**Continuous time simple random walk on a graph $\Gamma = (G, E)$.**

Let $G$ be an infinite (connected) graph. For $x, y \in G$ let

$$\mu_{xy} = \mu_{yx} = \begin{cases} 1 & \text{if } \{x, y\} \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Define the *vertex degree* $\mu_x = \mu(x) = \sum_y \mu_{xy}$.
Assume $\mu(x) < \infty$ for all $x \in G$, and extend $\mu$ to a measure on $G$. Set $V(x, r) = \mu(B(x, r))$.

*Dirichlet form:*

$$\mathcal{E}(f, f) = \tfrac{1}{2} \sum_{x,y} (f(x) - f(y))^2 \mu_{xy}.$$

The CTSRW on $\Gamma$ is the process

$$Y = (Y_t, t \in [0, \infty), P^x, x \in G)$$

associated with the Dirichlet form $(\mathcal{E}, L^2(G, \mu))$.

$Y$ waits at a point $x$ for an exponential time with mean 1, then moves to $y \sim x$ with probability $\mu_{xy}/\mu_x$.

*Heat kernel:*

$$q_t(x,y) = \frac{P^x(Y_t = y)}{\mu_y}.$$

*Laplacian:*

$$\mathcal{L}f(x) = \frac{1}{\mu_x}\sum_y \mu_{xy}(f(y) - f(x)).$$

*Heat equation:*

$$\frac{\partial}{\partial t} q_t(x_0, x) = \mathcal{L} q_t(x_0, x).$$

*Discrete Gauss-Green:*

$$\mathcal{E}(f,g) = (-\mathcal{L}f, g), \text{ where } (\cdot,\cdot) \text{ is the iner product in } L^2(G,\mu).$$

*Random walk on $\mathcal{C}_\infty$* This is just the CTSRW on the graph $(\mathcal{C}_\infty(\omega), \mathcal{O}(\omega)|_{\mathcal{C}_\infty(\omega)})$. More precisely, we have two probability spaces:
1. $(\Omega, \mathbb{P}_p)$ - the space for the percolation process.
2. $\Omega'$ ($= D([0,\infty), \mathbb{Z}^d)$ for example) – the space carrying the random walk. For each $\omega \in \Omega$, and $x \in \mathcal{C}_\infty(\omega)$ we then have the probabilty law $P^x_\omega$ for $Y$ started at $x$. Write $q^\omega_t(x,y)$ for the heat kernel of $Y$.

## Brief History

1. De Masi, Ferrari, Goldstein, Wick 1989: CLT and ('annealed') invariance principle.
2. Grimmett, Kesten, Zhang, 1993: $Y$ is recurrent if $d = 2$ and transient if $d \geq 3$.
3. Benjamini & Schramm (unpublished):

$$\mathbb{E}_p q_t^\omega(x,x) \geq ct^{-d/2}.$$

4. Benjamini, R. Lyons, Schramm 1999: bounded harmonic functions on $\mathcal{C}_\infty$ are constant. (Also results on general transitive graphs.)
5. Heicklen & Hoffmann (2001–2006):

$$\mathbb{E}_p(q_t^\omega(x,x)|x \in \mathcal{C}_\infty) \leq ct^{-d/2}(\log t)^{\delta_d}.$$

6. Mathieu & Remy (Ann Prob 2004): An isoperimetric inequality on $\mathcal{C}_\infty$ which implies that $\mathbb{P}_p$ a.s. on $\{\omega : x \in \mathcal{C}_\infty(\omega)\}$,

$$q_t^\omega(x,x) \leq ct^{-d/2}, \quad t \geq t_0(x,\omega).$$

7. Benjamini & Mossel (PTRF 2003): spectral gap inequalities for balls in $\mathcal{C}_\infty$.
8. MB (Ann Prob 2004): Gaussian upper and lower bounds on $q_t^\omega(x, y)$.
9. Sidoravicius & Sznitman (PTRF 2004), Berger & Biskup (PTRF 2007), Mathieu & Piatnitski (preprint 2005): quenched invariance principle.

**Theorem 3.1.** *Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exist r.v. $T_x(\omega)$ with*

$$\mathbb{P}_p(T_x \geq n; x \in \mathcal{C}_\infty) \leq c\exp(-n^{\varepsilon_d})$$

*and (non-random) constants $c_i = c_i(d,p)$ such that for $x, y \in \mathcal{C}_\infty(\omega)$:*

$$q_t^\omega(x,y) \leq \frac{c_1}{t^{d/2}} e^{-c_2|x-y|^2/t} \quad \textit{for } t \geq T_x(\omega) \vee |x-y|. \qquad (GUB)$$

$$q_t^\omega(x,y) \geq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t} \quad \textit{for } t \geq T_x(\omega) \vee |x-y|. \qquad (GLB)$$

Note also

$$q_t^\omega(x,y) \geq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t} \qquad \text{for } t \geq T_x(\omega) \vee |x-y|^{3/2}. \quad (GLB')$$

**Remarks.**
1. CTSRW on $\mathbb{Z}^d$ satisfies (GB)= (GUB)+(GLB) with $T_x \equiv 1$.
2. We need $t \geq D = |x-y|$ for Gaussian behaviour. If $t \ll D$ then (even for $\mathbb{Z}^d$)

$$P^x(Y_t = y) \approx e^{-cD} P(\texttt{Poisson}(t) = D) \approx e^{-cD} e^{-cD\log(D/t)}.$$

3. There exist 'long range' bounds (Carne-Varopoulos, Davies) which give good upper bounds for any infinite graph. These bounds are:

$$q_t(x,y) \le c \exp(-c_1 d(x,y)^2/t).$$

GUB follow if $D^2 \ge at \log t$, or $t < D^2/\log D$. For when $a$ is large enough

$$e^{-c_1 D^2/2t} \le e^{-(c_1 a)/2 \log t} = t^{-ac_1/2} < t^{-d/2}$$

4. The real zone of interest for these bounds is when

$$0 \le D \le (ct \log t)^{1/2}.$$

GLB′ is $D \le t^{2/3}$ so contains this zone.
5. Applications of the bounds often need control on $T_x(\omega)$ as well.
6. These bounds also hold (with minor modifications since $\mathcal{C}_\infty$ is bipartite) for the discrete time walk on $\mathcal{C}_\infty$.

A guide to how one proves this is:

**Theorem 3.2.** *(Delmotte, (1999), simplified.) Let* $\Gamma$ *be an infinite connected graph. The following are equivalent:*
*(a)* $\Gamma$ *satisfies* $(V_d)$ *and (famPI):*

$$c_1 r^d \leq \mu(B(x,r)) \leq c_2 r^d \quad \text{for } x \in G, r \geq 1.$$

*(b)* $q_t(x,y)$ *satisfies GB with* $T_x \equiv 1$,
*(c)* $\Gamma$ *satisfies Parabolic Harnack Inequality (PHI) and* $(V_d)$.

**Note:** 1. The more general version of this theorem, uses 'volume doubling' rather than $(V_d)$.
2. (famWkPI) works as well as (famPI).
3. $\Gamma$ satisfies (famPI) if for all $B = B(x,r)$ one has the PI:

$$\int_B |f - \overline{f}_B|^2 d\mu \leq C_P r^2 \mathcal{E}_B(f,f) \tag{PI}$$

4. The hard implication is (a) $\Rightarrow$ (b), (c). Delmotte proved (a) $\Rightarrow$ (c) using Moser's argument.
5. This extends to graphs earlier work on manifolds (Grigoryan, Saloff-Coste), metric spaces (Sturm).

**$(V_d)$ and (famWkPI) on $\mathcal{C}_\infty$**

*Volume.* We have

$$\theta(p) = \mathbb{P}_p(|\mathcal{C}(x) = \infty) = \mathbb{P}_p(x \in \mathcal{C}_\infty).$$

So by the ergodic theorem if $Q_n$ is the box side $n$ centre 0,

$$\frac{|\mathcal{C}_\infty \cap Q_n|}{|Q_n|} \to \theta(p).$$

Dividing a very big box into big boxes one deduces

$$\mathbb{P}_p(c_1 R^d \le |B_\omega(x, R)|) \ge 1 - e^{-cR^\delta}.$$

Also $|B_\omega(x, R)| \le cR^d$ always.

*Poincaré inequalities.* Using the isoperimetric inequality one gets

$$\mathbb{P}_p(\text{ (wkPI) holds for } B_\omega(x, R)) \ge 1 - e^{-cR^\delta}.$$

Hence by Borel-Cantelli for each $x$ one has $(V_d)$ and (wkPI) for $B(x, R)$ for all $R \ge R_x(\omega)$.