Remainder of proof of Theorem 3.1

Recall in a VG ball $B(x_0, R)$ all balls $B(y, r)$ are good for $N_B \leq r \leq R$.

Lemma 4.1. *Let $B = B(x_0, R)$ be very good. Then*

$$q_t(x,x) \leq ct^{-d/2} \quad 2N_B \log N_B \leq t \leq \frac{cR^2}{\log R}, \; x \in B(x_0, 8R/9).$$

Proof. (Argument of Kusuoka-Zhou.) Let $f_t(y) = q_t(x, y)$. Set

$$\varphi(t) = (f_t, f_t) = \sum_z q_t(x,z) q_t(z,x) \mu_z = q_{2t}(x,x).$$

Then

$$\varphi'(t) = 2(\frac{\partial}{\partial t} f_t, f_t) = 2(\mathcal{L} f_t, f_t) = -2\mathcal{E}(f_t, f_t).$$

Cover B by balls B_i of radius r, where r is to be chosen later.

Then

$$
\begin{aligned}
-\varphi'(t) = \mathcal{E}(f_t, f_t) &\geq \sum_i \mathcal{E}_{B_i}(f_t, f_t) \\
&\geq_? \sum_i \frac{1}{r^2} \int_{B_i} (f_t - \overline{f}_{t,i})^2 d\mu \qquad\qquad \text{(using PI)} \\
&= \frac{1}{r^2} \sum_i \int_{B_i} f_t^2 d\mu - \frac{1}{r^2} \sum_i \frac{1}{\mu(B_i)} \Big(\int_{B_i} f_t d\mu \Big)^2 \\
&\geq_? \frac{1}{r^2} \varphi(t) - \varepsilon(t) - \frac{1}{r^{2+d}} \qquad \text{(using } V_d\text{)} \\
&\geq_? \frac{1}{r^2} (\varphi(t) - r^{-d}) \qquad \text{(using Carne-Varopoulos).}
\end{aligned}
$$

If all balls were good we could choose r so that $\frac{1}{2}\varphi(t) = r^{-d}$ and this leads to an (easy) differential inequality which implies $\varphi(t) \leq ct^{-d/2}$.

For $\mathcal{C}_\infty$ one needs that all the balls $B_i = B(z_i, r)$ are good, and has to check that the values of r allowed by this lead to the bounds we want.

A similar pattern holds for the other parts of the proof.

Full Gaussian upper bounds – i.e. $q_t(x,y) \le ct^{-d/2}\exp(-cd(x,y)^2/t)$.

One usually finds that this is the hardest step in getting heat kernel estimates.

MB + R.F. Bass: (GUB) will follow by a chaining argument once one can prove

$$P^x(d(x,Y_t) \ge \lambda t^{1/2}) \to 0 \text{ uniformly in } x,t \text{ as } \lambda \to \infty. \qquad \text{(Ytight)}$$

Rich Bass realized that (YTight) can be proved using an argument of Nash. Define the entropy and mean distance by:

$$Q(t) = -\sum_y q_t(x_1,y)\log q_t(x_1,y)\mu_y, \quad M(t) = \sum_y d(x_1,y)q_t(x_1,y)\mu_y.$$

Lemma 4.2. *(Nash, 1959). (Real variable lemma). Let* $M,Q : [0,\infty) \to [-\infty,\infty)$*, with* $M(0)=0$*. If*

$$Q(t) \ge c_1 + \tfrac{d}{2}\log t, \quad M(t) \ge c_2 e^{Q(t)/d}, \quad Q'(t) \ge cM'(t)^2, \qquad (1)$$

then $c_1 t^{1/2} \le M(t) \le c_1 t^{1/2}$.

This bound on M implies (Ytight).

To prove the 3 inequalities in (1) all one needs is:

1. The upper bound $q_t(x_1, y) \le t^{-d/2}$,
2. The bound
$$\sum_{y \in G} e^{-\lambda d(x_1,y)} \le c\lambda^{-d},$$
which comes from the volume growth of G.

Note both of these relate to the structure of G as seen from x_1.

Lower bounds

Once one has (GUB), (GLB) follow by an argument of Fabes and Stroock, (also based on ideas of Nash). Basic idea: look at

$$H'(t) = \frac{d}{dt} \int_{B(x,R)} \log q_t(x, y)\mu(dy),$$

and obtain a differential inequality which forces $H(R^2) \ge -c$.

Input 'data': (GUB) plus (famPI).

Application of Gaussian bounds

1. One gets elliptic and parabolic Harnack inequalities.

2. Some parts of the upper bounds are used to obtain **invariance principles**. Recall that $Y = (Y_t, t \in [0,\infty), P^x_\omega)$ is the CTSRW on $\mathcal{C}_\infty$. Set

$$Y^{(n)}_t = n^{-1/2} Y_{nt}.$$

If $W = (W(t), t \geq 0)$ is a standard BM on $\mathbb{R}^d$ and $D \geq 0$ write $W_D(t) = W(Dt)$ for a BM run at speed D.

Invariance principle/ functional CLT:

$$Y^{(n)} \text{ converges weakly to } W_D. \tag{2}$$

Two kinds of functional CLT:
1. Averaged (annealed). For an event $F \subset \Omega' = D(\mathbb{R}_+, \mathbb{R})$ define the averaged law:

$$P^*(F) = \mathbb{E}_p(E^0_\omega(F)|0 \in \mathcal{C}_\infty).$$

Y is not Markov with respect to P^*.

Averaged FCLT: (2) holds on (Ω', P^*).
2. (Quenched). There exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega|0 \in \mathcal{C}_\infty) = 1$ such that for $\omega \in \Omega_0$ (2) holds on (Ω', P^0_ω).

Example. (For CLT).

Let $d = 1$ and consider the random graph obtained by taking $\mathcal{G}(\omega) = \mathbb{Z}_-$ with probability $\frac{1}{2}$, and $\mathcal{G}(\omega) = \mathbb{Z}_+$ with probability $\frac{1}{2}$. Let Y be the CTRSW on $\mathcal{G}$.

For each ω the process Y is a CTRSW with reflection at 0, either on $\mathbb{Z}_+$ or on $\mathbb{Z}_-$. So the quenched CLT obviously fails, but the averaged one holds.

Theorem 4.3. *(De Masi, Ferrari, Goldstein, Wick (1989)). AFCLT holds for CTSRW in a general stationary ergodic environment, provided* $\mathbb{E}\mu_e < \infty$*.*

Rather surprisingly, even for the case of an iid random environment given by taking $\mu_e \in [C_1, C_2]$ with $0 < C_1 < C_2$, the quenched FCLT was only proved in 2004 by Sidoravicius & Sznitman.

Theorem 4.4. *(Berger and Biskup, PTRF 2005, Mathieu and Piatnitski, to appear Proc. RS London, Sidoravicius and Sznitman, PTRF 2004). A quenched FCLT holds for the CTSWR on* $\mathcal{C}_\infty$*.*

Corrector and QFCLT

Most (all?) proofs of the QFCLT use the "corrector". We want to make $\mathbb{Z}^d$ into a 'harmonic graph' – i.e. we want a map $\varphi = \varphi_\omega : \mathbb{Z}^d \to \mathbb{R}^d$ such that $M_t = \varphi(Y_t)$ is a P_ω^0-martingale. So φ satisfies

$$L\varphi(x) = \sum_{y \sim x} \mu_{xy}(\varphi(y) - \varphi(x)) = 0,$$

or

$$\varphi(x) = \frac{1}{\mu_x} \sum_{y \sim x} \mu_{xy}\varphi(y). \tag{1}$$

Let $I : \mathbb{Z}^d \to \mathbb{Z}^d$ be the identity map: the corrector is

$$\chi(x) = \varphi(x) - I(x) = \varphi(x) - x.$$

Look at the slightly simpler problem of proving a CLT for Y. We have

$$\frac{Y_t}{t^{1/2}} = \frac{M_t}{t^{1/2}} - \frac{\chi(Y_t)}{t^{1/2}}. \tag{2}$$

Given control of $|\varphi(x) - \varphi(y)|^2$ when $x \sim y$ the martingale CLT gives that $t^{-1/2}M_t \Rightarrow N(0, D)$.

Recall that $\chi = \varphi - I$ where

$$\varphi(x) = P\varphi(x) = \frac{1}{\mu_x} \sum_{y \sim x} \mu_{xy} \varphi(y), \tag{1}$$

and

$$\frac{Y_t}{t^{1/2}} = \frac{M_t}{t^{1/2}} - \frac{\chi(Y_t)}{t^{1/2}}. \tag{2}$$

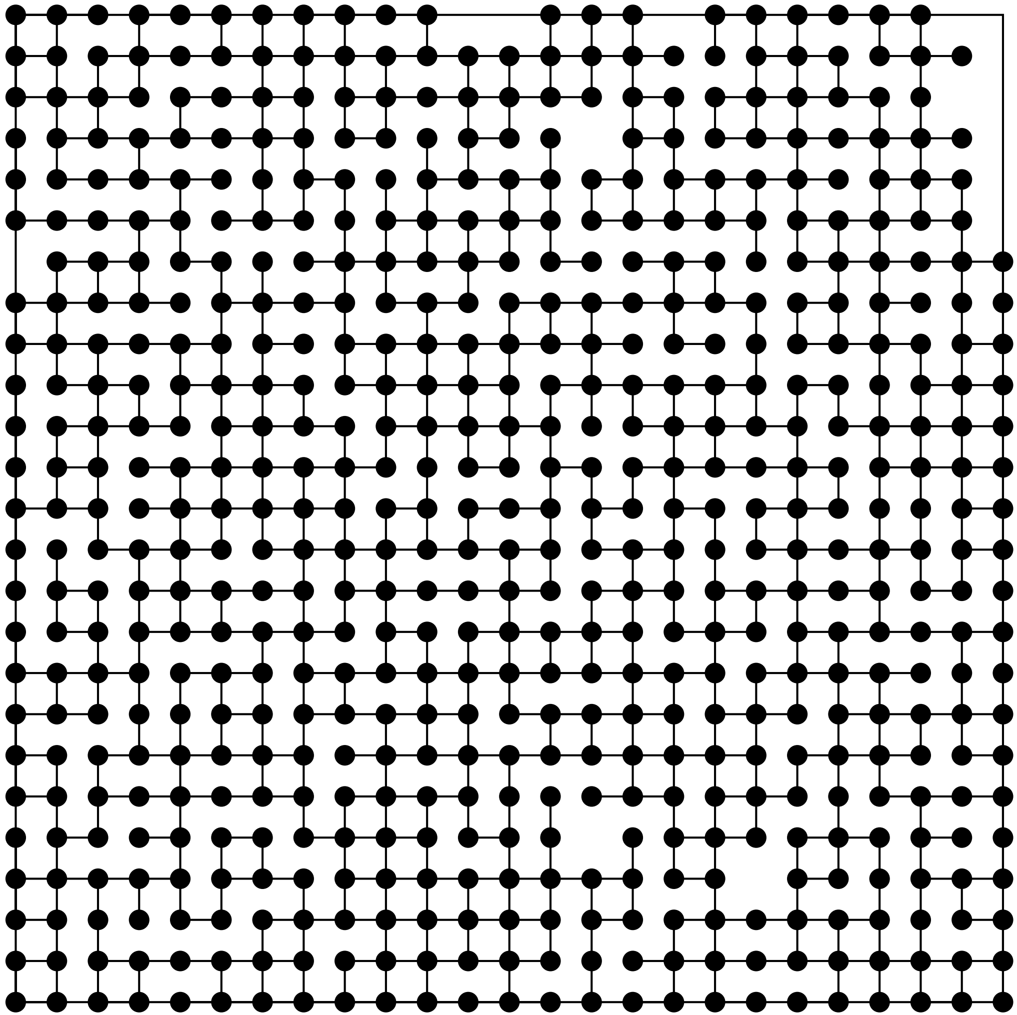
We have $t^{-1/2}\chi(Y_t) \to 0$ provided:

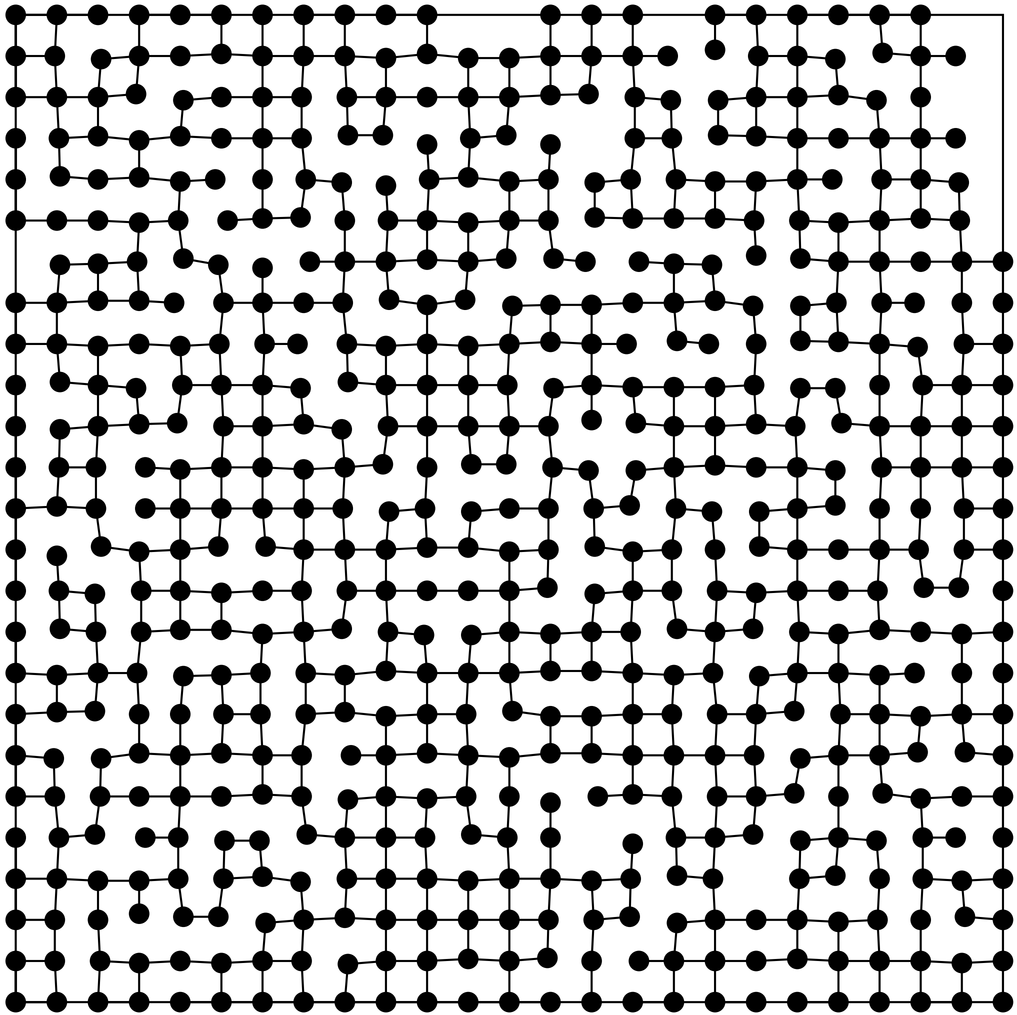
(a) $P_\omega^0(|Y_t| \geq \lambda t^{1/2})$ is small. (Follows from (GUB).)
(b) $|\chi(x)|/|x| \to 0$ as $|x| \to \infty$.

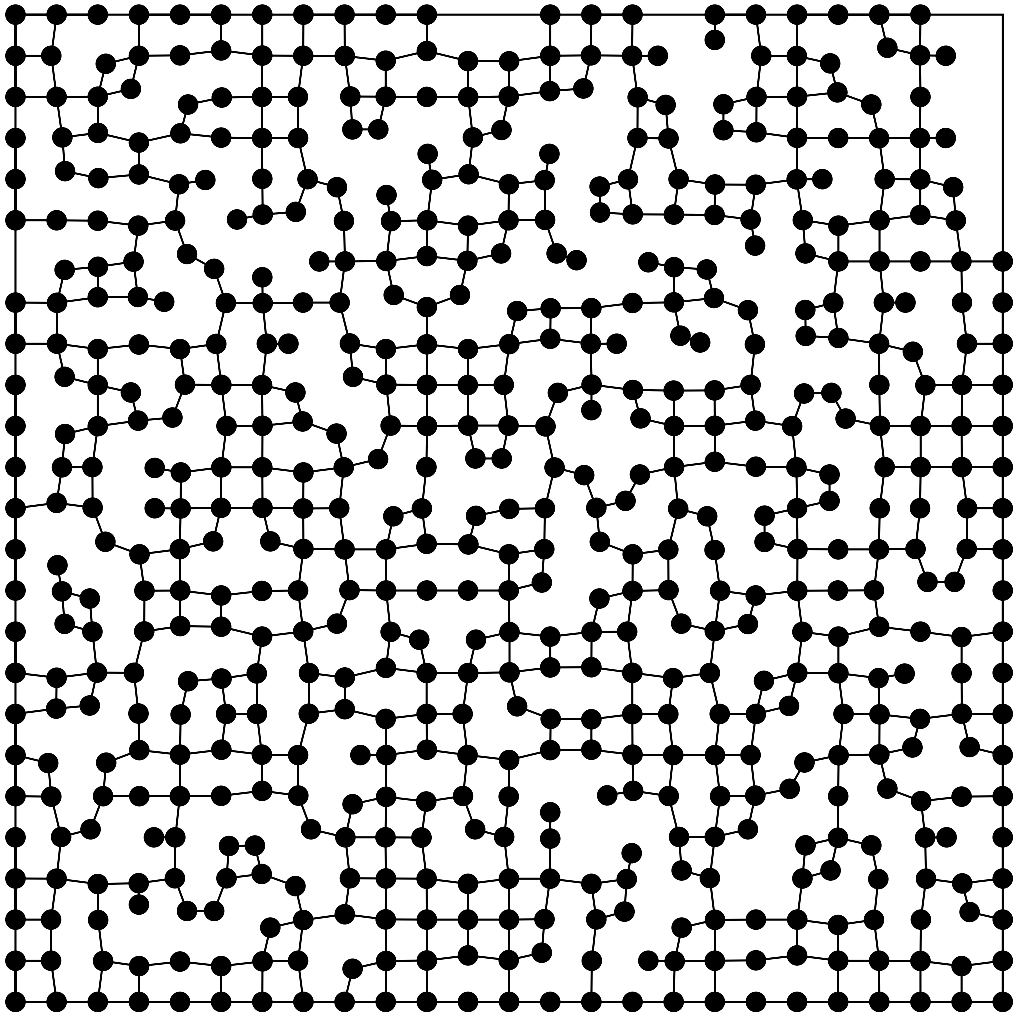
It is easy to solve (1) in a finite box, but hard to prove that $\chi = \varphi - I$ is small. Globally (1) has many solutions; we need one such that (b) holds.

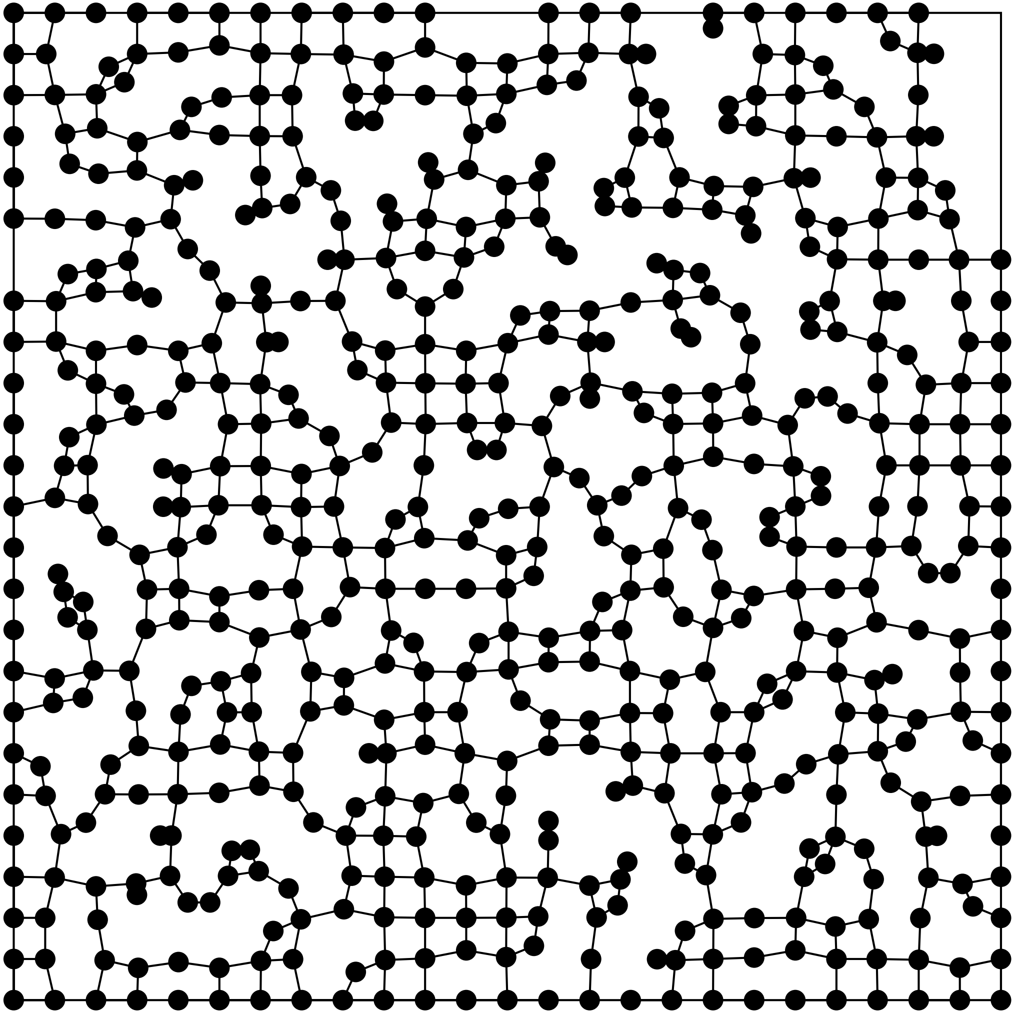
The following sequence shows how a numerical solution to (1) can be obtained in a finite box. Fixing the points on the boundary, one updates

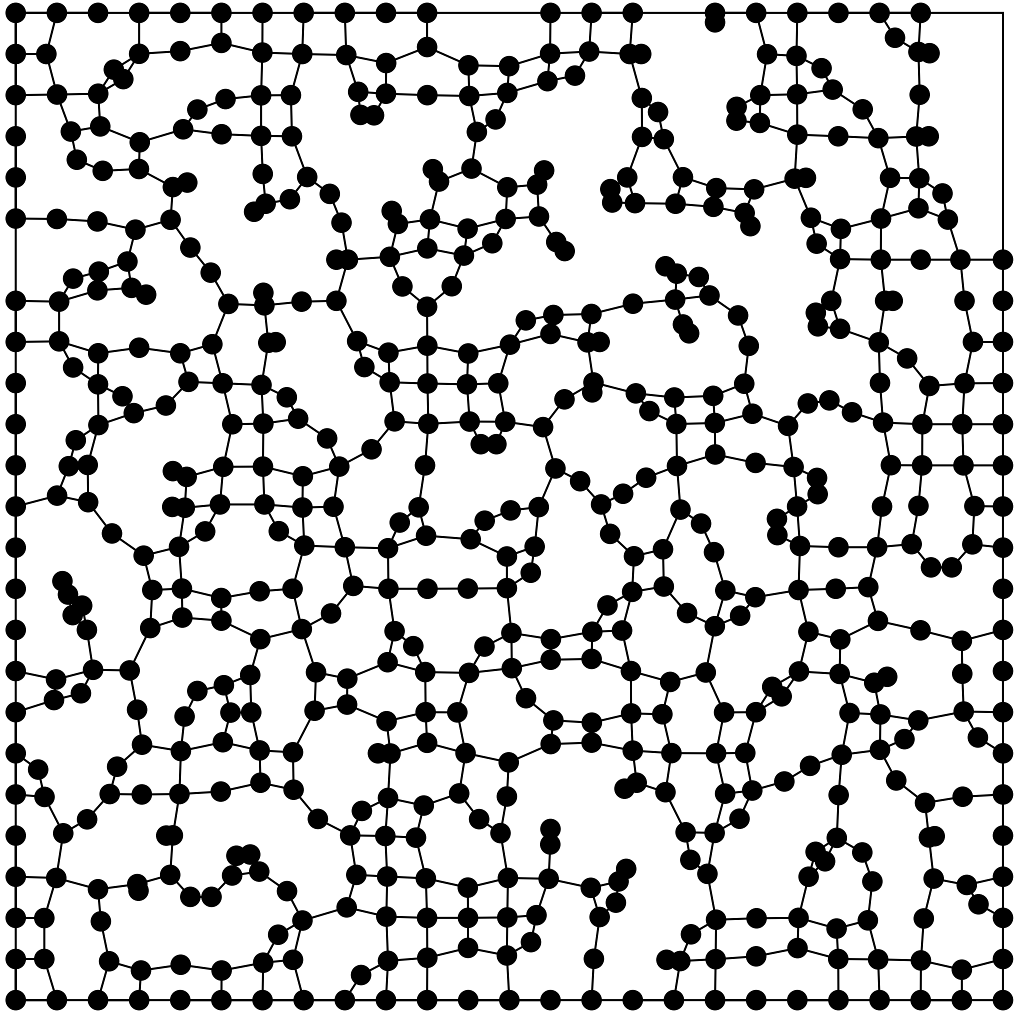
$$\varphi(x) \to 0.9\varphi(x) + 0.1P\varphi(x).$$

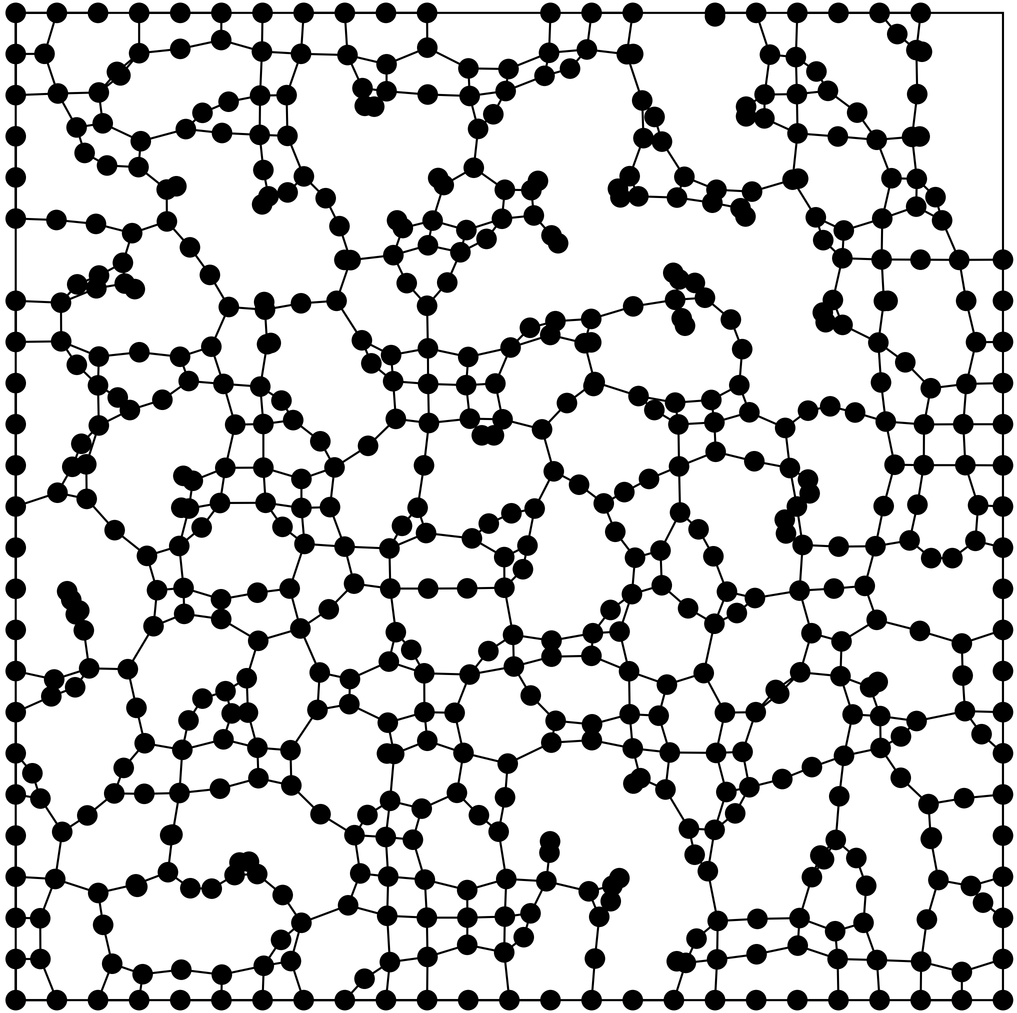


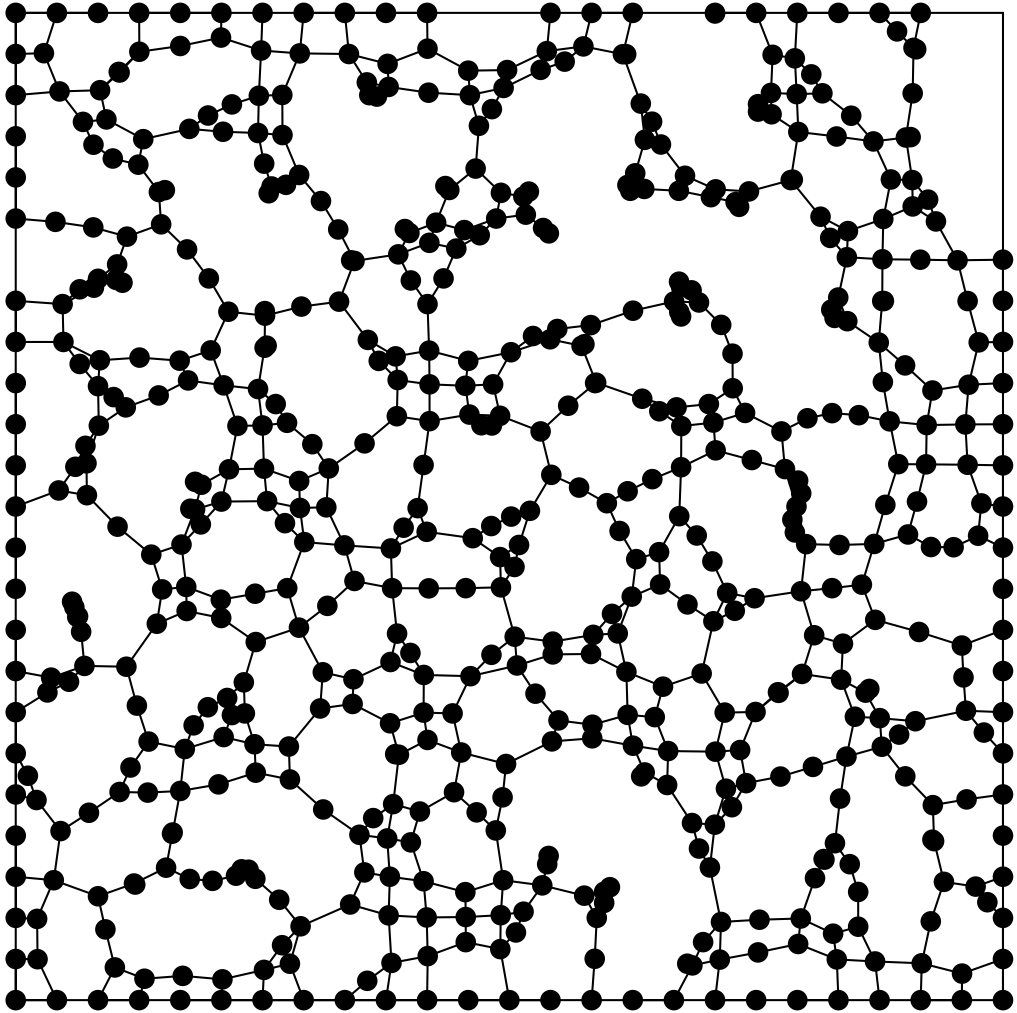


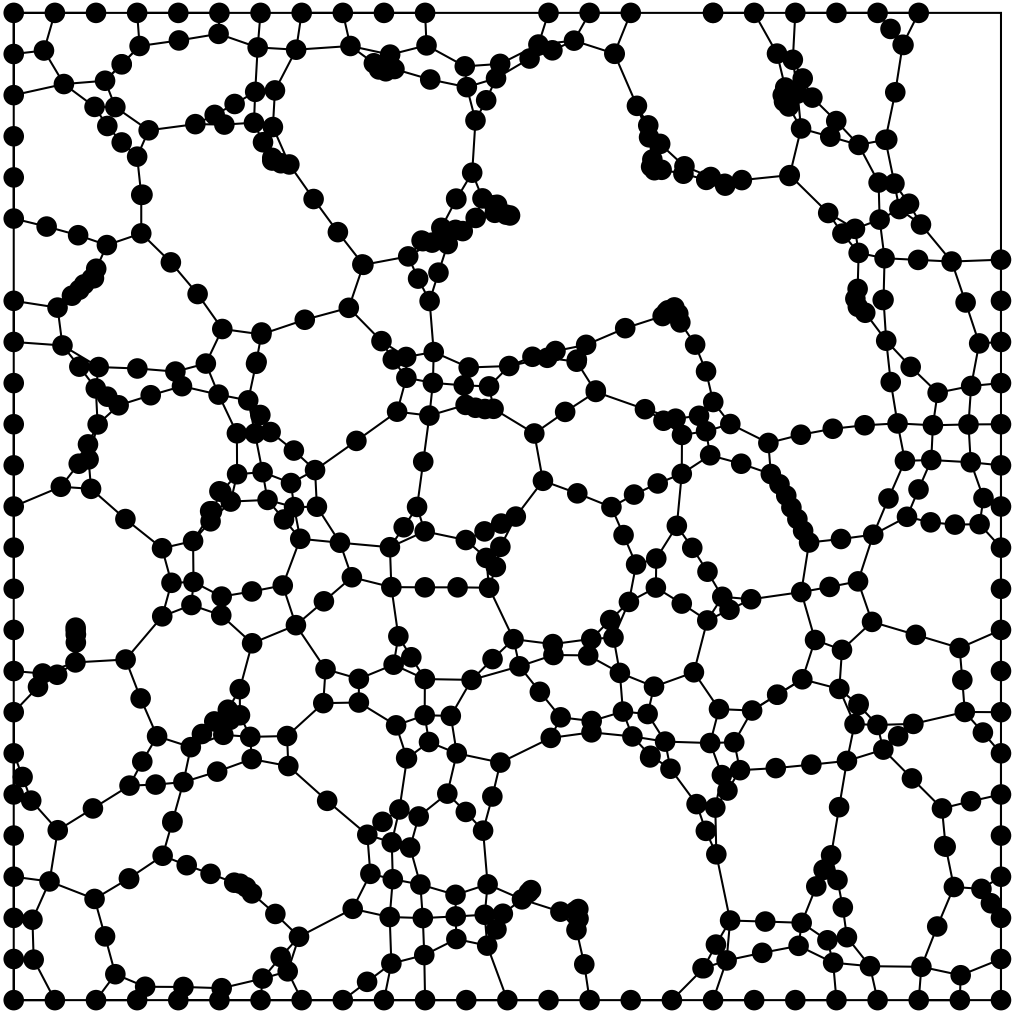


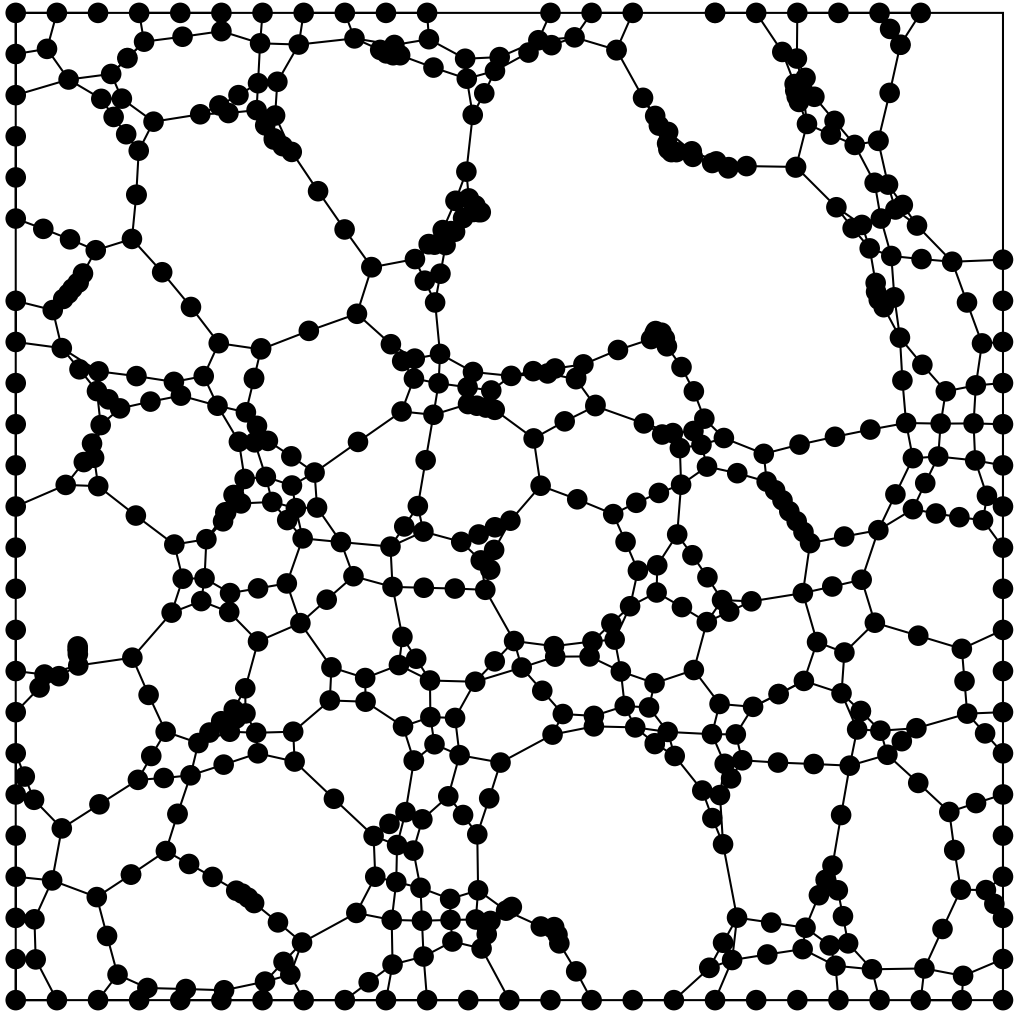


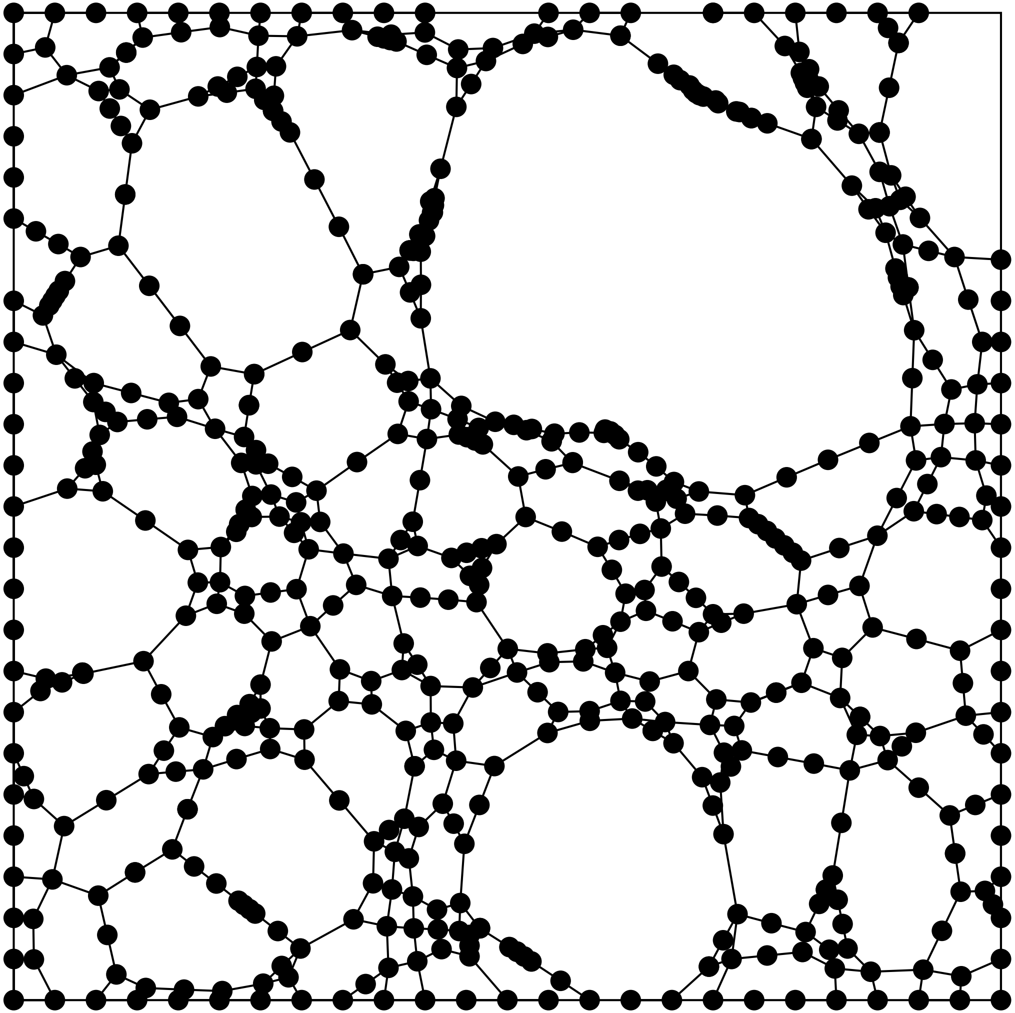


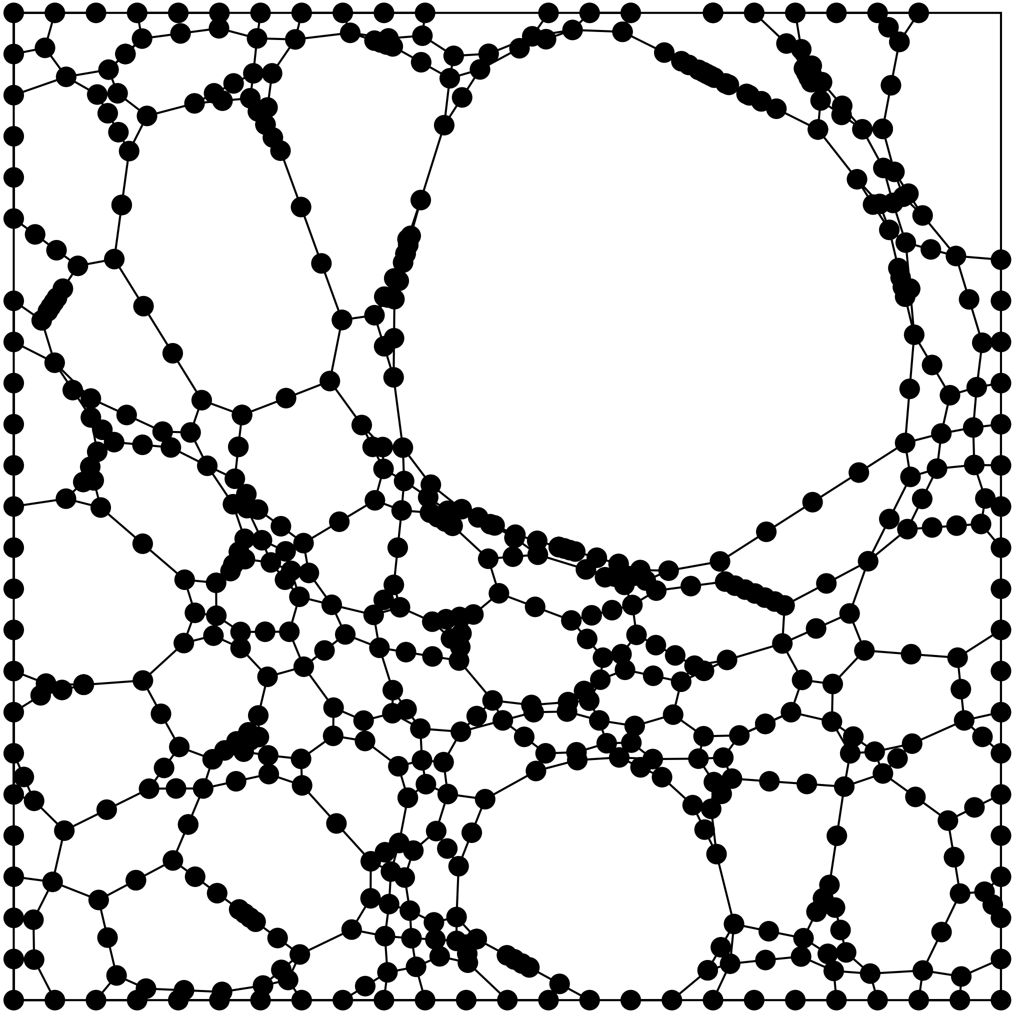












Key points in the proofs:

1. Construction of the corrector $\chi_\omega(x)$ by using the 'environment viewed from the particle' (Kipnis and Varadhan).

2. The increments of the corrector give a stationary ergodic process. So if z_n^k is the nth intersection of $\mathcal{C}_\infty$ with the kth coordinate axis, by the ergodic theorem

$$\frac{1}{n}\chi(z_n^k) \to C \text{ as } n \to \infty.$$

3. (Berger and Biskup). Subadditivity 'on average' of χ follows:

$$\lim_{n\to\infty} \frac{1}{n^d}|\{x \in \mathcal{C}_\infty : |x| \le n, |\chi(x)| > \varepsilon n\}| = 0.$$

4. (Biskup and Prescott). Subadditivity 'on average' plus the global upper bound $q_t^\omega(x,y) \le ct^{-d/2}$ implies pointwise sublinearity:

$$\lim_{n\to\infty} \max_{x\in\mathcal{C}_\infty, |x|\le n} \frac{|\chi(x)|}{n} = 0.$$

Hence one gets the QFCLT. One has $D = D(p,d) > 0$, but no effective way to calculate it.

Local Limit Theorem

Define the transition density for $Y_t^{(n)}$:

$$q_t^{(n,\omega)}(x,y) = n^{d/2} q_{nt}^{\omega}([n^{1/2}x],[n^{1/2}y]);$$

here [z] is the closest point in $\mathcal{C}_\infty$ to z.

The quenched CLT implies that in any small ball $U = B(x_0, \varepsilon)$

$$\int_U q_t^{(n,\omega)}(0,y)dy \to \int_U k_t(0,y)dy, \qquad (*)$$

where

$$k_t(x,y) = (2\pi Dt)^{-d/2} \exp(-|x-y|^2/2Dt).$$

Theorem 4.5. *(MB + Hambly)*

$$q_t^{(n,\omega)}(0,x) \to k_t(0,x).$$

Proof. The parabolic Harnack inequality implies Hölder continuity of $q_t^{(n,\omega)}(0,\cdot)$, and this allows one to get pointwise convergence from (*).

This is of interest for 2 reasons:
(a) In its own right.
(b) Understanding behaviour at $p = p_c$ should give insight into the the behaviour of (for example) $D(d, p)$ for p close to p_c.

Almost nothing is known!

The first difficulty is that (it is believed) $\mathbb{P}_{p_c}(|\mathcal{C}(0)| < \infty) = 1$, so all clusters are finite.

One can propose the study of RW on the 'incipient infinite cluster' (IIC). Roughly speaking one conditions on the event

$$F_n = \{\mathcal{C}(0) \text{ has size/diameter/... bigger than } n\},$$

and lets $n \to \infty$.

One would like that different choices for F_n all lead to the same limit. (This is not true for $p < p_c$.)

It seems hopeless to think about the IIC in those cases where it is not even known that $\theta(p_c) = 0$.

What has been done

1. Kesten (1986) constructed IIC for regular trees (quite easy) and bond percolation in $\mathbb{Z}^2$ (not so easy). Call the IIC $\widetilde{\mathcal{C}}$. Kesten then looked at RW Y^* on $\widetilde{\mathcal{C}}$ in these two cases.

For trees he proved an annealed FCLT, and showed that

$$n^{-1/3} Y^*_{nt} \Rightarrow Z_t,$$

where Z is a non-trivial process.

For $\mathbb{Z}^2$ he proved $\mathbb{E}^0 |Y^*_t|^2 \leq t^{1-\varepsilon}$.

2. Lace expansion methods give IIC for:

(a) Spread out oriented percolation in $\mathbb{Z}_+ \times \mathbb{Z}^d$ (van der Hofstad, den Hollander, Slade), when $d > 4$ and $L \geq L_0(d)$.

(b) Spread out percolation in $\mathbb{Z}^d$ (van der Hofstad, Jarai), $d > 6$, for $L \geq L_0(d)$.

(c) If d is large enough one also gets results for standard percolation/oriented percolation.

3. Aldous defined the 'continuum random tree' (CRT) – for trees, or in high dimensions, one expects that $\widetilde{\mathcal{C}}$ (rescaled) will converge to the CRT.

Random walk on the IIC $\widetilde{\mathcal{C}}$

Alexander-Orbach conjecture (1982): for IIC $\widetilde{\mathcal{C}}$ in $\mathbb{Z}^d$ for any $d \geq 2$ the transition density $q_t(x, y)$ satisfies

$$q_t(x, x) \approx t^{-2/3}.$$

If one defines the 'spectral dimension'

$$-\tfrac{1}{2} d_s(\widetilde{\mathcal{C}}) = \lim_{t \to \infty} \frac{\log q_t(x, x)}{\log t}$$

then the AO conjecture is that $d_s(\widetilde{\mathcal{C}}) = 4/3$.

1. Regular trees. MB, Kumagai obtained quenched heat kernel estimates, and showed that a quenched CLT does not hold. Proved AO conjecture in this context.

2. Averaged version proved earlier in: Jonsson, Wheater: The spectral dimension of the branched polymer phase of two-dimensional quantum gravity. *Nuclear Physics B* 1998.

3. Spread out oriented percolation in $\mathbb{Z}_+ \times \mathbb{Z}^d$. (MB, Jarai, Kumagai, Slade). Proved AO conjecture when $d > 6$.

What about $d = 2$ and SLE?

For the triangular lattice SLE methods give some precise information on critical percolation clusters. Camia and Newman have constructed the limit of critical percolation in a bounded region.

However, it is not clear if SLE methods give enough information on the structure of $\widetilde{\mathcal{C}}$ to give results on random walks.

Open problem. Is the AO conjecture true for $\widetilde{\mathcal{C}}$ for the triangular lattice in $d = 2$?

Computers have improved a lot since 1982, and I understand that numerical results now suggest the AO conjecture does not hold for $d = 2$.

But I do not know of any conjecture on what $d_s(\widetilde{\mathcal{C}})$ should be.