Moduli Spaces for Vector Bundles with Level Structures on Algebraic Curves

Summary

Kai A. Behrend

March 4, 1996

1 Introduction

This is a summary of the main results of my *Diplomarbeit*, written under the sepervision of Prof. Harder and completed in 1987.

In chapter I, the moduli problem for vector bundles with level structures over an algebraic curve is treated. Smooth and proper moduli spaces are constructed. (For interesting special cases, see remark 2.1.3 and corollary 2.1.6.)

In chapter II, the limit of the ℓ-adic cohomology algebras of the moduli spaces as 'level$\rightarrow \infty$' is constructed. In the case of finite ground field, the trace of the Frobenius on this cohomology algebra in the limit is calculated. In particular, Betti numbers of the moduli spaces are obtained.

1.1 Notation

Let k be a field, X an algebraic curve over k, i.e. a smooth proper geometrically connected one-dimensional scheme over k. Let g denote the genus of X.

For a coherent $\mathcal{O}_X$-module E, let the *rank* of E, $\operatorname{rk} E$, be the dimension of its generic fiber. The *degree* is defined for coherent $\mathcal{O}_X$-modules, by requiring it to be additive on short exact sequences, and to give the degree of a corresponding divisor for invertible sheaves. For coherent $\mathcal{O}_X$-modules E of non-zero rank, we define the *slope* μE by : $\mu E = \frac{\deg E}{\operatorname{rk} E}$.

A *torsion sheaf* is a coherent $\mathcal{O}_X$-module of rank zero. A *vector bundle* E is a torsion free (i.e. locally free) coherent $\mathcal{O}_X$-module. A *subbundle* L of E is a coherent submodule, that locally is a direct summand.

We will often formulate statements dealing simultaneously with stability and semi-stability. In such a case, the symbol $\substack{<\\ \leq}$ means that $<$ applies to the case of stability, $\leq$ to the case of semi-stability.

For a k-scheme U, $H^p(U, \mathbb{Q}_\ell)$ denotes the p^{th} étale cohomology group with values in $\mathbb{Q}_\ell$, of the base extension of U to an algebraic closure $\overline{k}$ of k.

Towards the end we will assume that $k = \mathbb{F}_q$ is a finite field. Then we will use the following notation: K is the function field of X, $\mathbb{A}_K$ the adéle ring of K. $GL_r^\circ(\mathbb{A}_K) \subset GL_r(\mathbb{A}_K)$ is the set of adéle valued matrices whose determinant has idéle norm 1. $\mathfrak{k} \subset GL_r^\circ(\mathbb{A}_K)$ is the standard compact subgroup $\mathfrak{k} = \prod_v GL_r(\mathcal{O}_v)$, where the product is taken over all places v of K, and $\mathcal{O}_v$ is the completion of the valuation ring of v. ω will denote a Tamagawa measure on $GL_r(\mathbb{A}_K)$.

2 The Results

2.1 The Moduli Spaces

Definition 2.1.1 Let $\mathcal{T}$ be a torsion sheaf on X. A non-zero homomorphism $f : E \to \mathcal{T}$ of coherent $\mathcal{O}_X$-modules is called a *level structure on E with respect to $\mathcal{T}$* or a *level-$\mathcal{T}$-structure* for short. The level structure $f : X \to \mathcal{T}$ is called *singular* if $\operatorname{Ext}^1_{\mathcal{O}_X}(E, \operatorname{cok} f) \neq 0$, *non-singular* otherwise.

Definition 2.1.2 Let $\vec{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ be an n-tuple of torsion sheaves on X. Let the supports of the $\mathcal{T}_i$ be pairwise disjoint. Let E be a coherent $\mathcal{O}_X$-module of positive rank r, and $\vec{f} = (f_1, \ldots, f_n)$, $f_i : E \to \mathcal{T}_i$, an n-tuple of level structures. Let $\vec{\Delta} = (\Delta_1, \ldots, \Delta_n)$ be an n-tuple of real numbers such that $0 < \Delta_i < \deg \mathcal{T}_i$ for all $i = 1, \ldots, n$.

$\vec{f} : E \to \vec{\mathcal{T}}$ is called *(semi-)stable with respect to* $\vec{\Delta}$ or $\vec{\Delta}$*-(semi-)stable* if the following conditions are satisfied:

i. $f = \sum f_i$ is injective on $\mathfrak{t}(E)$, where $\mathfrak{t}(E)$ is the torsion subsheaf of E.

ii. For every $i = 1, \ldots, n$ it holds that

$$\deg \operatorname{im} f_i \substack{>\\ \geq} \Delta_i.$$

iii. For every $i = 1, \ldots, n$ it holds that

$$\deg f_i(\mathfrak{t}(E)) \stackrel{<}{\leq} \Delta_i.$$

iv. For every subsheaf L of E, with $0 < \operatorname{rk} L < r$ it holds that

$$\mu L \stackrel{<}{\leq} \mu E + \sum_{i=1}^{n} \Delta_i \left(\frac{\epsilon(f_i, L)}{l} - \frac{1}{r} \right).$$

Here $l = \operatorname{rk} L$. $\epsilon(f, L)$ is defined as follows:

$$\epsilon(f, L) = \begin{cases} 0 & \text{if } f(L) = 0 \\ 1 & \text{if } f(L) \neq 0 \end{cases}.$$

Remark 2.1.3 In case E is a vector bundle, and $\vec{\Delta}$ satisfies $0 < \Delta_i < 1$ for all $i = 1, \ldots, n$, then an n-tuple $\vec{f} : E \to \vec{\mathcal{T}}$ of level structures on E with respect to $\vec{\mathcal{T}}$ is (semi-)stable, if and only if for every subbundle L of E, with $0 \neq L \neq E$, we have

$$\frac{\deg L - \sum_{f_i(L) \neq 0} \Delta_i}{\operatorname{rk} L} \stackrel{<}{\leq} \frac{\deg E - \sum_{f_i(E) \neq 0} \Delta_i}{\operatorname{rk} E}.$$

Definition 2.1.4 Let $r > 0$ and d be integers, let $\vec{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ be an n-tuple of torsion sheaves on X with pairwise disjoint supports, let $\vec{\Delta} = (\Delta_1, \ldots, \Delta_n)$ be an n-tuple of real numbers, satisfying $0 < \Delta_i < \deg \mathcal{T}_i$ for $i = 1, \ldots, n$. Let $\mathfrak{U}^{ss}(r, d, \vec{\mathcal{T}}, \vec{\Delta})$ (resp. $\mathfrak{U}^{s}(r, d, \vec{\mathcal{T}}, \vec{\Delta})$, resp. $\mathfrak{U}^{sns}(r, d, \vec{\mathcal{T}}, \vec{\Delta})$) denote the functor

$$(\text{locally noetherian } k\text{-schemes}) \longrightarrow (\text{sets}),$$

defined by taking a locally noetherian k-scheme T to the following set:

> {Families E_T of coherent $\mathcal{O}_X$-modules, parametrized by T, such that for every $t \in T$, E_t has rank r and degree d, together with n-tuples $\vec{L} = (L_{1,T}, \ldots, L_{n,T})$, where for each $i = 1, \ldots, n$, $L_{i,T}$ is an invertible submodule of $p_{T*}\mathcal{H}om(E_T, \mathcal{T}_{i,T})$ such that for every $t \in T$, $L_{i,t} \to \operatorname{Hom} E_t \mathcal{T}_{i,t}$ is injective, satisfying the following condition for every geometric point $t \in T(K)$: If $\vec{f} = (f_1, \ldots, f_n)$ is an n-tuple of level structures defined by $\vec{L}_t$, then $\vec{f}$ is $\vec{\Delta}$-semi-stable (resp. $\vec{\Delta}$-stable, resp. $\vec{\Delta}$-stable and non-singular).}$/ \sim$

Here '∼' denotes the following equivalence relation: $(E_T, \vec{L}_T) \sim (E'_T, \vec{L}'_T)$ if there exists an invertible sheaf L on T such that $E'_T \otimes p_T{}^*L \cong E_T$ as $\mathcal{O}_{X\times T}$-modules and $L'_{i,T} \otimes p_T{}^*L$ and $L_{i,T}$ are equivalent subobjects of $p_{T*}\mathcal{H}om(E_T, \mathcal{T}_{i,T})$ under the isomorphism induced by $E'_T \otimes p_T{}^*L \cong E_T$.

Theorem 2.1.5 *Let r, d, $\vec{\mathcal{T}}$ and $\vec{\Delta}$ be given as in definition 2.1.4. There is a projective k-scheme $U^{\rm ss}$ and open subschemes $U^{\rm s}$ and $U^{\rm sns}$:*

$$U^{\rm sns} \subset U^{\rm s} \subset U^{\rm ss}.$$

$U^{\rm sns}$ is non-singular of dimension $1 - n + r^2(g-1) + r\sum_{i=1}^n \deg \mathcal{T}_i$.

$U^{\rm ss}$, $U^{\rm s}$ and $U^{\rm sns}$ satisfy the following universal mapping properties:

i. There is a morphism of functors

$$\Phi^{\rm ss} : \mathfrak{U}^{\rm ss}(r, d, \vec{\mathcal{T}}, \vec{\Delta}) \longrightarrow U^{\rm ss}$$

such that $\Phi^{\rm ss}$ is an initial object in the category of all morphisms of functors

$$\Psi : \mathfrak{U}^{\rm ss}(r, d, \vec{\mathcal{T}}, \vec{\Delta}) \longrightarrow V,$$

where V runs through all k-schemes.

ii. There is a morphism of functors

$$\Phi^{\rm s} : \mathfrak{U}^{\rm s}(r, d, \vec{\mathcal{T}}, \vec{\Delta}) \longrightarrow U^{\rm s}$$

with the same universal mapping property as $\Phi^{\rm ss}$ and the additional property that $\Phi^{\rm s}(K)$ is bijective, for every algebraically closed field K/k.

iii. There is an isomorphism of functors

$$\Phi^{\rm sns} : \mathfrak{U}^{\rm sns}(r, d, \vec{\mathcal{T}}, \vec{\Delta}) \longrightarrow U^{\rm sns}.$$

So there is an element $(E_{U^{\rm sns}}, \vec{L}_{U^{\rm sns}})$ of $\mathfrak{U}^{\rm sns}(r, d, \vec{\mathcal{T}}, \vec{\Delta})(U^{\rm sns})$, such that $(U^{\rm sns}; E_{U^{\rm sns}}, \vec{L}_{U^{\rm sns}})$ represents the functor $\mathfrak{U}^{\rm sns}(r, d, \vec{\mathcal{T}}, \vec{\Delta})$.

Hence $U^{\rm s}$ is a coarse, and $U^{\rm sns}$ a fine moduli scheme.

For (i) and (ii) we need X to have sufficiently many k-rational points (sufficiently many meaning more than a certain constant number, depending only on r, g and $\deg \mathcal{T}_1, \ldots, \deg \mathcal{T}_n$). For (iii) we need r and d to be relatively prime.

Note that in some cases these moduli spaces might be empty.

Proof. (Sketch.) Construct, using Hilbert schemes, a certain locally universal scheme R, with a $G = PGL(h)$-operation on it, for some suitably chosen integer h. Linearize the action of G on R in a very carefully chosen way. Let R^{ss} (resp. R^{s}) be the open subscheme of semi-stable (resp. stable) points of R, for the G-action. Let R^{sns} be the subscheme of those points, that induce a non-singular stable n-tuple of level structures. (R^{sns} is, by the way, the non-singular locus of R^{s}.) Then $U^{ss} = R^{ss}/G$ is a 'good' quotient, $U^{s} = R^{s}/G$ is a geometric quotient, and $R^{sns} \to U^{sns} = R^{sns}/G$ is a principal G-bundle. □

Corollary 2.1.6 *Choose $\vec{\Delta} = (\Delta_1, \ldots, \Delta_n)$ in such a way, that*

i. *$\Delta_1, \ldots, \Delta_n$ are irrational and linearly independent over $\mathbb{Q}$.*

ii. *Every $\vec{\Delta}$-stable n-tuple of level structures is non-singular. (For example, let for every $i = 1, \ldots, n$, $\Delta_i < 1$ or $\Delta_i > \deg \mathcal{T}_i - 1$.)*

Then $U^{s}(r, d, \vec{\mathcal{T}}, \vec{\Delta})$ is representable by a smooth projective k-scheme of dimension $1 - n + r^2(g-1) + r\sum_{i=1}^{n} \deg \mathcal{T}_i$, in case it is not empty.

Proof. Follows from theorem 2.1.5, since the notions of semi-stability, stability, and non-singular stability coincide. □

2.2 Betti Numbers and the Trace of Frobenius

Consider the following situation: Fix $r > 0$ and $d \in \mathbb{Z}$ such that $(r, d) = 1$. Choose a sequence $x_1, x_2, \ldots$ of different (closed) points of X. For every i choose a 'weight' Δ_i such that the following conditions are satisfied:

i. For all $i > 0$: $0 < \Delta_i < 1$ and Δ_i is irrational.

ii. For all $n > 0$ we have: $(\Delta_1, \ldots, \Delta_n)$ is linearly independent over $\mathbb{Q}$.

iii. $\sum_{i=1}^{\infty} \Delta_i = \infty$.

For $i \in \mathbb{N}$ and $N \in \mathbb{N}$, let $\mathcal{T}_i(N) = \mathcal{O}^{r}_{Nx_i}$, i.e. $\mathcal{T}_i(N)$ is the torsion sheaf with support $\{x_i\}$ and module of sections $(\mathcal{O}_{X,x_i}/\mathbb{M}^{N}_{x_i})^{r}$. For fixed $n \in \mathbb{N}$, we consider level structures with respect to $\vec{\mathcal{T}}(N) = (\mathcal{T}_1(N), \ldots, \mathcal{T}_n(N))$. So let $U^{s}(n, N) = U^{s}(r, d, \vec{\mathcal{T}}(N), \vec{\Delta})$, where $\vec{\Delta} = (\Delta_1, \ldots, \Delta_n)$, be the smooth and proper moduli space of corollary 2.1.6.

Passing to the First Limit

For passing to the first limit we fix n and let N vary.

Proposition 2.2.1 *Let $M \leq N$ be positive integers. Then for $p \leq 2M-2$ there is a canonical isomorphism*

$$H^p(U^s(n,M),\mathbb{Q}_\ell) = H^p(U^s(n,N),\mathbb{Q}_\ell).$$

Definition 2.2.2 Let

$$H^p(U^s(n),\mathbb{Q}_\ell) = H^p(U^s(n,N),\mathbb{Q}_\ell)$$

for some N, with $p \leq 2N-2$. Note that $U^s(n)$ does not stand for any moduli space. The notation is purely symbolic. This also defines a $\mathbb{Q}_\ell$-algebra

$$H^*(U^s(n),\mathbb{Q}_\ell) = \bigoplus_{p=1}^{\infty} H^p(U^s(n),\mathbb{Q}_\ell).$$

Proposition 2.2.3 *Let n be large enough such that $U^s(n,N) \neq \emptyset$ for all N. (This is only a restriction if $g=0$.) There exist certain cohomology classes $c_1(L_1),\ldots,c_1(L_n) \in H^2(U^s(n),\mathbb{Q}_\ell)$ with the following properties:*

i. For every $i = 1,\ldots,n$, $c_1(L_i)$ is the first Chern class of a universal subbundle

$$L_i \hookrightarrow p_{U^s(n,N)_*}\mathcal{H}om(E_{U^s(n,N)}, \mathcal{T}_i(N)_{U^s(n,N)}).$$

ii. $\mathbb{Q}_\ell[c_1(L_1),\ldots,c_1(L_n)]$ is a free symmetric $\mathbb{Q}_\ell$-algebra on the generators $c_1(L_1),\ldots,c_1(L_n)$.

Definition 2.2.4 Let $G^*(U^s(n),\mathbb{Q}_\ell) = H^*(U^s(n),\mathbb{Q}_\ell)/(c_1(L_1),\ldots,c_1(L_n))$.

Passing to the Second Limit

Now we let n vary.

Proposition 2.2.5 *Let $m \leq n$ be positive integers. Then for*

$$p \leq 2\left(\sum_{i=1}^{m}\Delta_i + \gamma(r,g)\right) - 2$$

there is a canonical isomorphism

$$G^p(U^s(m), \mathbb{Q}_\ell) = G^p(U^s(n), \mathbb{Q}_\ell).$$

Here $\gamma(r,g)$ is a constant, depending only on r and g. More precisely:

$$\gamma(r,g) = \begin{cases} -r^2/2 & \text{if } g = 0 \\ (r-1)(g-1) & \text{if } g > 0 \end{cases}.$$

Definition 2.2.6 Let

$$G^p(U^s, \mathbb{Q}_\ell) = G^p(U^s(n), \mathbb{Q}_\ell)$$

for some n, with $p \leq 2\left(\sum_{i=1}^n \Delta_i + \gamma(r,g)\right) - 2$. Again, the notation U^s is purely symbolic. This also defines a $\mathbb{Q}_\ell$-algebra

$$G^*(U^s, \mathbb{Q}_\ell) = \bigoplus_{p=1}^{\infty} G^p(U^s, \mathbb{Q}_\ell).$$

Theorem 2.2.7 *Let $k = \mathbb{F}_q$ be a finite field. Then the Poincaré series P_t of the cohomology algebra constructed above is as follows:*

$$P_t(G^*(U^s, \mathbb{Q}_\ell)) = (1-t^2)(1-t^{2r}) \prod_{i=1}^{r} \frac{(1+t^{2i-1})^{2g}}{(1-t^{2i})^2} \tag{1}$$

I.e. the coefficient of t^p in this series is the dimension of $G^p(U^s, \mathbb{Q}_\ell)$.

For the trace of the arithmetic Frobenius Φ_q on the cohomology algebra we have:

$$q^{1+r^2(g-1)} \mathrm{trace}\Phi_q | G^*(U^s, \mathbb{Q}_\ell) = (q-1)\frac{\omega(GL_r(K)\backslash GL_r^\circ(\mathbb{A}_K))}{\omega(\mathfrak{k})} \tag{2}$$

This means that the trace of Φ_q on $G^*(U^s, \mathbb{Q}_\ell)$ is absolutely convergent and converges to the number on the right.

Proof.(Sketch.) First prove (2), by estimating the behavior of the number of level structures as '$(n, N) \to \infty$'. Then using the fact that $U^s(n, N)$ is smooth and proper, (1) follows from (2) using the Weil conjectures. □

Corollary 2.2.8 *Let $n, N \in \mathbb{N}$, and assume that $U^s(n, N) \neq \emptyset$, which is automatically the case if $g > 0$. For those $p \geq 0$ that satisfy* *i.* $p \leq 2\left(\sum_{i=1}^{n} \Delta_i + \gamma(r,g)\right) - 2$

ii. $p \leq 2N - 2$

the p^{th} *Betti number of* $U^s(n,N)$ *is the coefficient of* t^p *in the series*

$$\frac{(1-t^2)(1-t^{2r})}{(1-t^2)^n} \prod_{i=1}^{r} \frac{(1+t^{2i-1})^{2g}}{(1-t^{2i})^2}.$$

Remark 2.2.9 For the case $r = 2$, more precise information about the Betti numbers can be obtained using the above method: For all $p \leq 2N-2$, the p^{th} Betti number of $U^s(n,N)$ is the coefficient of t^p in the series

$$\frac{1}{(1-t^2)^n} \frac{(1+t)^{2g}}{(1-t^2)(1-t^4)} \left((1+t^3)^{2g} - t^{2(g+\delta)}(1+t)^{2g}\right).$$

Here δ is defined as follows:

$$\delta = \begin{cases} [\sum_{i=1}^{n} \Delta_i] + 1 & \text{in case } d + [\sum_{i=1}^{n} \Delta_i] \text{ is even} \\ [\sum_{i=1}^{n} \Delta_i] & \text{in case } d + [\sum_{i=1}^{n} \Delta_i] \text{ is odd} \end{cases}.$$

For example, if $n = 1$ and x_1 is a k-rational point, we can calculate approximately 50% of the Betti numbers.