Range and local times of Lévy processes

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Consider a 1-dimensional Lévy processes with

$$Ee^{i\theta X_t} = E^0 e^{i\theta X_t} = e^{t\psi(\theta)},$$

$$\psi(\theta) = -ia\theta + \sigma^2\theta^2 - \int \left(e^{i\theta x} - 1 - i\theta x 1_{(|x|<1)}\right)\nu(dx).$$

Exclude the compound Poisson case when $\nu(\mathbb{R}) < \infty$.
The *range of* X is the set

$$R_t = R_t(X) = \{X_s, X_{s-}, 0 < s \le t\} \cup \{X_0\}.$$

Problem. What are the properties of R_t?

Easy properties:
(a) R_t is closed.
(b) As X is cadlag R_t is bounded.
(c) As X is non-constant on every interval, R_t is perfect.

Lebesgue measure. We have

$$E|R_t| = E\int 1_{(|x|\in R_t)}dx = \int P^0(T_x < \infty)dx.$$

Here

$$T_A = \inf\{t > 0 : X_t \in A\}, \quad T_x = T_{\{x\}}.$$

Definition. X *hits points* if $P^0(T_x < \infty) > 0$ for some x.
0 *is regular for* $\{0\}$ if $P^0(T_0 = 0) = 1$.
From now on I assume:
(A1) X hits points and 0 is regular for $\{0\}$.
Kesten (1969) found conditions for this:

$$X \text{ hits points} \iff \int \mathbb{R}e\frac{1}{1+\psi(\theta)}d\theta < \infty.$$

$$0 \text{ is regular for } \{0\} \iff X \text{ hits points, and}$$

$$\sigma > 0 \text{ or } \int (|x| \wedge 1)\nu(dx) = \infty.$$

(The last line is equivalent to requiring the that paths of X are of unbounded variation.)

For processes satisfying (A1) the potential kernel density $u_\lambda(x)$ is bounded and continuous. Set

$$w(x) = u_1(0) - u_1(x);$$

then

$$w(x) + w(-x) = \frac{1}{\pi}\int (1-\cos\theta x)\mathbb{R}e\frac{1}{1+\psi(\theta)}d\theta.$$

Note. 1. If X doesn't hit points then R_t is a perfect set of Lebesgue measure 0.
2. See Mountford and Port (Ann. Prob. 1991) for the case when X has paths of finite variation.

Under (A1), R_t is a set of positive Lebesgue measure. Does it contain intervals, or is it nowhere dense?

Using the Blumenthal 01 law and the Baire category theorem if

$$S_I = \inf\{t > 0 : R_t \text{ contains an interval}\},$$

then $P^0(S_I > 0) = 1$ implies that $P^0(S_I = \infty) = 1$, so R_t is nowhere dense for all t, a.s. ("Either intervals form immediately, or they never form".)

Local Times. Under (A1), X has local times $(L^x_t, x \in \mathbb{R}, t \ge 0)$. For each x the process $t \to L^x_t$ is a continuous additive functional which increases only on $\{t : X_t = x\}$. One has $P^0(L^0_t > 0) = 1$ for all $t > 0$.

Kesten (Indiana J. Math, 1976) observed that if $(x,t) \to L^x_t$ is jointly continuous then, P^0-a.s., $L^x_t > 0$ in an interval around 0, and so

$$P^0(\ R_t \text{ contains an interval around } 0) = 1.$$

Conditions for joint continuity were already known (Boylan, Blumenthal-Getoor); in particular processes with a Gaussian component ($\sigma^2 > 0$) and stable processes with $\alpha > 1$ have jointly continous local times.

Kesten also gave examples of symmetric Lévy processes satisfying (A1) with R_t nowhere dense. Let $NCLP(\kappa, \beta, \gamma)$ have $\sigma = 0$, $a = 0$, and $\nu(dx) = g(x)dx$ where

$$g(x) = x^{-2}(\log \frac{1}{|x|})^{\beta}(\log\log \frac{1}{|x|})^{\gamma}(1 + \kappa - 2\kappa 1_{(x<0)}).$$

Then if $\kappa = 0$ (i.e. symmetric case)

$$\psi(\theta) \asymp |\theta|(\log|\theta|)^{\beta}(\log\log|\theta|)^{\gamma},$$

so these processes satisfy (A1) when $\beta > 1$ or $\beta = 1$, $\gamma > 1$.

Kesten proved that if $\kappa = 0$, $\beta = 1$, $\gamma \in (1, 2)$ then these processes have a nowhere dense range.

His proof relied on the fact that one can find N points $y_1, \dots y_N$ such that

$$|w(y_i - y_j) - \delta| \leq \frac{c\delta}{N}, \quad i \neq j.$$

Recall

$$g(x) = x^{-2}(\log \frac{1}{|x|})^{\beta}(\log\log \frac{1}{|x|})^{\gamma}(1 + \kappa - 2\kappa 1_{(x<0)}).$$

If $\beta = \gamma = 0$ one gets the *Cauchy process (CP)*:
$\kappa = 0$ symmetric CP (SCP)
$0 < \kappa < 1$ (partially) asymmetric CP (PACP)
$\kappa = 1$ completely asymmetric CP (CACP).
One finds:

$$\psi(\theta) = c_1|\theta| + ic_2\kappa\theta\log|\theta|,$$

so if $\kappa \neq 0$

$$\mathbb{R}e\frac{1}{1 + \psi(\theta)} \asymp \frac{1}{|\theta|(\log|\theta|)^2}.$$

So the PACP and CACP satisfy (A1).
Kesten asked if the PACP has nowhere dense range. (It was already known it had a discontinuous local time.)
The CACP must have intervals in its range, since it has no downward jumps.

Theorem 1. *(W. Pruitt, S. J. Taylor, ZfW 1985). The range of the PACP is nowhere dense.*

Notation. Write $\xi(a) = 1/\log(1/|a|)$.

One needs to find gaps in $R = R_\infty$ close to 0. However looking at just one point is not enough, since

$$P^0(x \notin R) \asymp w(x) \asymp \xi(x) \to 0 \quad \text{as } x \to 0.$$

One has

$$w(x) \sim \begin{cases} c(1+\kappa)\xi(x), & x > 0, \\ c(1-\kappa)\xi(x), & x < 0, \end{cases}$$

so one cannot use Kesten's argument.

Sketch proof.

$N(a) = \#$ 'passes' by X over $[-a, -a/2] = I(a)$.

Then $N(a)$ is roughly geometric, with $EN(a) \asymp \xi(a)^{-1}$.

Take M evenly spaced points $\{y_1, \dots y_M\}$ in $I(a)$.

Let $F_i = \{y_i \notin R_\infty\}$. The key bounds are, that there exists $\rho = \rho(M) < 1$ such that

$$P(F_i, N(a) = r) \geq c_1\xi(a)\rho^r$$
$$P(F_i \cap F_j, N(a) = r) \leq c_2\xi(a)\rho^{2r}.$$

So by the inclusion-exclusion formula,

$$P(\cup F_i, N(a) = r) \geq \xi(a)M\rho^r(c_1 - c_2M\rho^r).$$

So, taking $M = c\rho^{-r}$ (one can!) one gets

$$P(I(a) \not\subset R_\infty, N(a) = r) \geq c_3\xi(a).$$

Hence

$$P(I(a) \not\subset R_\infty) \geq \delta > 0,$$

and one deduces that

$$p_I = P(R_\infty \text{ contains an interval around } 0) < 1 - \delta.$$

One would now like to use a 01 law to conclude that $p_I = 0$, but this does not seem to follow from the Blumental 01 law. (The range might initially contain intervals, but not around 0... In fact, this *is* what happens if we look at the range due to a single excursion from 0.)

By working rather harder Pruitt and Taylor were able to get δ up to 1.

Let X^a be X time changed to remove the time spent outside $[-a, a]$. ('Trace of X on $[-a, a]$'). Set

$$\mathcal{E}_a = \sigma(X_t^a, t \geq 0), \qquad \mathcal{E} = \cap_{a>0}\mathcal{E}_a.$$

If X is transient then L_∞^0 is $\mathcal{E}$-measurable.
Also, for $a > 0$

$$A = \{R_\infty \text{ contains an interval around } 0\} \in \mathcal{E}_a,$$

so $A \in \mathcal{E}$. $\mathcal{E}_a$ contains information about how the excursions of X leave 0:

Millar (ZfW, 1977): If $\sigma^2 = 0$ then "X only leaves 0 in one way". (This can be 'continuous upward', 'continuous downward', or 'oscillatory'.)

Theorem 2. *(MB, ZfW 1981). If X is recurrent and $\sigma^2 = 0$ then $\mathcal{E}$ is P^0-trivial.*

Corollary 3. *Let X be a Lévy process satisfying (A1). Then either:*
(1) R_t is nowhere dense for all $t > 0$, P^0-a.s.
or else
(2) R_t contains an interval around 0 for all $t > 0$, P^0-a.s.

So there are 3 possibilities:

CI – continous local times, interval type range, e.g. stable with $\alpha > 1$,
DI – discontinous local times, interval type range, e.g. CACP,
DN – discontinous local times, nowhere dense range, e.g. Kesten's examples, PACP.

Problem 1. When is R_t nowhere dense?

Recall $w(x) = u_1(0) - u_1(x)$, and set

$$\varphi(x)^2 = w(x) + w(-x) \asymp w(x) \vee w(-x).$$

Let $\overline{\varphi}$ be the monotone rearrangement of φ: $x \to \overline{\varphi}(x)$ is monotone on $\mathbb{R}_+$ and for all $\lambda > 0$

$$|\{y \in \mathbb{R}_+ : \overline{\varphi}(y) \leq \lambda\}| = |\{y \in \mathbb{R}_+ : \varphi(y) \leq \lambda\}|.$$

Set

$$I(\overline{\varphi}) = \int_{0+} \frac{\overline{\varphi}(t)dt}{t(\log(1/t))^{1/2}}.$$

Theorem 3. *(MB, J. Hawkes.) Let X satisfy (A1). Then X has continuous local times $\iff$*

$$I(\overline{\varphi}) < \infty \qquad \iff \sum_{n=1}^{\infty} n^{-1/2}\overline{\varphi}(2^{-n}) < \infty.$$

Conjecture. X has interval type range if and only if

$$I(\overline{\chi}) < \infty,$$

where

$$\chi(x)^2 = w(x) \wedge w(-x) \leq \varphi(x)^2.$$

For the PACP one has ($\xi(x) = 1/\log(1/|x|)$)

$$w(x) \sim \begin{cases} c(1+\kappa)\xi(x), & x > 0, \\ c(1-\kappa)\xi(x), & x < 0. \end{cases}$$

So in this case $\varphi \asymp \chi$.
For the CACP ($\kappa = 1$, no downward jumps)

$$w(x) \sim \begin{cases} c'\xi(x), & x > 0, \\ c(1 - e^{-|x|}), & x < 0. \end{cases}$$

Since $\xi(x) \gg |x|$ we have $\chi(x) \ll \varphi(x)$ for the CACP. If the conjecture is true then for symmetric Lévy processes discontinuous local time is equivalent to nowhere dense range.

Problem 2. Can one use 'Dynkin's isomorphism' to get any information about R_t in the symmetric case?

Let $Z_t^x = \{t : X_t = x\}$, and

$$R_t^* = \{x : Z_t^x \text{ is infinite }\} \subset R_t.$$

The same 01 laws hold for R^* as for R.
For the CACP R^* is nowhere dense. (Remark of James Taylor to me.)

Problem 3. If X has continuous local times is R^* of interval type?

The difficulty is that prove that $L_t^x > 0$ for all $x \in (-\varepsilon, \varepsilon)$, implies that $(-\varepsilon, \varepsilon) \subset R_t^*$.
For a fixed x there are characterisations of L_t^x, in terms of the set Z_t^x which imply that if $L_t^x > 0$ then, with probability 1, Z_t^x is infinite. Set

$$F_x = \{L_t^x > 0\}, \quad G_x = \{x \in R_t^*\};$$

we have $P(F_x \cap G_x^c) = 0$.
But one might still have

$$P(\cup_x (F_x \cap G_x^c)) > 0.$$

This cannot occur if we have a characterisation of the local time which holds *simultaneously* for all levels x.

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Theorem 4. *(MB, Perkins, Taylor 1986). If*
(1) X has continuous local times
and
(2) additional regularity holds
then characterisations of the local time in terms of Hausdorff measure of Z_t^x, and $\lim_{\varepsilon\downarrow 0} |(Z_t^x)^\varepsilon|/g(\varepsilon))$ hold for all x, t, ω except for $\omega \in N$ where $P(N) = 0$.

Here $Z^\varepsilon = \{t : |t - s| < \varepsilon$ for some $s \in Z\}$.

Problem 4. Can one remove the additional regularity assumption (2)?

(There may be much easier ways of solving Problem 3.)

If X has discontinuous local times then there is no obvious definition of $(L_t^x, x \in \mathbb{R}, t \geq 0)$. (This is not a problem in proving discontinuity, since one has $L_t^p, p \in \mathbb{Q}$ discontinuous.)

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But suppose we have a functional of X, $J_t^x(\varepsilon)$ such that

$$P(\lim_{\varepsilon\downarrow 0} J_t^x(\varepsilon) = L_t^x \text{ for all } t) = 1.$$

We can still ask if

$$P(\lim_{\varepsilon\downarrow 0} J_t^x(\varepsilon) \text{ exists for all } t, x) = 1,$$

and if it does we could define $L_t^x = \lim_{\varepsilon\downarrow 0} J_t^x(\varepsilon)$.

Problem 5. When X has discontinuous local times is there any approximation which converges simultaneously for all x?

Higher dimensions.

(Consider only properly d-dimensional processes.) In this case $|R_t| = 0$, so R_t contains no open set.

Problem 6. If $\nu(\mathbb{R}^d) = \infty$ does R_t contain non-trivial connected components?