Random walks on random graphs

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Random walks in random environment on $\mathbb{Z}^d$

There are two common models for random walk in a random environment in $\mathbb{Z}^d$:

'Random walk in random environment' (RWRE). i.i.d. random transition probabilities out of each point $x \in \mathbb{Z}^d$. This model is hard, because in general it is not symmetric (i.e. reversible). There are still many open problems. I will not talk about this model.

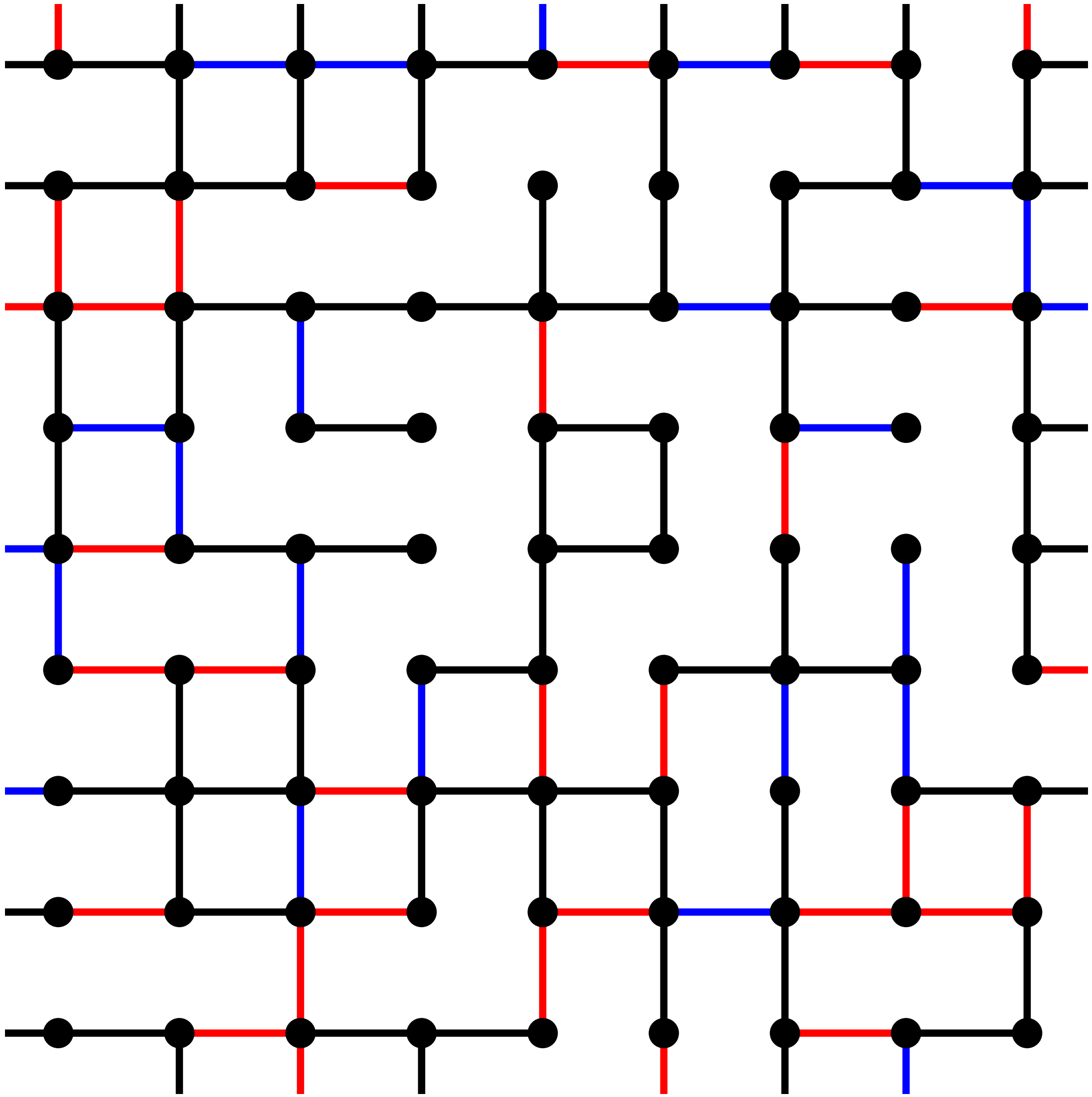
'Random conductance model' (RCM). As we'll see, now quite well understood.

Intuitive description: put i.i.d. random conductances (or weights) $\omega_e \in [0, \infty)$ on the edges of the Euclidean lattice $(\mathbb{Z}^d, E_d)$. Look at a continuous time Markov chain X_t with jump probabilities proportional to the edge conductances.

So if $X_0 = x$ then the jump probability from x to $y \sim x$ is

$$P_{xy} = \frac{\omega_{xy}}{\sum_{z \sim x} \omega_{xz}}.$$

Example



Bond conductivities: blue $\ll 1$, black ≈ 1, red $\gg 1$.

Definitions

Environment. Fix a probability measure κ on $[0,\infty)$. Let $\Omega = [0,\infty]^{E_d}$ be the space of environments, and let $\mathbb{P}$ be the probability law on Ω which makes the coordinates $\omega_e, e \in E_d$ i.i.d. with law κ.
Choose a 'speed measure' $\nu_x(\omega)$, $x \in \mathbb{Z}^d$. (How? See next slide....)

Random walk. Let $\Omega' = D([0,\infty), \mathbb{Z}^d)$ be the space of (right cts left limit) paths. For each $\omega \in \Omega$ let P^x_ω be the probability law on Ω' which makes the coordinate process $X_t = X_t(\omega')$ a Markov chain with generator

$$\mathcal{L}_\nu f(x) = \frac{1}{\nu_x} \sum_{y \sim x} \omega_{xy}(f(y) - f(x)).$$

Write $\omega_{xy} = 0$ if $x \not\sim y$, and $\mu_x = \sum_y \omega_{xy}$.
Then ν is a stationary measure for X, X is reversible (symmetric) with respect to ν, and the overall jump rate out of x is μ_x/ν_x.

Choice of the 'speed measure' ν

Any choice of ν is possible, but there are two particularly natural ones:

1. $\nu_x = \mu_x = \sum_y \omega_{xy}$. This makes the times spent at each site x before a jump i.i.d. $\mathtt{exp}(1)$. (Provided $\mu_x > 0$). Call this the **constant speed random walk (CSRW)**.
Minor nuisance: if $\mu_x = 0$ then each edge out of x has conductivity zero. Note that X never jumps into such a point. We will soon remove these points from the state space.

2. $\nu_x = \theta_x \equiv 1$ for all x. This makes the times spent at x i.i.d. $\mathtt{exp}(\mu_x)$. Call this the **variable speed random walk or (VSRW).**

For either choice $\nu = \mu$ or $\nu = 1$ define the heat kernel (transition density with respect to ν) by

$$q_t^\omega(x,y) = \frac{P_\omega^x(X_t = y)}{\nu_y} = q_t^\omega(y,x).$$

Questions. Long time behaviour of X and q_t^ω.

Percolation and positive conductance bonds

If $\omega_e = 0$ then X cannot jump across the edge e. Define a percolation process associated with the environment ω by taking

$$\{x, y\} \text{ is open } \Leftrightarrow \omega_{xy} > 0,$$

and let $\mathcal{C}_x = \mathcal{C}_x(\omega)$ be the open cluster containing x – i.e. the set of $y \in \mathbb{Z}^d$ such that x and y are connected by a path of open edges. So

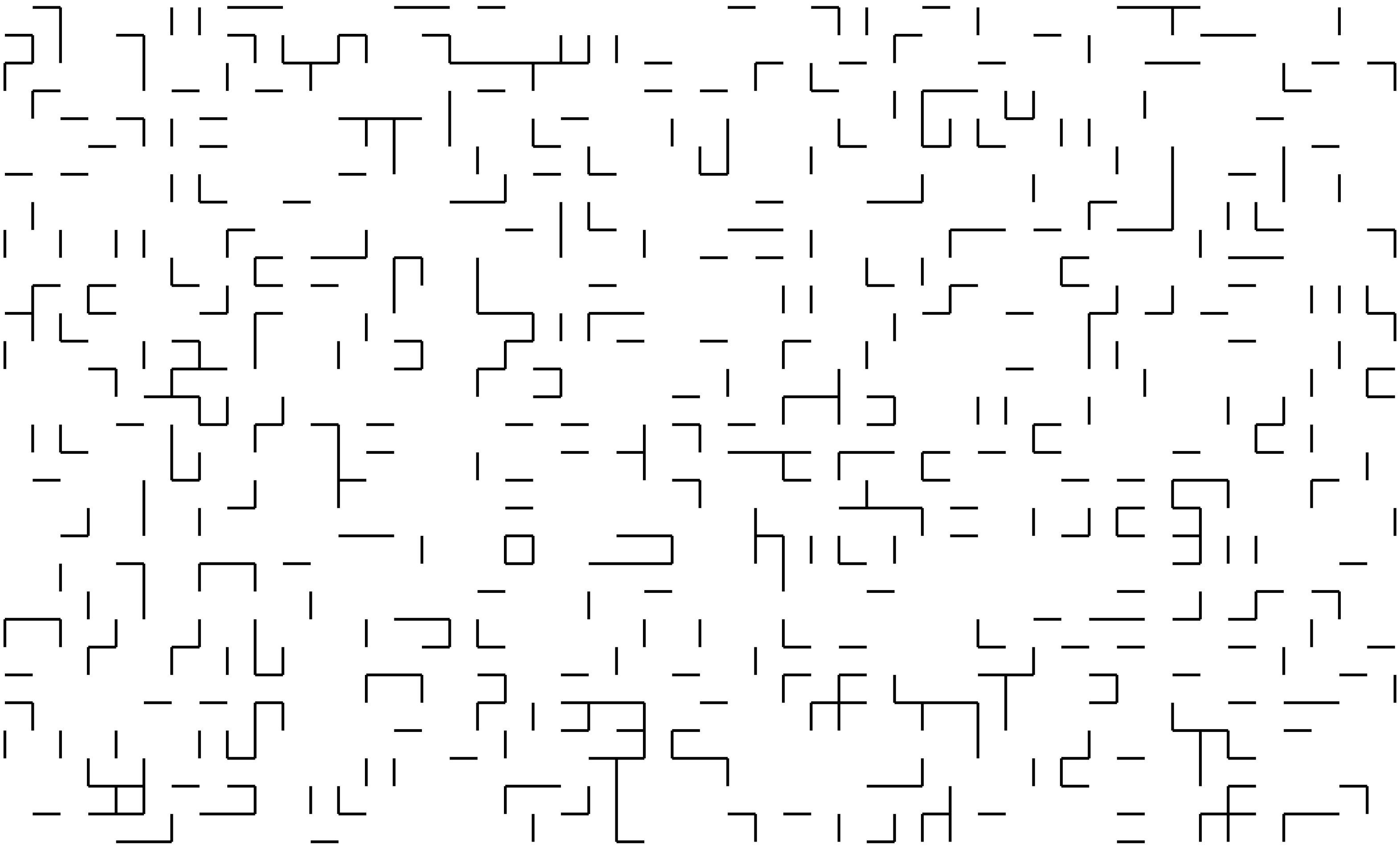
$$P^x_\omega(X_t \in \mathcal{C}_x \text{ for all } t \geq 0) = 1.$$

Let

$$p_+ = \mathbb{P}(\omega_e > 0).$$

If $p_+ < p_c(d)$, the critical probability for bond percolation in $\mathbb{Z}^d$, then all the open clusters are finite, so X is trapped in a small finite region.

Bond percolation with $p_+ = 0.2$.



GB and QFCLT

Assumption. From now on assume

$$p_+ > p_c(d).$$

Write $\mathcal{C}_\infty(\omega)$ for the ($\mathbb{P}$-a.s. unique) infinite cluster of the associated percolation process, and consider X started only at points in $\mathcal{C}_\infty(\omega)$.

Problems. What we would like to have (but may not):

Gaussian bounds (GB) on $q_t^\omega(x, y)$.

Quenched functional CLT with diffusivity σ^2: Let $X_t^{(n)} = n^{-1/2} X_{nt}$, and W be a BM($\mathbb{R}^d$). Then for $\mathbb{P}$-a.a. ω, under P_ω^0,

$$X^{(n)} \Rightarrow \sigma W.$$

(In particular is $\sigma^2 > 0$?)

Gaussian bounds in the random context

For GB one wants:

For each $x \in \mathbb{Z}^d$ there exist r.v. $T_x(\omega) \geq 1$ with

$$\mathbb{P}(T_x \geq n, x \in \mathcal{C}_\infty) \leq c \exp(-n^{\varepsilon_d}) \qquad (T)$$

and (non-random) constants $c_i = c_i(d,p)$ such that the transition density of X satisfies,

$$\frac{c_1}{t^{d/2}} e^{-c_2|x-y|^2/t} \leq q_t^\omega(x,y) \leq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t}, \qquad (GB)$$

for $x, y \in \mathcal{C}_\infty(\omega)$, $t \geq \max(T_x(\omega), c|x-y|)$.

1. The randomness of the environment is taken care of by the $T_x(\omega)$, which will be small for most points, and large for the rare points in large ‘bad regions’.
2. Good control of the tails of the r.v. T_x, as in (T), is essential for applications.

Consequences of GB and QFCLT

GB lead to Harnack inequalities, which imply Hölder continuity of harmonic functions, solutions of the heat equation on $\mathcal{C}_\infty$, and Green's functions.

However: define the *slab* $S_N = \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : |x_1| < N\}$. We would guess that

$$\lim_{N\to\infty} P^0_\omega(X \text{ leaves } S_N \text{ upwards, i.e. in } \{x : x_1 > 0\}\ ) = \tfrac{1}{2}, \qquad (*)$$

but (GB) do not imply this.

QFCLT does imply (*), but on its own does not give good control of harmonic functions.

If one has both GB and QFCLT then one obtains a local limit theorem for $q^\omega_t(x, y)$ (MB + B. Hambly) and good control of Green's functions.

Averaged FCLT

Recall that

$$X_t^{(n)} = n^{-1/2} X_{nt}.$$

For events $F \subset \Omega \times \Omega'$ define

$$\mathbb{E}^*(1_F) = \mathbb{E}E_\omega^0 1_F, \qquad \text{and write } \mathbb{P}^* \text{ for the associated probability.}$$

Theorem A. (De Masi, Ferrari, Goldstein, Wick 1989). Let $(\omega_e, e \in E_d)$ be a general stationary ergodic environment. Assume that $\mathbb{E}\omega_e < \infty$. Under $\mathbb{P}^*$, $X^{(n)} \Rightarrow \sigma W$, where W is a BM, and $\sigma^2 \geq 0$.

This gives a FCLT, but one where we average over all environments.

This paper used the Kipnis-Varadhan approach of ‘the environment seen from the particle’.

We have no example of a environment satisfying the conditions of Theorem A for which the QFLCT fails. But work in progress (MB-Burdzy-Timar) shows that more generally it is possible to have AFCLT without QFCLT.

Special cases for the law of ω_e.

1. "Elliptic": $0 < C_1 \le \omega_e \le C_2 < \infty$.
2. "Percolation": $\omega_e \in \{0, 1\}$ (and $p_+ > p_c$.)
3. "Bounded above": $\omega_e \in [0, 1]$ (and $p_+ > p_c$.)
4. "Bounded below": $\omega_e \in [1, \infty)$.

In Cases 1–3 there is little difference between the CSRW and the VSRW.

1. *Elliptic case.* GB follow from general results of Delmotte (1999). QFCLT was only proved in full generality by Sidoravicius and Sznitman (2004).

2. *Percolation.* GB proved by MB (2004). QFCLT proved by Sidoravicius and Sznitman (2004), Berger and Biskup (2007), Mathieu and Piatnitski (2007).

3. *Bounded above.* Berger, Biskup, Hoffmann, Kozma (2008) showed GB may fail. (We will see why later.) QFCLT holds with $\sigma^2 > 0$: Biskup and Prescott (2007), Mathieu (2007).

4. *Bounded below.* GB for VSRW, and QFCLT for both VSRW and CSRW proved by MB, Deuschel (2010).

Call the VSRW Y and the CSRW X.

General case

Theorem 1. *(S. Andres, MB, J-D. Deuschel, B.M. Hambly). Assume that* $p_+ > p_c$*. Then a QFLT holds both for the VSRW and for the CSRW (with diffusion constants* σ_V^2*,* σ_C^2*). Further* $\sigma_V^2 > 0$ *always, and if* $\mathbb{E}\omega_e < \infty$ *then*

$$\sigma_C^2 = \frac{\sigma_V^2}{\mathbb{E}_1 \mu_0}.$$

Here $\mathbb{E}_1(\cdot) = \mathbb{E}(\cdot | 0 \in \mathcal{C}_\infty)$.

In general GB fail, for the reason identified by Berger, Biskup, Hoffmann, Kozma. However, when $d \geq 3$ we do have bounds on Green's functions.

Green's functions

Let $d \geq 3$ and define the Green's function

$$g_\omega(x,y) = \int_0^\infty q_t^\omega(x,y)dt.$$

($g_\omega(x,\cdot)$ is harmonic on $\mathbb{Z}^d - \{x\}$ and is the same for the CSRW and the VSRW.)

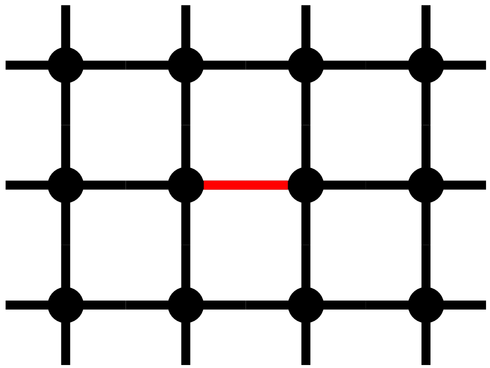
Theorem 2. *(ABDH) $d \geq 3$, and assume $\omega_e \geq 1$. There exists a constant C such that for any $\varepsilon > 0$ there exists $M = M(\varepsilon,\omega)$ with $\mathbb{P}(M < \infty) = 1$ such that*

$$\frac{(1-\varepsilon)C}{|x|^{d-2}} \leq g_\omega(0,x) \leq \frac{(1+\varepsilon)C}{|x|^{d-2}} \quad \textit{for } |x| > M(\varepsilon,\omega).$$

Why are the CSRW and VSRW different ?

Recall the jump rates from x to y are:

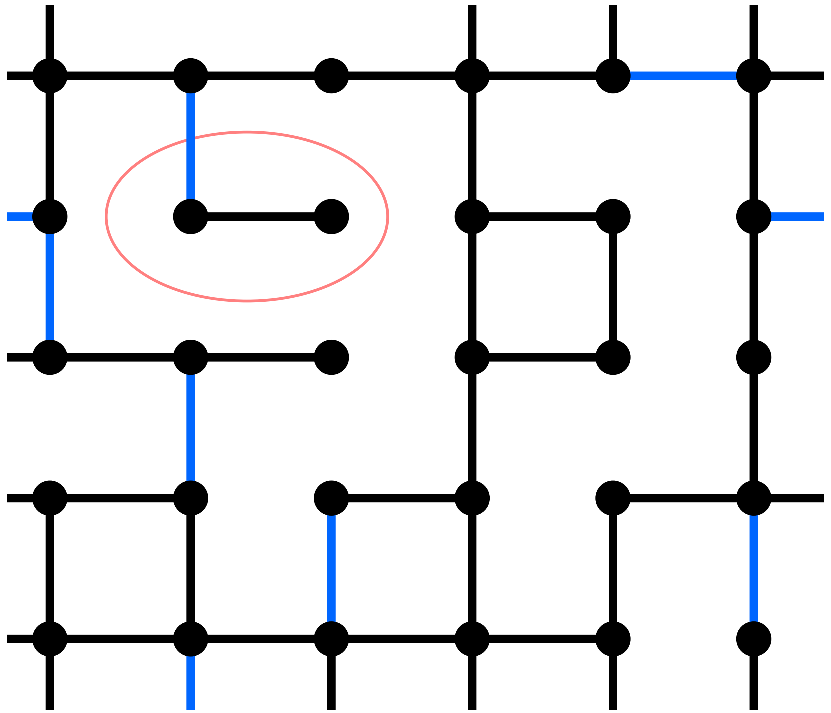
(1) ω_{xy} for the VSRW,
(2) $\omega_{xy}/\sum_z \omega_{xz}$ for the CSRW.
Consider a configuration like this, where black bonds have $\omega_e \approx 1$ and the red bond $\{x, y\}$ has $\omega_{xy} = K \gg 1$.



The red bond acts as a 'trap'. Both walks jump across the red bond $O(K)$ times before escaping. Since each jump takes time $O(1)$, the CSRW takes time $0(K)$. Each jump takes the VSRW time $O(K^{-1})$, so the total time is only $O(1)$.

Why may GB fail?

Consider a configuration like this:



The blue bond with $0 < \omega_e << 1$ attached to the black bond acts as a kind of 'battery', and can hold the random walk for a long period.

Overview of proofs

1. The basic idea is to follow Kipnis-Varadhan and construct the ‘corrector’ $\chi(\omega, x) : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ such that for $\mathbb{P}$-a.a. ω

$$M_t = Y_t - \chi(\omega, Y_t) \text{ is a } P_\omega^0\text{-martingale .}$$

Then one proves:

$$M_t^{(n)} = n^{-1/2} M_{nt} \Rightarrow \sigma W_t, \tag{1}$$

$$n^{-1/2}\chi(\omega, Y_{nt}) \to 0, \text{ in } P_\omega^0 \text{ probability.} \tag{2}$$

2. A general CLTs for martingales (Helland, 1982) gives convergence in (1).

3. To control the corrector one needs something like:

$$\lim_{n\to\infty} \max_{|x|\le n} \frac{\chi(\omega, x)}{n} = 0. \tag{3}$$

In $d \geq 3$ proving this requires upper bounds on $q_t^\omega(x, y)$.

Heat kernel bounds – 1

As we have seen, in the general case one cannot expect to have GB for $q_t^\omega(x,y)$, due to the bounds with very small conductivity:

$$S_\varepsilon = \{e \in E_d : \omega_e \in (0,\varepsilon)\}.$$

However, we can follow a strategy introduced by Biskup-Prescott and Mathieu. Choose $\varepsilon > 0$ small enough so that S_ε is a very sparse set. Then the percolation process

$$\mathcal{O}' = \{e : \omega_e \geq \varepsilon\}$$

is still supercritical. Call the infinite cluster $\mathcal{C}'_\infty$.

If Y_t is the original process, let Y' denote Y looked at only when Y is on $\mathcal{C}'_\infty$. Then one *does* have GB for Y', and hence can obtain control of the corrector for Y', and so for Y.

Heat kernel bounds - 2

A general guide to proving heat kernel bounds is given by the following theorem.

Theorem B. *(T. Delmotte, 1999). Let $G = (V, E)$ be a (locally finite) graph, with distance $d(x, y)$. The following are equivalent:*
(a) The heat kernel $q_t(x, y)$ satisfies (GB).
(b) G satisfies Volume Doubling and Poincaré inequality: VD+PI.
[(c) Solutions of the heat equation on G satisfy a PHI.]

Here (GB) means that if $t \geq 1 \vee d(x, y)$

$$\frac{\exp(-c_1 d(x, y)^2/t)}{|B(x, c_1 t^{1/2})|} \leq q_t(x, y) \leq \frac{\exp(-c_2 d(x, y)^2/t)}{|B(x, c_2 t^{1/2})|}.$$

Note that $|B(x, t^{1/2}|$ can be replaced by $ct^{d/2}$ if, as is the case on $\mathbb{Z}^d$,

$$cr^d \leq |(B(x, r)| \leq c' r^d.$$

VD and PI for graphs

Poincaré inequality (PI): For every ball $B = B(x,r)$, and $f : B \to \mathbb{R}$,

$$\sum_{x \in B} (f(x) - \overline{f}_B)^2 \mu_x \le C_P r^2 \sum_{x \sim y, x,y \in B} (f(y) - f(x))^2$$

Here $\overline{f}_B$ is the real number which minimises the LHS.

An example of a graph for which the PI fails is two copies of $\mathbb{Z}^d$ (with $d \ge 2$) connected at their origins.

Volume doubling (VD): for each $x \in V$, $r \ge 1$,

$$|B(x,2r)| \le C_D |B(x,r)|.$$

This holds for example on $\mathbb{Z}^d$ where $|B(x,r)| \asymp r^d$.

Heat kernel bounds - 3

Delmotte's theorem was based on the characterization of PHI on manifolds due to Grigoryan and Saloffe-Coste. This in turn was ultimately based on work on Moser in the early 1960s.

In the random graph case, it has (so far) proved difficult to adapt Moser's methods. But other techniques due to Nash can be used, and give a version of Theorem B which holds for supercritical percolation clusters.

A similar approach also works in for the truncated cluster $\mathcal{C}'_\infty$, and gives (GB) for Y'.

This then leads to good control of the corrector χ and hence to a FLCT for the VSRW.

CSRW

Once we have the QFCLT for the VSRW Y it is easy to get it for the CSRW X. Set

$$A_t = \int_0^t \mu_{Y_s} ds,$$

and let τ_t be the inverse of A. Then the CSRW X is given by

$$X_t = Y_{\tau_t}, \quad t \geq 0.$$

By the ergodic theorem

$$A_t/t \to C = 2d\, \mathbb{E}_1 \omega_e \in [1, \infty],$$

so $\tau_t/t \to C^{-1} \in [0, 1]$. So we get the QFCLT for X, and the diffusivity σ_X^2 is positive if and only if $C < \infty$, i.e. if and only if

$$\mathbb{E}\omega_e < \infty.$$

CSRW: beyond the CLT

If $\mathbb{E}\omega_e = \infty$ then we just have

$$X_t^{(n)} = n^{-1/2} X_{nt} \to 0.$$

The reason is that X spends long periods in the 'traps', i.e. jumping across 'red bonds' $e = \{x, y\}$ with $\omega_e \gg 1$.
'Fractional kinetic motion' FK(α). Let $d \geq 1$, W be BM($\mathbb{R}^d$) and ξ_t be an independent stable subordinator index $\alpha \in (0, 1)$. Let L_t be the inverse of ξ: $L_t = \inf\{s : \xi_s > t\}$. Then the FK($\alpha$) is given by:

$$R_t^{(\alpha)} = W_{L_t}, \quad t \geq 0.$$

R is not Markov, moves like a BM, but has long periods of remaining constant.
MB and J. Černý: if $d \geq 3$, and $\mathbb{P}(\omega_e > t) \sim t^{-\alpha}$ then

$$n^{-\alpha/2} Y_{nt} \Rightarrow c R_t^{(\alpha)}.$$

Fractional Kinetic motion