(N,L_i)}$.

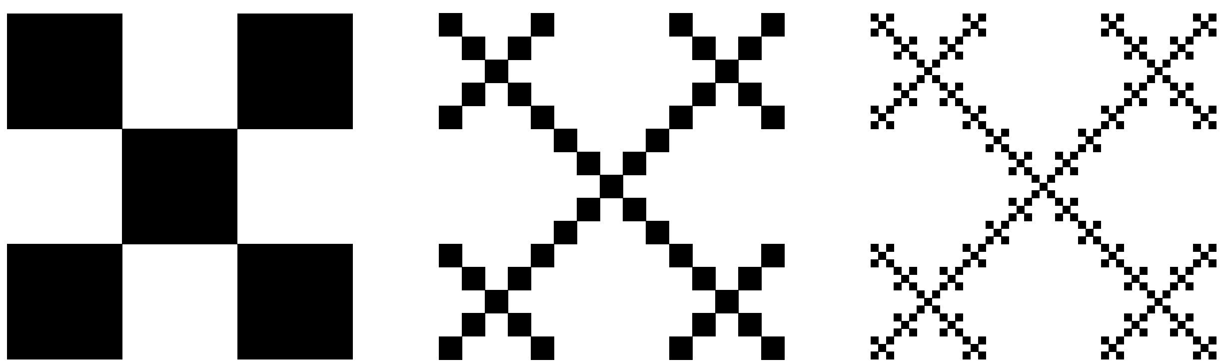

Figure 2. The sets F_1^ξ, F_2^ξ, F_3^ξ when $N=2$, $L_1=1$, $L_2=2$, $\xi=(1,2,1,\dots)$.

It is straightforward to check the following properties of H_n^ξ and H^ξ.

Lemma 4.3.
(a) $H_n^\xi \subset H_m^\xi$ *if* $n \le m$.
(b) $H_n^\xi \subset [0,a_n]^N$.
(c) If $m > n$ *then* $H_m^\xi \cap [0,a_n]^N = H_n^\xi$.
(d) $|H_n^\xi| = b_n$.
(e) H_n^ξ *and* H^ξ *are cubical sets.*

We call an *n-block* of H^ξ any subset of H^ξ isomorphic to H_n^ξ. The form of the n-blocks is determined by the elements $\xi_1, \dots \xi_n$ in the environment sequence: ξ_1 determines the smallest scale structure, then ξ_2 determines how these 1-blocks are pieced together to form the 2-blocks, and so on.

We define the graph $\Gamma^\xi = (G^\xi, E^\xi)$ to be the graph induced by the cubical set H^ξ. It is clear that Γ^ξ is a tree. We work with the natural weights on Γ^ξ, write d for the usual graph distance, and $B(x,r)$, $V(x,r)$ for balls and the volume function.

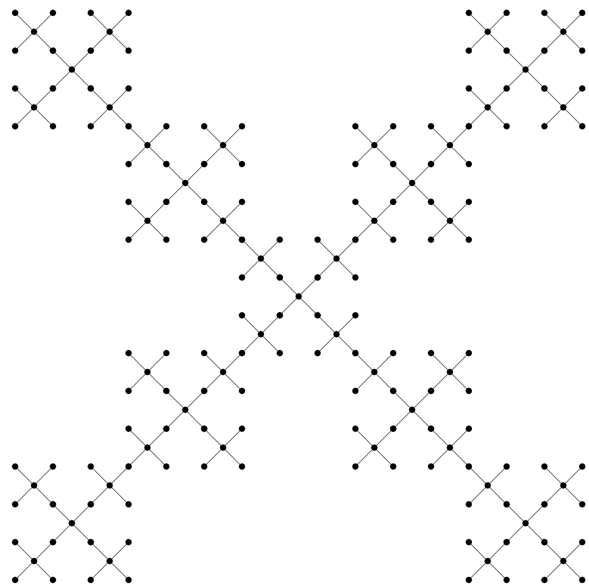

Figure 3. Part of the graph Γ^ξ, with $N = 2$, $L_1 = 1$, $L_2 = 2$, $\xi = (2, 1, 1, \dots)$.

For the remainder of this section we will allow the constants c_i to depend on N and L_2, but not on the environment sequence ξ. (Since $2L_1 + 1 \geq 3$ we do not need to include explicit dependence on L_1.) Note that the sequences a_n, b_n satisfy

$$3a_n \leq a_{n+1} \leq (2L_2 + 1)a_n, \qquad 5b_n \leq (2^N + 1)b_n \leq b_{n+1} \leq (2^N L_2 + 1)b_n. \tag{4.2}$$

For $x \in G^\xi$ let $D_k(x)$ be a k-block containing x. For some x there will be more than one of these – if so then we choose the one closest to the origin. We abuse notation and will also write $D_k(x)$ for the subgraph of Γ^ξ induced by the cubical set $D_k(x)$ – note that this subgraph contains $2^N b_k$ edges. We have $|D_k(x)| = b_k$, and that the diameter of $D_k(x)$ is a_k (in the L_∞ metric), and $2a_k$ in the graph metric d.

We now consider the volume growth of Γ^ξ.

Lemma 4.4. *(a) Let $n \geq 1$ and $a_n \leq R \leq a_{n+1}$. Then for $x \in G^\xi$,*

$$c_1 b_n \leq V(x, R) \leq c_2 b_n.$$

(b) Γ^ξ satisfies the volume doubling condition (VD).

Proof. (a) If $y \in D_{n-1}(x)$ then $d(x, y) \leq 2a_{n-1} < a_n \leq R$. So $D_{n-1}(x) \subset B(x, R)$, and

$$V(x, R) \geq 2^N |D_{n-1}(x)| = 2^N b_{n-1} \geq c_1 b_n.$$

The upper bound is proved in a similar way. $B(x, R)$ is contained in an L_∞ ball in $\mathbb{R}^N$ of radius $\frac{1}{2}a_{n+1}$, and so cannot intersect more than 2^N $(n + 1)$-blocks. Hence

$$V(x, R) \leq 2^{2N} |D_{n+1}(x)| \leq c_2 b_n.$$

(b) is immediate from (a) and (4.2). □

The exit times of balls are obtained in a similar way, but require a little more work.

Lemma 4.5. *Let $x \in G^\xi$, let $n \geq 1$, and let x_0 be the centre of $D_n(x)$. Let A be the set of 2^N corners of $D_n(x)$. Then (for the discrete time simple random walk on Γ^ξ)*

$$\mathbb{E}^{x_0} T_A = \tfrac{1}{2} a_n b_n.$$

Proof. Let $y_0 \in A$. The graph $D_n(x_0)$ consists of 2^N identical subgraphs, each containing one corner of $D_n(x_0)$ and connected at the centre x_0. By symmetry each of these subgraphs has b_n edges. Write Γ' for the subgraph graph containing y_0. Let X' be the simple random walk on Γ'. As Γ' is a tree, the effective resistance between any two points is just the graph distance between them, so $R_e(x_0, y_0) = d(x_0, y_0) = a_n$. Thus by (1.4) we have (for X')

$$\mathbb{E}^{x_0} T_{y_0} + \mathbb{E}^{y_0} T_{x_0} = a_n b_n,$$

and so by symmetry $\mathbb{E}^{x_0} T_{y_0} = \frac{1}{2} a_n b_n$. Finally, again using the symmetry of $D_n(x)$, we have (for X) $\mathbb{E}^{x_0} T_A = \frac{1}{2} a_n b_n$. □

Proposition 4.6. *Let $n \geq 1$ and $a_n \leq R \leq a_{n+1}$. Then for $x \in G^\xi$*

$$c_1 a_n b_n \leq \mathbb{E}^x \tau_{x,R} \leq c_2 a_n b_n.$$

Proof. Since $V(x, R) \leq c b_n$, and the effective resistance from x to $B(x, R)^c$ is at most a_{n+1}, the upper bound is clear from (1.5).

For the lower bound note first that $D_{n-1}(x)$, and every $(n-1)$-block touching $D_{n-1}(x)$, are in $B(x, R)$. If we write C for the set of centers of these $(n-1)$-blocks, any path from x to $B(x, R)^c$ must pass through a point in C. So, using Lemma 4.5, $\mathbb{E}^x \tau_{x,R} \geq \frac{1}{2} a_n b_n$. □

Proposition 4.7. *Γ^ξ satisfies (EHI).*

Proof. We begin by proving the elliptic Harnack inequality for k-blocks, rather than balls. Call two k-blocks *adjacent* if they meet at a point, and write $N_k(x)$ for the union of $D_k(x)$ and the k-blocks adjacent to $D_k(x)$. (There will be between 1 and 2^N of these.)

Suppose $x \in G^\xi$, $k \geq 1$ and $h > 0$ is harmonic on $N_k(x)$. We prove that there exists c_3, depending only on d and L_2 such that

$$\sup_{D_k(x)} h \leq c_3 \inf_{D_k(x)} h. \tag{4.3}$$

Write $B = D_k(x)$, and $\Lambda = \partial(B^c)$; this is the sets of corner points in B which are also in a k-block adjacent to B. By the maximum principle h attains its maximum and minimum on B at points in Λ – at x_1 and x_0 say. Write B' for the k-block adjacent to B which contains x_0, and Λ' for the set of all the corners of B and B' except x_0. Consider the simple random walk X started at x_0 and run until it first hits a point in Λ'. As $x_1 \in \Lambda'$, by symmetry

$$\mathbb{P}^{x_0}(X_{T_{\Lambda'}} = x_1) = \frac{1}{\#(\Lambda')} = \frac{1}{2^{N+1} - 2}.$$

So, since $h(X)$ is a martingale, we obtain $h(x_0) = \sum_{y \in \Lambda'} h(y) \mathbb{P}^{x_0}(X_{T_{\Lambda'}} = y) h(y) \geq c h(x_1)$, proving (4.3).

A chaining argument now gives the elliptic Harnack inequality in its standard version, for balls in Γ^ξ. □

Remark 4.8. By looking at Green's functions, as in the proof of Proposition 3(c), we could also prove that Γ^ξ satisfies (VSR) for any $\xi \in \Xi$. (We cannot use Proposition 3(c) directly here, since Γ^ξ need not satisfy (V_α).)

We now investigate the conditions on ξ under which Γ^ξ satisfies (V_α). Fix $\xi \in \Xi$, and set

$$h_j(n) = n^{-1} \sum_{r=1}^{n} 1_{(\xi_r = j)}, \quad j = 1, 2.$$

Note that $h_1(n) + h_2(n) = n$. Write

$$l_j = 2L_j + 1, \quad m_j = 2^N L_j + 1, \quad j = 1, 2.$$

Then we have

$$a_n = l_1^{h_1(n)} l_2^{h_2(n)}, \quad \text{and } b_n = m_1^{h_1(n)} m_2^{h_2(n)}. \tag{4.4}$$

Elementary calculations show that the function $f(x) = \log(2^N x + 1)/\log(2x + 1)$ is decreasing on $[1, \infty)$, so that we have

$$\frac{\log m_2}{\log l_2} < \frac{\log m_1}{\log l_1}. \tag{4.5}$$

Proposition 4.9. *(a) A necessary and sufficient condition that Γ^ξ satisfies $V(\alpha)$ for some $\alpha > 1$ is that there exists $c_1 < \infty$ and $p_1 \in [0, 1]$ such that*

$$|h_1(n) - np_1| \le c_1, \quad \textit{for all } n \ge 1. \tag{4.6}$$

(b) If (4.6) holds, then, writing $p_2 = 1 - p_1$, Γ^ξ satisfies (V_α) and $(E_{1+\alpha})$ with

$$\alpha = \frac{p_1 \log m_1 + p_2 \log m_2}{p_1 \log l_1 + p_2 \log l_2}. \tag{4.7}$$

Proof. Suppose first Γ^ξ satisfies $V(\alpha)$. Then there exist positive constants c_2, c_3 such that for all $x \in G^\xi$, $n \ge 1$,

$$c_2 b_n \le V(x, a_n) \le c_3 b_n, \quad c_2 a_n^\alpha \le V(x, a_n) \le c_3 a_n^\alpha.$$

Hence $(c_2/c_3) a_n^\alpha \le b_n \le (c_3/c_2) a_n^\alpha$ and so there exists $c_4 < \infty$ such that $|\log(b_n) - \alpha \log(a_n)| \le c_4$ for all $n \ge 1$. So, using (4.4), the function (in n)

$$\big| h_1(n) \big(\log m_1 - \alpha \log l_1 - \log m_2 + \alpha \log l_2 \big) + n \big(\log m_2 - \alpha \log l_2 \big) \big\| \tag{4.8}$$

is bounded by c_4. If $\log m_1 - \alpha \log l_1 - \log m_2 + \alpha \log l_2 = 0$ then we would have $\log m_2 - \alpha \log l_2 = 0$, which would imply that

$$\frac{\log m_2}{\log l_2} = \frac{\log m_1}{\log l_1},$$

contradicting (4.5). So writing

$$p_1 = -\frac{\log m_2 - \alpha \log l_2}{\log m_1 - \alpha \log l_1 - \log m_2 + \alpha \log l_2}$$

we deduce from the boundedness of (4.8) that (4.6) holds.

Now suppose that (4.6) holds. Then by (4.4) we obtain

$$\begin{aligned} l_1^{np_1} l_2^{np_2} (l_1 l_2)^{-c_1} &\le a_n \le l_1^{np_1} l_2^{np_2} (l_1 l_2)^{c_1} \\ m_1^{np_1} m_2^{np_2} (m_1 m_2)^{-c_1} &\le b_n \le m_1^{np_1} m_2^{np_2} (m_1 m_2)^{c_1}. \end{aligned}$$

This implies that

$$c_5 a_n^\alpha \le b_n \le c_6 a_n^\alpha, \quad n \ge 1,$$

with α given by (4.7). The conditions (V_α) and $(E_{1+\alpha})$ now follow using Lemma 4.4 and Proposition 4.6. □

Remark 4.10. See [BH, Theorem 6.2] for a similar result in the fractal context. The condition (4.6) on ξ is extremely strong, and shows (for example) that if the components ξ_i of ξ are chosen to be independent (non-trivial) random variables, then Γ^ξ fails to satisfy (V_α) for any α. Volume doubling, on the other hand, holds for all $\xi \in \Xi$ by Lemma 4.4(b).

Proof of Proposition 4. If $\alpha = 1$ we take Γ_α to be $\mathbb{Z}$. So assume $\alpha > 1$. Let $L_1 = 1$ and choose $d \ge 2$ large enough so that $\log m_1 / \log l_1 = \log(2^N + 1)/\log(2+1) > \alpha$. Then choose L_2 large enough so that $\log m_2 / \log l_2 = \log(2^N L_2 + 1)/\log(2L_2 + 1) < \alpha$. Therefore there exists $p_1 \in (0,1)$ such that

$$\alpha = \frac{p_1 \log m_1 + (1-p_1)\log m_2}{p_1 \log l_1 + (1-p_1)\log l_2}.$$

We can choose $\xi \in \Xi$ so that $|h_1(n) - np_1| \le 1$, so that the condition (4.6) of Proposition 4.9 is satisfied. Thus Γ^ξ satisfies (V_α) and $(E_{1+\alpha})$. □

5. Some additional examples

Let $\Gamma_i = (G_i, E_i)$, $i = 1, 2$ be two infinite connected graphs. Let $z_i \in G_i$. A *join* of Γ_1 and Γ_2 (at z_1, z_2) is the graph Γ obtained by identifying z_1 and z_2. Thus, formally, Γ has vertex set $G_1 \cup G_2 - \{z_2\}$ and edge set

$$\begin{aligned} \big\{\{x_i, y_i\} : \{x_i, y_i\} \in E_i,\ x_i, y_i \ne z_i,\ i = 1, 2\big\} \cup \big\{\{z_1, x_1\} : \{z_1, x_1\} \in E_1\big\} \\ \cup \big\{\{z_1, x_2\} : \{z_2, x_2\} \in E_2\big\}. \end{aligned}$$

If $(\Gamma_i, a^{(i)})$ are weighted graphs, then we define the weights a_{xy} for Γ by taking $a_{xy} = a_{xy}^{(i)}$ if $x, y \in G_i$.

From [D2] we have:

Lemma 5.1. *Let Γ_i be infinite connected weighted graphs, satisfying (V_{α_i}) and (E_{β_i}), with $\alpha_i < \beta_i$, $i = 1, 2$. Let Γ be a join of Γ_1 and Γ_2.*
(a) If Γ_i both satisfy (VSR), and $\beta_1 - \alpha_1 = \beta_2 - \alpha_2$, then Γ satisfies (EHI).
(b) If $\alpha_1 \neq \alpha_2$ then Γ does not satisfy (VD).

Lemma 5.2. *There exists a transient graph which satisfies (EHI) but not (VD).*

Proof. Examples of recurrent graphs satisfying (EHI) but not (VD) were given in [D2]. Let $1 \leq \alpha_1 < \alpha_2 < 2$, $\beta_1 - \alpha_1 = \beta_2 - \alpha_2$, and let Γ_i be graphs satisfying (EHI), (E_{β_i}) and (V_{α_i}). Such graphs exist by Theorem 2, and satisfy (VSR) by Proposition 3(b). Let Γ be a join of Γ_1 and Γ_2; by Lemma 5.1, Γ satisfies (EHI) but fails (VD).

To construct a transient graph choose $b = 9$, and take $M \geq 2$ sufficiently large so that $\lambda = \log M / \log b > \beta_1 - \alpha_1$. Let $\widetilde{\Gamma}$ be a graph constructed from Γ by the procedure of Section 2. Then $\widetilde{\Gamma}$ fails (VD), but satisfies (EHI).

Write $\widetilde{\Gamma}_i$ for the subgraphs of $\widetilde{\Gamma}$ with vertex sets $G_i \times U$. Then each of these subgraphs satisfies $(V_{\alpha_i+\lambda})$, (E_{β_i}) and (EHI), and so is transient by Proposition 3(d). Since $\widetilde{\Gamma}$ contains a transient subgraph, it is transient. □

Proof of Theorem 6. Let $\Gamma = (G, E)$ be the graph constructed in Lemma 5.2. Write x_0 for the common point of the components G_1 and G_2. Note that $\beta_1 < \beta_2$. Now consider the product graph $\Gamma^{(2)}$ with vertex set $G \times G$ and edge set

$$E^{(2)} = \big\{\{(x, y_1), (x, y_2)\} : \{y_1, y_2\} \in E\big\} \cup \big\{\{(x_1, y), (x_2, y)\} : \{x_1, x_2\} \in E\big\}.$$

If $X^{(1)}$ and $X^{(2)}$ are independent copies of the continuous time random walk on Γ then $Z_t = (X_t^{(1)}, X_t^{(2)})$, $t \geq 0$ is a continuous time random walk on $\Gamma^{(2)}$.

We now show that (EHI) fails for $\Gamma^{(2)}$. We use B to denote balls in Γ. Let $R \gg 1$, and choose $z_i \in G_i$ with $d(z_i, x_0) > 4R$. Write $Q = B(z_1, 2R) \times B(z_2, 2R)$, and τ_Q for the exit time of Z from Q. Let $\delta = (\beta_2 - \beta_1)/4$, and for simplicity suppose that $\beta_2 \leq 3$. Write

$$\tau_1 = \inf\{t \geq 0 : X_t^{(1)} \notin B(z_1, R)\},$$
$$\tau_2(y) = \inf\{t \geq 0 : X_t^{(2)} \notin B(y, R^{1-\delta/\beta_2})\}.$$

The estimates in Lemma 1.3(b) and (c) also hold for the continuous time random walks $X^{(i)}$, and can be applied to the exit times τ_i above as long as the relevant balls do not contain the join point x_0. So, if $t = R^{\beta_1+\delta}$ we deduce that

$$\mathbb{P}^{x_1}(\tau_1 > t) \leq c \exp(-c' R^\delta), \tag{5.1}$$

and if $y \in B(z_2, R)$ then

$$\mathbb{P}^y(\tau_2(y) \leq t) \leq c \exp\left(-c' \Big(\frac{R^{\beta_2(1-\delta/\beta_2)}}{t}\Big)^{1/(\beta_2-1)}\right) \leq c \exp(-c' R^\delta). \tag{5.2}$$

Let $D = \partial B(z_1, R) \times B(z_2, R^{1-\delta/\beta_2}) \subset \partial Q$ and let h be the harmonic function in Q with boundary values 1 on D and 0 on $\partial Q - D$, so that $h((x_1, x_2)) = \mathbb{P}^{(x_1,x_2)}(\tau_Q \in D)$. Note that h is harmonic on the ball in $\Gamma^{(2)}$ with centre (z_1, z_2) and radius $2R$. The estimate (5.1) implies that

$$h(z_1, z_2) > 1 - c\exp(-c' R^\delta),$$

while by (5.2) if $R^{1-\delta/\beta_2} < d(y, z_2) < R$ then

$$h(z_1, y) < c\exp(-c' R^\delta).$$

So we deduce that

$$\sup_{B(z,R)} h > ce^{c' R^\delta} \inf_{B(z,R)} h,$$

and since R can be as large as we like this shows that (EHI) fails on $\Gamma^{(2)}$. $\square$

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