

Enumerative Geometry of Calabi-Yau threefolds

Kai Behrend

The University of British Columbia

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A *Calabi-Yau threefold* is a complex algebraic variety X , such that

- ① X is projective, i.e., $X \subset \mathbb{P}^n(\mathbb{C})$ is the common zero locus of a number of homogeneous equations in $n + 1$ variables
- ② X is non-singular, hence a compact complex manifold,
- ③ $\dim_{\mathbb{C}} X = 3$,
- ④ there exists a global, nowhere vanishing holomorphic differential 3-form, in other words, a global section of $\omega_X = \Omega_X^3$ (a holomorphic orientation),
- ⑤ (strict Calabi-Yau) $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

Given a CY3 X , we are interested in

$$n_{g,\beta}(X) = \#\{\text{curves } C \text{ in } X, \text{ of genus } g(C) = g \text{ and class } [C] = \beta\},$$

$$g \in \mathbb{Z}$$

$$\beta \in H_2(X, \mathbb{Z}).$$

The most famous example of a CY3 is the *quintic*. Defined by a single equation of degree 5 in 5 variables, defining a hypersurface X in $\mathbb{P}^4(\mathbb{C})$.

$$F(x_0, x_1, x_2, x_3, x_4) = \sum_{\substack{i_0, i_1, i_2, i_3, i_4 \geq 0 \\ i_0 + i_1 + i_2 + i_3 + i_4 = 5}} \alpha_{i_0, i_1, i_2, i_3, i_4} x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} = 0.$$

For this hypersurface to be non-singular, the coefficients need to be generic (and non-singular implies strict).

The quintic has $\binom{9}{5} = 126$ coefficients, up to scaling only 125. The group $PGL(5)$ of dimension 24 acts on these equations by projective linear coordinate changes, and so up to linear projective coordinate changes there is a 101-dimensional family of quintics.

This is reflected in

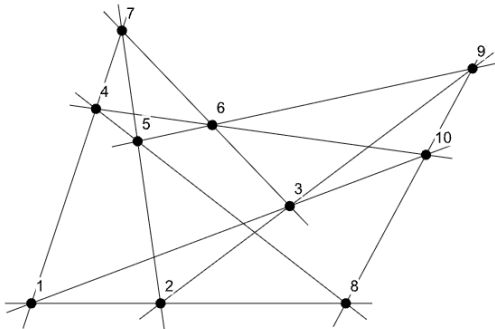
$$h^{1,2} = \dim H^1(X, T_X) = \dim H^2(X, \Omega_X) = 101.$$

The holomorphic volume form: $\text{res} \left(\frac{\sum_i \pm x_i dx_0 \wedge \dots \widehat{dx_i} \wedge \dots \wedge dx_4}{F(x_0, x_1, x_2, x_3, x_4)} \right).$

$$X_\psi : x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0$$

$$G = \left\{ (\alpha_0, \dots, \alpha_4) \in (\mu_5)^5 \mid \prod_{i=0}^4 \alpha_i = 1 \right\} / \mu_5,$$

Singular locus of X_ψ/G :



Blow-up X_ψ/G to desingularize: get the 1-parameter *mirror quintic*, with 101 curve classes β .

The Hodge diamond.

The Hodge diamond of the smooth quintic is $(\chi = 4 - 204 = -200)$

$$\begin{array}{ccccc}
 & & & & 1 \\
 & & & 0 & & 0 \\
 & & 0 & & 1 & & 0 \\
 1 & & 101 & & 101 & & 1, \\
 & & 0 & & 1 & & 0 \\
 & & & 0 & & 0 \\
 & & & & & & 1
 \end{array}$$

for the resolved Dwork quotient, it is $(\chi = 204 - 4 = 200)$

$$\begin{array}{ccccc}
 & & & & 1 \\
 & & & 0 & & 0 \\
 & & 0 & & 101 & & 0 \\
 1 & & 1 & & 1 & & 1. \\
 & & 0 & & 101 & & 0 \\
 & & & 0 & & 0 \\
 & & & & & & 1
 \end{array}$$

Why are CY3s special with regards to curve enumeration?

X : complex manifold of arbitrary dimension n .

$C \subset X$: a (smooth, proper) curve in X .

The space of infinitesimal deformations of C in X is

$\Gamma(C, N_{C/X})$, normal vector fields on C .

This can be approximated, using Riemann-Roch for the curve C , as:

(note the ses. $0 \rightarrow T_C \rightarrow T_X|_C \rightarrow N_{C/X} \rightarrow 0$)

$$\begin{aligned} \chi(C, N_{C/X}) &= \deg N_{C/X} + (n-1)(1-g) \\ &= \deg(T_X|_C) - \deg(T_C) + (n-1)(1-g) \\ &= \deg(\Lambda^n T_X|_C) - (2-2g) + (n-1)(1-g) \\ &= -\deg(\omega_X|_C) + (n-3)(1-g). \end{aligned}$$

This equals zero (so that C does not deform), independently of C , if

- ① $n = \dim X = 3$,
- ② ω_X is trivial, the Calabi-Yau condition.

So for a CY3 X , in generic situations, expect a reasonable answer to the

Question

How many curves of genus g and class $\beta \in H_2(X, \mathbb{Z})$ exist in X ?

Conjecture (Clemens)

Let X be a sufficiently general/generic quintic threefold. The number of rational curves ($g = 0$) of fixed degree is finite.

For example the number of lines is

$$n_{0,1} = 2875$$

The number of conics is

$$n_{0,2} = 609\,250$$

The conjecture has only been proved for $d \leq 11$.

These lines and conics all look like $\mathbb{P}^1 \rightarrow \mathcal{O}(-1) + \mathcal{O}(-1)$, if X is generic enough. Ideally, all rational curves look like this, but this (the strong version of Clemens conjecture) is false.

In general, the normal bundle of $C \rightarrow X$ is a vector bundle of rank 2, $N \rightarrow C$, such that $\det N = \omega_C$. For example $N = L_1 \oplus L_2$, such that $L_1 \otimes L_2 \xrightarrow{\sim} \omega_C$. This kind of open CY3 is called a *local curve*, and is very useful for heuristic study of curves in CY3s.

Candelas, de la Ossa, Green, and Parkes in 1991. Form the generating function for $n_{0,d}$, number of rational curves of degree d on the quintic

$$\partial_t F(t) = \frac{5}{2}t^2 + \frac{25}{6}t + \frac{25}{6} + \frac{1}{(2\pi i)^2} \sum_{d=1}^{\infty} d n_{0,d} \operatorname{Li}_2(e^{2\pi i d t})$$

and consider the two periods: $y_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$

$$y_1(z) = y_0(z) \log(z) + 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left(\sum_{j=n+1}^{5n} \frac{1}{j} \right) z^n$$

Form the mirror map

$$t(z) = y_1(z)/y_0(z)$$

Then the above generating function after changing coordinates from t to z is a solution to the Picard-Fuchs equation

$$\left[\left(z \frac{dz}{z} \right)^4 - 5^5 z \left(z \frac{dz}{z} + \frac{1}{5} \right) \left(z \frac{dz}{z} + \frac{2}{5} \right) \left(z \frac{dz}{z} + \frac{3}{5} \right) \left(z \frac{dz}{z} + \frac{4}{5} \right) \right] \left(y_0(z) \partial_t F(t(z)) \right) = 0$$

This determines the $n_{0,d}$ recursively. (Mirror symmetry, X and \hat{X} .)

To make sense of the numbers, need

- ① a moduli space $M_{g,\beta}(X)$,
- ② its virtual fundamental class $[M_{g,\beta}(X)]^{\text{vir}} \in H_0^{BM}(M_{g,\beta}(X))$.

Consider the Fermat quintic

$$X : x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0,$$

and its moduli space of lines $M_{0,1}(X)$.

X is not generic with respect to lines. In fact, it has 50 pencils of lines:

Given $\zeta^5 = -1$ and $\langle a, b, c \rangle \subset \mathbb{P}^2$ lying on the Fermat curve

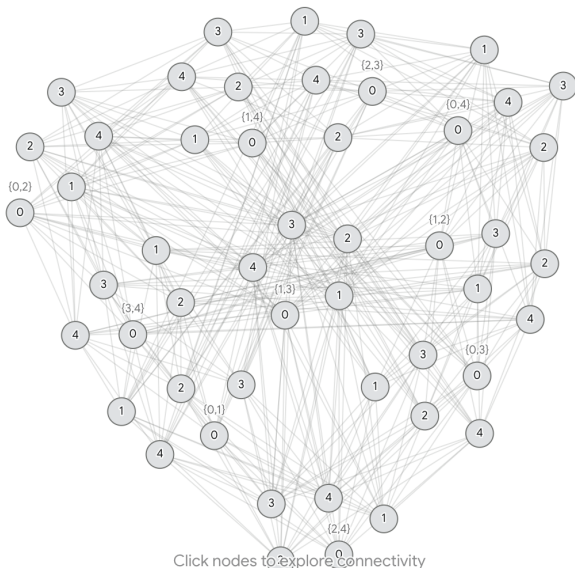
$a^5 + b^5 + c^5 = 0$, we have

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^4 \\ \langle u, v \rangle &\longmapsto \langle u, \zeta u, av, bv, cv \rangle. \end{aligned}$$

There are 10 choices of where to put the u part vs. the v part, and 5 choices for ζ .

There are 375 lines which are members of two of these pencils, they are given by setting one of a, b, c equal to zero.

The moduli space of lines on the Fermat quintic



The moduli space of lines $M_{0,1}(X)$ on the Fermat quintic consists of 50 genus 6 smooth curves, which intersect in 375 ordinary double points, each component intersects 15 others. Away from the intersections this moduli space is not smooth, but carries a scheme structure transversally given by $\mathbb{C}[t]/t^2$, in other words the critical locus of t^3 . At the nodes, the scheme structure is the critical locus of $t_1^2 t_2^2$.

This is quite complicated, but we can describe it a different way, namely as the space of lines in $\mathbb{P}^4(\mathbb{C})$, which happen to lie on $X \subset \mathbb{P}^4(\mathbb{C})$.

Let Y be the space of lines in $\mathbb{P}^4(\mathbb{C})$. (This is a Grassmannian variety, it has dimension 6.)

Over Y we have a vector bundle whose fibre over a line $L \subset \mathbb{P}^4(\mathbb{C})$ (given by a 2-dimensional subspace $W \subset \mathbb{C}^5$) is the space of quintic equations on W . This vector bundle is $\text{Sym}^5 W^*$ over Y (it has rank 6).

Restricting the Fermat equation to W gives us a global section $s \in \Gamma(Y, \text{Sym}^5 W^*)$, and the zero locus of this section is the space of lines on the Fermat quintic

$$\begin{array}{ccc} M_{0,1}(X) & \longrightarrow & Y \\ \downarrow & & \downarrow s \\ Y & \xrightarrow{0} & \text{Sym}^5 W^* \end{array}$$

For a generic quintic this section will have a discrete set of zeros, because $\text{rank}(\text{Sym}^5 W^*) = \dim Y$.

For the Fermat quintic, it has the complicated structure described above.

The *virtual fundamental class* of $M_{0,1}(X)$ is the localized top Chern class of this bundle $\text{Sym } W^*$.

$$[M_{0,1}(X)]^{\text{vir}} = c_{\text{top}}(\text{Sym}^5 W^*) \cap [Y] \in H_0^{BM}(M_{0,1}(X)).$$

It is a homology class of degree 0, which pushes forward into Y to give the Poincaré dual of the top Chern class of the bundle $\text{Sym}^5 W^*$.

The *virtual number of points* on the Fermat quintic is

$$\int_{[M_{0,1}(X)]^{\text{vir}}} 1 = \deg c_{\text{top}}(\text{Sym}^5 W^*) \cap [Y] = 2875. \quad (\text{Schubert})$$

(This requires $M_{0,1}(X)$ to be compact.)

Main advances: Fulton-MacPherson style intersection theory, upgraded in the 1990s by Li-Tian and B.-Fantechi to be intrinsic to the moduli space $M_{0,1}(X)$ without embedding into an ambient Y , which may not exist, or maybe useless for the definition of a virtual class.

Consider again the tangent space to the space of curves in CY3 X at a given curve $C \rightarrow X$.

This is $H^0(C, N_{C/X})$, the normal vector fields to C in X .

The existence of $H^1(C, N_{C/X})$ is the main obstruction to the moduli space being of the expected dimension, i.e., just a finite set of points.

$$\begin{aligned} H^1(C, N_{C/X})^\vee &= H^0(C, N_{C/X}^\vee \otimes \Omega_C) \\ &= H^0(C, N_{C/X} \otimes \det N_{C/X}^\vee \otimes \Omega_C) \\ &= H^0(C, N_{C/X} \otimes \det \Omega_X) \\ &= H^0(C, N_{C/X}) \end{aligned}$$

Observation

The tangent space to the space of curves in X is the dual of the obstruction space.

In fact, the moduli space of curves in a CY3 is *derived symplectic*, with a symplectic form of degree -1 .

Theorem (shifted Darboux theorem)

The local geometry of a derived space with -1 -shifted symplectic structure is given by the (derived) scheme theoretic critical locus of a holomorphic or polynomial function.

Given a function $f : \mathbb{C}^n \rightarrow \mathbb{C}$, the critical locus $\text{Crit}(f)$ is defined as the common zero locus of the n functions

$$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} : \mathbb{C}^n \rightarrow \mathbb{C}.$$

The tangent space and the obstruction space at the origin are, respectively, the kernel and the cokernel of the matrix of partial derivatives of these functions

$$0 \longrightarrow T_{\text{Crit } f|_0}^0 \longrightarrow \mathbb{C}^n \xrightarrow{\frac{\partial^2 f}{\partial x_i \partial x_j}|_0} \mathbb{C}^n \longrightarrow T_{\text{Crit } f|_0}^1 \longrightarrow 0.$$

Key: the Hessian matrix is symmetric.

Therefore: T^0 (tangents) and T^1 (obstructions) are duals of each other.

Hence: a critical locus is a -1 -shifted symplectic (derived) scheme.

Let M be a smooth, non-compact complex manifold, and let $f : M \rightarrow \mathbb{C}$ be a holomorphic function. We consider the critical locus $Y = \text{Crit}(f) \subset M$. For the purposes of this discussion, let us assume that Y is compact. The space Y is the intersection of the zero section and the graph of the differential df

$$\begin{array}{ccc} Y & \longrightarrow & M \\ \downarrow & & \downarrow \Gamma_{df} \\ M & \xrightarrow{0} & \Omega_M \end{array}$$

This is the intersection of two Lagrangian submanifolds within the cotangent bundle Ω_M (which is a complex symplectic manifold). Because Y is compact, the virtual intersection number is well-defined:

$$\#^{\text{virt}}(Y) = \mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \int_{[Y]^{\text{virt}}} 1$$

where $[Y]^{\text{virt}} \in H_0(Y, \mathbb{Q})$ is the virtual fundamental class of the intersection scheme Y .

Theorem (Singular Gauß-Bonnet)

The intersection number is given by the Euler characteristic of Y weighted by a constructible function $\mu : Y \rightarrow \mathbb{Z}$, namely:

$$\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \chi(Y, \mu).$$

For a point $P \in Y$, the value $\mu(P)$ is the Milnor number of f at P :

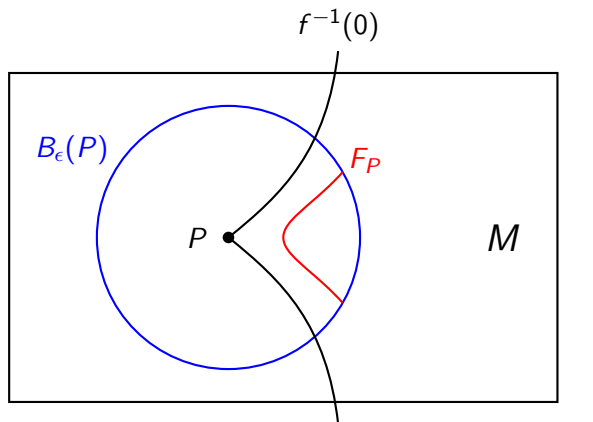
$$\mu(P) = (-1)^{\dim M} (1 - \chi(F_P)).$$

Here F_P is the Milnor fibre of f at the point P , which is the intersection of a nearby fibre of f with a small ball around P .

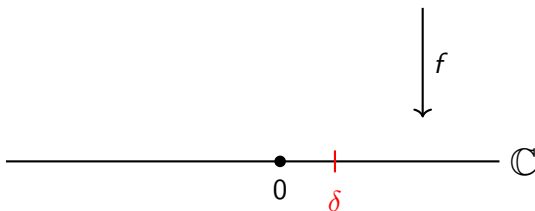
Thus, each point P of the Lagrangian intersection (critical locus) contributes $\mu(P)$ to the intersection number.

Theorem is a result of microlocal geometry, from the 1970s. (Microlocal index theorem of Kashiwara and MacPherson, determination of the characteristic variety of the perverse sheaf of vanishing cycles.)

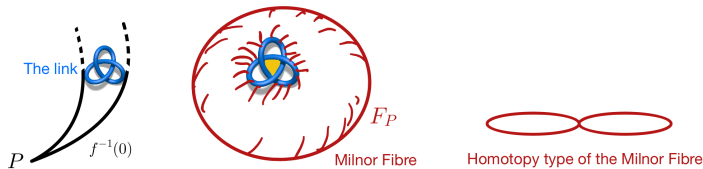
The Milnor fibre



$$\mu(P) = \pm(1 - \chi(F_P))$$



Example: $f(x, y) = x^2 + y^3$.



The partial derivatives are $2x = 0$, and $3y^2 = 0$, so this is an isolated point of multiplicity 2. The intersection number is 2.

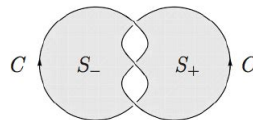
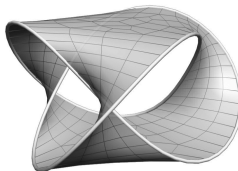
The link is a trefoil knot. The singularity of $f^{-1}(0)$ at the origin P is the cone over the trefoil knot. The Milnor fibre is a surface with boundary, the boundary is a circle, in the shape of a trefoil knot. (Seifert surface of the knot) The homotopy type of the Milnor fibre is that of a sphere with three points removed, or a bouquet of two circles. The Euler characteristic of the Milnor fibre is -1 , and the Milnor number at P is

$$\mu(P) = (-1)^2(1 - (-1)) = 2,$$

indicating that there are 2 vanishing cycles.

The Milnor number reproduces the intersection number.

Example: $f(x, y) = x^2 + y^3$.



In fact, for isolated singularities ($\dim Y = 0$), it is a classical theorem of Milnor that the Milnor fibre has the homotopy type of a bouquet of spheres, and the multiplicity of the singularity equals the number of these spheres.

In the case where the actual dimension of the critical locus equals the expected dimension, the singular Gauß-Bonnet theorem is Milnor's theorem.

Suppose instead that the Lagrangian intersection/critical locus Y is smooth. Then, always:

$$[Y]^{\text{vir}} = c_{\text{top}}(\Omega_Y) \cap [Y]$$

$$\mu(P) = (-1)^{\dim Y}$$

Hence,

$$\begin{aligned} \mathcal{I}_{\Omega_M}(M, \Gamma_{df}) &= \int_{[Y]^{\text{vir}}} 1 = \int_{[Y]} c_{\text{top}}(\Omega_Y) \\ &= (-1)^{\dim Y} \int_{[Y]} c_{\text{top}}(T_Y) = (-1)^{\dim Y} \chi(Y) = \chi(Y, \mu), \end{aligned}$$

and so the singular Gauß-Bonnet theorem is the classical one.

Theorem (B.)

The constructible function $\mu_Y : Y \rightarrow \mathbb{Z}$ is intrinsic to the scheme Y , if Y is smooth it takes the value $\mu_Y(P) = (-1)^{\dim Y}$, for all $P \in Y$.

The singular Gauß-Bonnet theorem can be stitched together to give rise to

Theorem (B.)

For any proper scheme Y with an obstruction theory, which is dual to the deformation theory, we have

$$\#^{\text{virt}}(Y) = \int_{[Y]^{\text{vir}}} 1 = \chi(Y, \mu_Y).$$

One of the main implications of this theorem is that the intersection number is *motivic*, i.e.,

- ① it makes sense for non-compact schemes: $\#^{\text{virt}}(Y) = \chi(Y, \mu_Y)$,
- ② it is additive over stratifications:

$$\chi(Y, \mu_Y) = \chi(Y \setminus Z, \mu_Y) + \chi(Z, \mu_Y), \quad \text{if } Z \hookrightarrow Y \text{ is closed.}$$

This is unusual for intersection numbers, it is only true for Lagrangian intersections.

Let $K(\text{Var})$ be the group of *motivic weights*, i.e., the Grothendieck group of \mathbb{C} -varieties modulo scissor relations: $[Y] = [Y \setminus Z] + [Z]$, whenever $Z \rightarrow Y$ is a closed immersion.

There exists a lift, the motivic virtual count of critical loci:

$$\begin{array}{ccc}
 & & K(\text{Var}) \\
 & \nearrow \Phi & \downarrow \chi \\
 \text{critical loci} & \xrightarrow{\#^{\text{virt}}} & \mathbb{Z}
 \end{array}$$

Here, Φ is defined by

$$\Phi(M, f) = -q^{-\frac{\dim M}{2}} [\phi_f],$$

where $q = [\mathbb{C}]$ is the motivic weight of the affine line, and $[\phi_f]$ are the *motivic vanishing cycles* of Denef-Loeser (2000), from their work on motivic integration, a motivic version of Milnor fibres.

e.g. $\Phi(\text{Crit}(x^2 + y^3)) = (2 - [E])/q$, where $E : x^2 + y^3 = 1$.

Example: $\text{Hilb}^n(\mathbb{C}^3)$.

The Hilbert scheme $\text{Hilb}^n(\mathbb{C}^3)$ is the scheme parametrizing subschemes of length n of affine 3-space. The affine 3-space \mathbb{C}^3 is a non-compact Calabi-Yau threefold, so that fits with our theme, although counting point configurations does not. Nevertheless, $\text{Hilb}^n(\mathbb{C}^3)$ is a critical locus, so it is an example of a (-1) -shifted symplectic scheme.

In fact, we can view $\text{Hilb}^n(\mathbb{C}^3)$ as the scheme of three commuting $n \times n$ -matrices, namely, respectively, the x , y and z coordinates of the subscheme, together with a generating vector. Then $\text{Hilb}^n(\mathbb{C}^3)$ is the critical locus of

$$f : (M_{n \times n}(\mathbb{C})^3 \times \mathbb{C}^n)^{\text{stab}} / GL_n \longrightarrow \mathbb{C}$$

$$(A, B, C, v) \longmapsto \text{tr}([A, B]C).$$

Example: $\text{Hilb}^n(\mathbb{C}^3)$.

Theorem

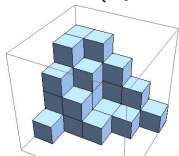
For the motivic weight of $\text{Hilb}^n(\mathbb{C}^3)$ we have

$$\sum_{n=0}^{\infty} \Phi(\text{Hilb}^n(\mathbb{C}^3)) t^n = \prod_{m=1}^{\infty} \prod_{k=1}^m \frac{1}{1 - q^{k+1-\frac{m}{2}} t^m}.$$

If we specialize to $q^{\frac{1}{2}} \rightarrow -1$, we get

$$\sum_{n=0}^{\infty} \#^{\text{virt}}(\text{Hilb}^n(\mathbb{C}^3)) t^n = \prod_{m=1}^{\infty} \left(\frac{1}{1 - (-t)^m} \right)^m.$$

This is (up to signs) the generating function for 3-dimensional partitions.



$$\sum_{n=0}^{\infty} \#\{3\text{D partitions of } n\} t^n = \prod_{m=1}^{\infty} \left(\frac{1}{1 - t^m} \right)^m$$

Using Donaldson-Thomas theory, we can make sense of motivic curve counts in Calabi-Yau threefolds. Our moduli scheme will be the Hilbert scheme of 1-dimensional proper subschemes $Z \subset X$ of fixed genus and holomorphic Euler characteristic. Call this scheme $\mathcal{I}_{g,n}(X)$. It makes sense for any Calabi-Yau threefold X , proper or not.

The motivic DT invariant of $\mathcal{I}_{g,n}(X)$ is constructed as follows.

First, the moduli space is locally the critical locus of a function. We have motivic vanishing cycles for these critical loci, they get combined into global invariants by an orientation of the moduli space. In the extreme case where the curve part of the subscheme is empty, we are back in the case of $\text{Hilb}^n(X)$.

Theorem (B.-Bryan-Szendrői)

The motivic weight of the Hilbert scheme of n points of the Calabi-Yau threefold X , compact or not, is given by

$$\sum_{n=0}^{\infty} \Phi(\mathrm{Hilb}^n Y) t^n = \left(\prod_{m=1}^{\infty} \prod_{k=1}^m \frac{1}{1 - q^{k-2-\frac{m}{2}} t^m} \right)^{[Y]}.$$

Uses the power structure on $K(\mathrm{Var})$, to make sense of the exponent.

If we specialize $q^{\frac{1}{2}} = -1$, we get

$$\sum_{n=0}^{\infty} \#^{\mathrm{vir}}(\mathrm{Hilb}^n Y) t^n = \left(\prod_{m=1}^{\infty} \frac{1}{1 - (-t)^m} \right)^{\chi(Y)}.$$

For example, the quintic gives:

$$\sum_{n=0}^{\infty} \#^{\mathrm{vir}}(\mathrm{Hilb}^n Y) t^n = \left(\prod_{m=1}^{\infty} \frac{1}{1 - (-t)^m} \right)^{-200}.$$

Thanks!