# Yor's conjectures on the structure of filtrations

Martin Barlow[1]

[1]University of British Columbia

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# Background

1950–1965: Development of rigorous theory of Markov processes, especially the precise formulation of the strong Markov property. (Meyer, Getoor)

1965–1976: Development of stochastic calculus (integration of predictable processes w.r.t. semimartingales) by Meyer and the 'Strasbourg school'.

Context of Meyer's theory: a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Here $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, and $(\mathcal{F}_t) = (\mathcal{F}_t, t \in [0, \infty))$ is a filtration satisfying the usual conditions:

right continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s$,
$\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$.

Very often a filtration arises from a stochastic process $X$, and we write $(\mathcal{F}_t^X)$ for the (usual) filtration generated by $X$:

$$\begin{aligned}
\mathcal{F}_t^0 &= \sigma(X_s, s \le t), \\
\mathcal{N} &= \{F \in \mathcal{F} : \mathbb{P}(F) = 0\}, \\
\mathcal{F}_t^X &= \bigcap_{s>t} \sigma(\mathcal{F}_s, \mathcal{N}).
\end{aligned}$$

Let $\mathcal{M}^2((\mathcal{F}_t))$ be the space of $L^2$ martingales w.r.t. $(\mathcal{F}_t)$.

**Martingale representation.** We say $\{M^1, \ldots, M^k\} \subset \mathcal{M}^2((\mathcal{F}_t))$ has the martingale representation property (MRP) if for any $N \in \mathcal{M}^2((\mathcal{F}_t))$ there exist predictable processes $H^i$ such that

$$N_t = \sum_{i=1}^{k} \int_0^t H_s^i dM_s^i.$$

Jacod (1976): connection between MRP and extremal solutions to a martingale problem.

# Martingale dimension

Analogous with dimension of vector space in linear algebra: $dM_t$ are vectors and $H_t$ scalars.

**Definition.** (Davis and Varaiya (1974)). The multiplicity of $(\mathcal{F}_t)$ is the smallest $k$ such that there exists a set of $k$ real valued martingles with the martingale representation property. Write this as $\dim((\mathcal{F}_t))$.

**Examples.**
(1) (Itô). If $X$ is $d$-dimensional Brownian motion then $\dim((\mathcal{F}_t^X)) = d$.
(2) If $\mathcal{F}_0 \subset \mathcal{F}$, $F \notin \mathcal{F}_0$ with $0 < \mathbb{P}(F) < 1$ and $\mathcal{F}_1 = \sigma(\mathcal{F}_0, F)$ then

$$\dim((\mathcal{F}_0, \mathcal{F}_1)) = 1.$$

More generally if $\mathcal{F}_1$ is generated by $\mathcal{F}_0$ and a partition of $\Omega$ into $n$ sets then $\dim((\mathcal{F}_0, \mathcal{F}_1)) = n - 1$.

# Early work on filtrations

1. Stricker (1978). If $X$ is a semimartingale wrt $(\mathcal{G}_t)$ and $X$ is adapted to $(\mathcal{F}_t)$ then $X$ is a semimartingale wrt $(\mathcal{F}_t)$.

He defined 'Condition (R)' for a filtration $(\mathcal{G}_t)$ to be a 'good extension' of a filtration $(\mathcal{F}_t)$. It holds if either of the following two equivalent conditions holds:

(1) every $(\mathcal{F}_t)$ martingale is a $(\mathcal{G}_t)$ martingale,
(2) for all $s < t$ $\mathcal{F}_t$ and $\mathcal{G}_s$ are conditionally independent wrt $\mathcal{F}_s$.
(i.e. $\mathcal{G}_s$ contains no information on the future of $\mathcal{F}_t$ after time $s$.)

2. $\tau$ is an end of optional set if there exists an optional process (e.g. adapted, right continous) $A$ such that

$$\tau = \sup\{t : A_t = 0\}.$$

Following ideas of David Williams, Barlow, Jeulin and Yor studied the expansion of a filtration $(\mathcal{F}_t)$ so as to make an end of optional time into a stopping time, by setting $\mathcal{G}_t = \sigma(\mathcal{F}_t, \{\tau \le t\})$.

# Fundamental example (Lévy, Skorokhod)

Set $\text{sgn}(x) = -1_{(x \leq 0)} + 1_{(x>0)}$.

Let $X$ be a Brownian motion (BM). Set

$$Y_t = \int_0^t \text{sgn}(X_s) dX_s = |X_t| - \tfrac{1}{2} L_t(|X|).$$

Then $Y$ is a martingale with $\langle Y \rangle_t = t$, so is a Brownian motion.
$L_t(|X|)$ is the local time of $|X|$ at 0. Since $|X_s| \geq 0$, we have

$$\inf_{s \leq t} Y_s = -\tfrac{1}{2} L_t(|X|), \text{ and so } |X_t| = Y_t - \inf_{s \leq t} Y_s.$$

Thus $\mathcal{F}^Y = \mathcal{F}^{|X|}$; further this filtration is strictly smaller than the filtration $\mathcal{F}^X$ since it is missing the information about the signs of the excursions of $X$.

Note that $X_t = \int_0^t \text{sgn}(X_s) dY_s$, so that $Y$ (as well as $X$) has the martingale representation property for $\mathcal{F}^X$.

# 3 problems posed by Yor

Three problems posed by Marc in November 1978.

1. (A problem relating to Gilat's theorem that every submartingale is equal in law to the absolute value of a martingale.)

2. Let $X = X^0$ be a BM. As we saw, if

$$X^1 = \Psi(X^0) = \int_0^t \mathrm{sgn}(X_s)dX_s,$$

then $\mathcal{F}^1 = \mathcal{F}^{X^1}$ is strictly smaller than $\mathcal{F}^0 = \mathcal{F}^{X^0}$. (Also $\mathcal{F}^1$ is a good extension of $\mathcal{F}^0$, ie they satisfy (R)). Set $X^n = \Psi(X^{n-1})$ for $n \geq 1$, and let $\mathcal{F}^n$ be the associated filtrations.

**Question.** Is

$$\bigcap_{n=0}^{\infty} \mathcal{F}_t^n \quad \text{trivial?}$$

# Yor's third question

We have seen that if $X$ is a BM then every martingale wrt $(\mathcal{F}_t^X)$ is a stochastic integral of $X$.

**Definition.** (a) A filtration $(\mathcal{F}_t)$ is Brownian if there exists a (one dimensional) BM $X$ such that $(\mathcal{F}_t) = (\mathcal{F}_t^X)$.
(b) A filtration $(\mathcal{F}_t)$ has the Brownian representation property (BRP) if there exists a BM $Y$ such that every $(\mathcal{F}_t)$ - martingale is a stochastic integral of $Y$. (Thus $(\mathcal{F}_t)$ is one dimensional in the sense of Davis and Varaiya.)

The Kunita-Watanabe theorem shows that if $(\mathcal{F}_t)$ is Brownian then it has the BRP.

**Conjecture 1.** If $(\mathcal{F}_t)$ has the BRP then it is Brownian.

**Example.** Let $X$ be a BM, $\mathcal{F} = \mathcal{F}^X$ and $Y = \int \mathrm{sgn}(X)dX$. Then every $(\mathcal{F}_t)$ – martingale is a stochastic integral of $Y$, but $\mathcal{F}^X \neq \mathcal{F}^{|X|} = \mathcal{F}^Y$.

# Adding information to a filtration

Let $(\mathcal{F}_t)$ be a one-dimensional filtration and $U \sim \mathcal{U}(0,1)$ be a r.v. independent of $\mathcal{F}_1$. Suppose we want to build a good extension $(\mathcal{G}_t)$ of $(\mathcal{F}_t)$ such that $\mathcal{G}_1 = \sigma(\mathcal{F}_1, U)$, and also $\dim((\mathcal{G}_t)) = 1$.

We cannot 'add the information at a fixed time $t$', by setting

$$\mathcal{G}_s = \mathcal{F}_s, \;\; s \leq t, \qquad \mathcal{G}_s = \sigma(\mathcal{F}_s, U), \;\; s > t,$$

since then we will introduce new martingales which jump at time $t$.

The fundamental example shows one way of doing this if $\mathcal{F}$ carries a BM $Y$. Write $U = \sum \xi_n 2^{-n}$ where $(\xi_n)$ are iid $\mathtt{Ber}(\frac{1}{2})$ r.v., Let $Z_t = Y_t - \inf_{s \leq t} Y_s$, so $Z$ has the law of the absolute value of a BM. Use $\xi_n$ to 'flip' the excursions of $Z$ from 0, to create a BM $X$. Then

$$\mathcal{F}_1^X = \sigma(\mathcal{F}_1^Y, U),$$

and $\mathcal{F}^X$ is still one-dimensional.

With this procedure the information is added at the last exit times of $Z$ from 0, which are special cases of end of optional times.

For an end of optional time $\tau$ (wrt a filtration $(\mathcal{F}_t)$) we define

$$\mathcal{F}_\tau = \sigma(V_\tau : \ V \text{ is an } (\mathcal{F}_t) \text{ optional process}),$$
$$\mathcal{F}_{\tau+} = \bigcap_{t>0} \mathcal{F}_{\tau+t}.$$

For a filtration satisfying the usual conditions we have $\mathcal{F}_T = \mathcal{F}_{T+}$ at a stopping time, but this need not be true at end of optional times.

**Example.** Let $B$ be a BM, and $\tau = \sup\{t < 1 : B_t = 0\}$. Then the sign of the excursion starting at time $\tau$ is given by

$$\xi = \lim_{h \downarrow 0} \operatorname{sgn}(B_{\tau+h}) = \operatorname{sgn}(B_1),$$

and we have $\xi \in \mathcal{F}_{\tau+}$ but $\xi \notin \mathcal{F}_\tau$.

# Walsh Brownian motion (WBM)

Introduced by Walsh (1978). This is a Markov process, state space

$$A = \{0\} \cup \bigcup_{k=1}^{N} \{re^{i\theta_k}, r > 0\}.$$

Here $N \geq 3$, and $\theta_j$ are distinct; we may as well take $\theta_j = 2j\pi/N$. Call $A_j = \{re^{i\theta_k}, r > 0\}$ the ray with angle $\theta_j$.

The WBM $Z_t = R_t e^{i\Theta_t}$ moves like a standard Brownian motion on each ray; and when at 0 makes excursions with probability $p_j$ on ray $A_j$. (We will take $p_j = 1/N$.)

Let $\tau = \sup\{t < 1 : Z_t = 0\}$, so that $\tau$ is an end of optional time.

Then $\Theta_1 = \Theta_t$ for $\tau < t \leq 1$, and is $\mathcal{F}_{\tau+}$ but not $\mathcal{F}_\tau$ measurable.

In fact $\mathcal{F}_{\tau+}$ is obtained from $\mathcal{F}_\tau$ and the events $\{\Theta_1 = \theta_k\}$, $1 \leq k \leq N$, so

$$\dim(\mathcal{F}_\tau, \mathcal{F}_{\tau+}) = N - 1.$$

Let $Z$ be a WBM with 3 rays. Then $R = |Z|$ is the absolute value of a BM, and it is not hard to prove that

$$Y = R - \tfrac{1}{2}L(R)$$

is a BM with the martingale representation property for $\mathcal{F}^Z$, so $\mathcal{F}^Z$ has the BRP.

Is there a BM $W$ which generates $\mathcal{F}^Z$? We could not prove this.

If there were, then the time $\tau$ would be a random time such that $W$ has 3 possible types of behaviour immediately after $\tau$.

We could not find such times. (Even for BM in $\mathbb{R}^d$).

If one is to prove that $\mathcal{F}^Z$ is not Brownian then one has to find an 'invariant' of filtrations which is different for $\mathcal{F}^Z$ and $\mathcal{F}^{BM}$.

**Definition.** Call the **splitting multiplicity** of a filtration $(\mathcal{F}_t)$

$$\operatorname{sp}\dim((\mathcal{F}_t)) = 1 + \sup_\tau \dim(\mathcal{F}_\tau, \mathcal{F}_{\tau+}),$$

where the sup is taken over all end of optional times $\tau$. For a WBM with $N$ branches, the last exits from 0 show that

$$\operatorname{sp}\dim((\mathcal{F}_t)) \geq N.$$

**Conjecture 2.** $\operatorname{sp}\dim((\mathcal{F}_t^{BM})) = 2.$

At least one of Conjectures 1 and 2 is false.

For a construction of WBM, and discussion of Conjectures 1-2, see:

M.T. Barlow, J. Pitman and M. Yor. On Walsh's Brownian Motions. Sem. Prob. XXIII , 275-293 (1989).

# Question for Lévy processes

Conjecture 2 states that for BM the splitting multiplicity is the same for general end of optional times as for last exits from sets, i.e. 2.

One can also ask this for a real Lévy process. Let $X$ be a symmetric stable process, index $\alpha \in (1,2)$. Then 0 is regular for $\{0\}$, so that $\{t : X_t = 0\}$ is $\mathbb{P}^0$-a.s. uncountable. Set $\tau = \sup\{t < 1 : X_t = 0\}$. Millar proved that

$$\mathcal{F}^X_\tau = \mathcal{F}^X_{\tau+}.$$

($X$ jumps over 0 infinitely often immediately after $\tau$, so one cannot assign a sign to its excursions from 0.)

**Conjecture 3.** For the stable process $X$, $\operatorname{sp}\dim((\mathcal{F}^X_t)) = 1$.

**Remarks.** (1) In fact all Lévy processes with infinitely many non-atomic jumps have isomophic filtrations.
(2) The Davis-Varaiya dimension satisfies $\dim((\mathcal{F}^X_t)) = \infty$.

Dubins, Feldman, Smorodinsky, Tsirelson (1994) disproved Conjecture 1, giving an example of filtration with BRP but which is not Brownian.

Tsirelson (1997) proved that one cannot construct a WBM (with $N \geq 3$) on the filtration of a BM.

He also proved that if $W$ is a BM($\mathbb{R}^d$), $A \subset \mathbb{R}^d$ and $\tau = \tau_A = \sup\{t < 1 : W_t \in A\}$ then

$$\operatorname{sp}\dim(\mathcal{F}^W_\tau, \mathcal{F}^W_{\tau+}) = 2.$$

(So $W$ can 'leave a set in at most 2 ways'.)

Barlow, Emery, Knight, Song, Yor (1998) proved that $\operatorname{sp}\dim((\mathcal{F}^{BM}_t)) = 2$.

# Tsirelson's argument

**Joining 2 copies of a filtration**. Let $B = B$ and $B'$ be two independent BM (on a probability space). Set

$$B^{\varepsilon} = \cos(\varepsilon)B + \sin(\varepsilon)B'.$$

So for each $\varepsilon$ the process $B^{\varepsilon}$ is also a BM.

Let $\mathcal{F}_t^B$ be the filtration of a BM. If $\xi \in L^0(\mathcal{F}_\infty^B)$ then there exists a measurable function $f : C([0,\infty)) \to \mathbb{R}$ such that $\xi = f(B_\cdot)$. Write $\xi^{\varepsilon} = f(B_\cdot^{\varepsilon})$.

**Lemma 1.** Let $\xi \in L^0(\mathcal{F}_\infty^B)$. Then $\xi^{\varepsilon} \to \xi$ in probability as $\varepsilon \to 0$. (The Brownian filtration is 'cosy'.)

**Lemma 2.** Let $\xi \in L^0(\mathcal{F}_\infty^B)$ have a continuous distribution function. Then $\mathbb{P}(\xi = \xi^{\varepsilon}) = 0$.

# Proof of Lemma 2

**Lemma 2.** Let $\xi \in L^0(\mathcal{F}_1)$ have a continuous distribution function. Then $\mathbb{P}(\xi = \xi^\varepsilon) = 0$.

Neveu's hypercontractivity property: there exists $p = p(\varepsilon) < 2$ such that if $X \in L^0(\mathcal{F}_1^B)$, $Y \in L^0(\mathcal{F}_1^{B^\varepsilon})$ then

$$E|XY| \leq ||X||_p ||Y||_p.$$

(Proof – stochastic calculus.)

Choose sets $A_i$ such that $\mathbb{P}(\xi \in A_i) = 1/n$. Then

$$\begin{aligned}\mathbb{P}(\xi \in A_i, \xi^\varepsilon \in A_i) &= \mathbb{E}1_{A_i}(\xi)1_{A_i}(\xi^\varepsilon)\\ &\leq (1/n)^{1/p}.(1/n)^{1/p} = n^{-2/p} = n^{-1-\delta}.\end{aligned}$$

So

$$\mathbb{P}(\cup_i\{\xi \in A_i, \xi^\varepsilon \in A_i\}) \leq n^{-\delta}.$$

# WBM filtration is not Brownian

Argument by contradiction. Suppose it is, let $B$ and $B^\varepsilon$ be Brownian motions as above, let $Z = F(B)$ be a WBM (with $N \geq 3$ branches), and let $Z^\varepsilon = F(B^\varepsilon)$. Stop $Z$ and $Z^\varepsilon$ when they hit $\{|z| = 1\}$.

Write

$$Z_t = R_t \exp(i\Theta_t), \quad Z_t^\varepsilon = R_t^\varepsilon \exp(i\Theta_t^\varepsilon).$$

These processes are all on the filtration $(\mathcal{F}_t)$ generated by $B$ and $B'$, and $\lim_{\varepsilon\to 0} Z_1^\varepsilon = Z_1$ (in $\mathbb{P}$) by Lemma 1.
Let $\tau = \sup\{t \leq 1 : Z_t = 0\}$, and define $\tau^\varepsilon$ analogously. Then $\tau$ and $\tau^\varepsilon$ have a continuous distribution function, so by Lemma 2,

$$\mathbb{P}(\tau = \tau^\varepsilon) = 0.$$

Essential idea: if $\tau \neq \tau^\varepsilon$ then $Z$ and $Z^\varepsilon$ make their last exits from 0 at different times, so $Z_1$ and $Z^\varepsilon$ cannot not be close.

Let $d(x,y)$ be the shortest path metric on the state space $A$ for the WBM. Stochastic calculus gives:

$$\begin{aligned} d(Z_t, Z_t^\varepsilon) = &\ \text{martingale} + \tfrac{1}{2} L_t(Z, Z^\varepsilon) \\ &+ \tfrac{(N-2)}{2N} \int_0^t 1_{(Z_s \neq 0)} dL_s(R^\varepsilon) + \tfrac{(N-2)}{2N} \int_0^t 1_{(Z_s^\varepsilon \neq 0)} dL_s(R). \end{aligned}$$

Also

$$\mathbb{P}(Z_{\tau^\varepsilon} \neq 0) = \mathbb{E} \int_0^\infty 1_{(Z_s \neq 0)} d1_{[\tau^\varepsilon, \infty)} = \mathbb{E} \int_0^\infty 1_{(Z_s \neq 0)} dL_t(R^\varepsilon).$$

Taking expectations (recall $Z, Z^\varepsilon$ are stopped when they hit the unit circle)

$$\begin{aligned} \mathbb{E} d(Z_\infty, Z_\infty^\varepsilon) &\geq \tfrac{(N-2)}{2N} \Big( \mathbb{P}(Z_{\tau^\varepsilon} \neq 0) + \mathbb{P}(Z_\tau^\varepsilon \neq 0) \Big) \\ &\geq \tfrac{(N-2)}{2N} \mathbb{P}(\tau \neq \tau^\varepsilon) = \tfrac{(N-2)}{2N}. \end{aligned}$$

So we cannot have $\lim_\varepsilon Z_\infty^\varepsilon = Z_\infty$, which contradicts Lemma 1.

Marc Yor in Helsinki, 1985