Enumerative Geometry of Calabi-Yau threefolds

Kai Behrend

The University of British Columbia

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Definition.

A *Calabi-Yau threefold* is a complex algebraic variety X, satisfying the following conditions:

1. X is projective. This means it is the common zero locus of a finite number of homogeneous equations in $n+1$ variables

   $$F_1(x_0,\dots,x_n),\dots,F_k(x_0,\dots,x_n)\,,$$

   in $\mathbb{P}^n(\mathbb{C})$,
2. X is smooth, or non-singular, i.e., a compact complex manifold,
3. $\dim_{\mathbb{C}} X = 3$,
4. there exists a global, nowhere vanishing holomorphic differential 3-form, in other words, a global section of $\omega_X = \Omega^3_X$ (a holomorphic orientation), this is considered part of the structure,
5. (strict Calabi-Yau) $H^1(X,\mathscr{O}_X) = H^2(X,\mathscr{O}_X) = 0$, or equivalently, $H^0(\Omega) = H^0(\Omega^2) = 0$.

The Hodge diamond.

For a CY3, with $\Omega^3_X = \mathscr{O}_X$, we have $\quad \Omega^2_X = T_X \quad$ and $\quad \Omega^1_X = \Lambda^2_X$.
By definition,

$$h^{p,q} = \dim H^q(X, \Omega^p)\,,$$

the q-th cohomology of the sheaf of holomorphic or algebraic p-forms.
We arrange these dimensions in a diamond:

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

The sum of numbers in row n is the dimension of $H^n(X, \mathbb{Q})$.
So the alternating sum of all numbers in the Hodge diamond is the Euler characteristic of X.

$$\chi(X) = \textstyle\sum_{pq} (-1)^{p+q} h^{p,q}\,.$$

The Hodge diamond of the smooth quintic is $(\chi = 6 - 202 = -196)$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1\,, \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

for the resolved Dwork quotient, it is $(\chi = 6 + 202 = 208)$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1\,. \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Conjecture (Clemens)

Let X be a sufficiently general/generic quintic threefold. The number of rational curves ($g = 0$) of fixed degree is finite.

For example the number of lines is

$$n_1 = 2875$$

The number of conics is

$$n_2 = 609\,250$$

The conjecture has only been proved for $d \leq 11$.

We have been vague about the actual moduli spaces used to define curve counts in CY3s. Briefly, some of the more important ones are the following.

Here we model curves in X by *maps* $f : C \to X$ from an abstract curve C to X. To compactify the moduli space we need to allow C to acquire nodal singularities. This gives rise to $\overline{M}_g(X, \beta)$, the space of stable maps of genus g and curve class $\beta \in H_2(X, \mathbb{Z})$. Maps can have symmetries, so $\overline{M}_g(X, \beta)$ is actually an algebraic stack. The moduli stack has a virtual fundamental class $[\overline{M}_g(X, \beta)]^{\mathrm{vir}} \in H_0^{BM}(\overline{M}_g(X, \beta))$, and if X is compact, the integral or degree of the virtual fundamental class make sense and we get the Gromov-Witten invariants

$$GW_g(X, \beta) = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{vir}}} 1 \quad \in \mathbb{Q}\,.$$

These are rational numbers, because of the symmetries. The stable map spaces have expected dimension zero, but they are not (-1)-shifted symplectic. Things go wrong at the boundary. If X is not compact, we can still make sense of GW invariants if there is a $\mathbb{C}^*$-action on X, which induces a $\mathbb{C}^*$-action on $\overline{M}_g(X, d)$ with compact fixed locus. One uses the Atiyah-Bott localization formula to define invariants in this case.

Here we model curves as subschemes $Z \subset X$, and we consider the moduli space of ideal sheaves of 1-dimensional subschemes $\mathcal{I}_{g,n}(X)$. Degenerate curves acquire 'point fuzz'.
This moduli space is (-1)-shifted symplectic, and therefore invariants make sense whether or not X is compact, because in the non-compact case we can use the weighted Euler characteristic as definition.
These invariants may be made motivic, and may be categorified.
They are also integers by construction.

Pandharipande-Thomas or PT-invariants.

These are a variation of DT invariants, which remove the 0-dimensional fuzz which 1-dimensional subschemes tend to acquire under deformations. Technically, these are defined via moduli spaces of certain derived category objects on X.

$$[\mathscr{O}_X \longrightarrow \mathcal{F}] \quad \in D(X, \mathscr{O}_X)$$

Again, the moduli space are -1 shifted symplectic, and make sense whether or not X is compact.

Gopakumar-Vafa invariants

These are certain delicate recombinations of Gromov-Witten invariants, which turn out to be integers (Ionel-Parker).

Let X be a Calabi-Yau threefold. Let $\beta \in H_2(X, \mathbb{Z})$ be a curve class. There is an associated GV-invariant $n_g(X, \beta)$ for every genus g. In genus 0,

$$GW_0(X, \beta) = \sum_{m|\beta} \frac{1}{m^3} n_0(X, \tfrac{\beta}{m}) \,.$$

For example, if X is the quintic $\beta = d$ is the degree, and we have

$$GW_0(X, d) = \sum_{m|d} \frac{1}{m^3} n_0(X, \tfrac{d}{m}) \,.$$

$$GW_0(X, 1) = n_0(X, 1) \,,$$

$$GW_0(X, 2) = n_0(X, 2) + \tfrac{1}{8} n_0(X, 1) \,.$$

(The double covers of each line contribute a value of $\frac{1}{8}$ to the GW invariants of degree 2.)

The various curve counting theories

1. Gromov-Witten
2. Donaldson-Thomas
3. Pandharipande-Thomas
4. Gopakumar-Vafa
5. there are more

should all contain the same information. (The GV-invariants should be the true virtual curve count.) In fact, there are precise conjectures stating how to convert between the numbers of the different theories, for example, relating GW and PT are the MNOP-conjecures. These are very subtle correspondences.
John Pardon ("Universal Counting Curves...") has recently provided a framework to classify the various curve counting theories. In particular, he proves the MNOP conjecture. I would like to explain some of this. It is extremely technical, I will be simplifying a lot.

Every compact CY3 X, with a curve class $\beta \in H_2(X, \mathbb{Z})$ poses an *enumerative problem*. ('Count curves in X of class β.')

$$1_{\mathcal{Z}(X,\beta)} \,.$$

If (X', β') is a deformation of (X, β) the two enumeration problems are *equal*.

$$1_{\mathcal{Z}(X,\beta)} = 1_{\mathcal{Z}(X',\beta')} \,.$$

Better: If $X \to S$ is a proper CY3-family, and $\beta \in H_2(X, \mathbb{Z})$ a curve class, then we get an enumerative problem for the family, which is equal to the one posed by any of its fibres.

For a CY3 X, compact or not, we define $\mathcal{Z}(X,\beta)$ to be the space of effective 1-cycles

$$\sum_{\text{finite}} m_i C_i\,, \qquad m_i \in \mathbb{Z}_{\geq 0}, \quad C_i \subset X \text{ irreducible compact subvariety,}$$

$$\sum m_i [C_i] = \beta\,.$$

The space $\mathcal{Z}(X,\beta)$ serves as moduli space of curves in X.
Write $\mathcal{Z}(X) = \coprod_\beta \mathcal{Z}(X,\beta)$.
We consider *families* of Calabi-Yau theefolds

$$\pi : X \longrightarrow S$$

morphism of complex manifolds, all fibres are (not necessarily compact) Calabi-Yau threefolds.
For each such family, we have a *relative* cycle space of 1-cycles:

$$\mathcal{Z}(X/S) \longrightarrow S$$

The fibre over $s \in S$ is $\mathcal{Z}(X_s)$.

Axiom

Given a CY3-family $X \to S$, and a cohomology class

$$\alpha \in H_c^{2\dim S}(\mathcal{Z}(X/S)),$$

we get an associated enumerative problem, denoted $(X/S, \alpha)$.

For example, $X \to *$ compact, then $\mathcal{Z}(X, \beta)$ is also compact, and we have, $1_{\mathcal{Z}(X,\beta)} \in H_c^0(\mathcal{Z}(X))$.

Axiom

Given a pullback diagram of families, with induced pullback diagram of cycle spaces:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "u"] & S \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Z}(X'/S') \arrow[r] \arrow[d] & \mathcal{Z}(X/S) \arrow[d] \\ S' \arrow[r, "u"] & S \end{tikzcd}$$

and $\alpha' \in H_c^{2\dim S'}(\mathcal{Z}(X'/S'))$ *we have the equality of enumeration problems*

$$(X'/S', \alpha') = (X/S, u_! \alpha') \,.$$

The Pardon algebra of CY3 enumeration problems:

$$H_c^0(\mathcal{Z}/CY3) := \varinjlim_{X/S} H_c^{2\dim S}(\mathcal{Z}(X/S)) \,.$$

Dually, we have:

Axiom

An enumerative theory *e assigns to every CY3-family $X \to S$ a homology class*

$$e(X/S) \in H^{BM}_{2\dim S}(\mathcal{Z}(X/S))\,.$$

It is required that for every pullback diagram of CY3-families $u : X'/S' \to X/S$,

$$u^! e(X/S) = e(X'/S')\,.$$

The Pardon coalgebra of enumerative theories is

$$H^{BM}_0(\mathcal{Z}/CY3) = \varprojlim_{X/S} H^{BM}_{2\dim S}(\mathcal{Z}(X/S))\,.$$

Examples: GW_0, or $\sum_g GW_g u^{2g-2}$.

Enumeration problems: $H^0_c(\mathcal{Z}/CY3) = \varinjlim_{X/S} H^{2\dim S}_c(\mathcal{Z}(X/S))\,.$

Enumerative theories: $H^{BM}_0(\mathcal{Z}/CY3) = \varprojlim_{X/S} H^{BM}_{2\dim S}(\mathcal{Z}(X/S))\,.$

By their nature, there is a pairing

$$H^{BM}_0(\mathcal{Z}/CY3) \otimes H^0_c(\mathcal{Z}/CY3) \longrightarrow \mathbb{Q}\,.$$

for example

$$\langle GW_0, 1_{\mathcal{Z}(X,\beta)}\rangle = GW_0(X,\beta)$$

Technically, the enumerative theory GW_0 is defined as follows:
Start with the virtual fundamental class $[\overline{M}_0(X,\beta)]^{\mathrm{vir}} \in H^{BM}_0(\overline{M}_0(X,\beta))$, push it forward via the proper morphism (stable map $\mapsto$ image cycle)

$$\overline{M}_0(X,\beta) \longrightarrow \mathcal{Z}(X,\beta)$$

to get the class

$$GW_0(X/*) \in H^{BM}_0(\mathcal{Z}(X))\,.$$

Start with a smooth and proper curve C, of genus g, and a vector bundle $E \to C$ of rank 2 over C. Assume $\det E = \omega_C$. Then $\operatorname{tot} E$ is a non-compact Calabi-Yau threefold. 1-cycles $\Sigma \to \operatorname{tot} E$ have a degree m, the degree of the associated ramified covering $\Sigma \to C$.
Then $\mathcal{Z}(\operatorname{tot} E, m)$ is a vector space, if we choose E carefully.
We will construct a twisted form of $\operatorname{tot} E$ over $\mathbb{P}^N$.

$$E_{\mathbb{P}^N} \quad := \quad [E \boxtimes \mathscr{O}(1) \longrightarrow C \times \mathbb{P}^N \longrightarrow \mathbb{P}^N] .$$

This is a CY3-family parametrized by $\mathbb{P}^N$, all fibres are isomorphic to E.
The associated relative cycle space is then a vector bundle

$$\mathcal{Z}(E, m)_{\mathbb{P}^N}$$

with typical fibre the vector space $\mathcal{Z}(E, m)$. The Thom class of this vector bundle is an element of

$$H_c^{2\dim \mathcal{Z}(E,m)}(\mathcal{Z}(E, m)_{\mathbb{P}^N}) = H_c^{2\dim \mathbb{P}^N}(\mathcal{Z}(E, m)_{\mathbb{P}^N})$$

if we choose $N = \dim \mathcal{Z}(E, m)$. This defines an enumeration problem, denoted $\quad x_{g,m}$.

Theorem (Pardon)

The space of enumeration problems $H^0_c(\mathcal{Z}/CY3)$ *is a commutative* $\mathbb{Q}$*-algebra, freely generated by the* equivariant local curve elements $x_{g,m}$ *for* $g \geq 0$, $m \geq 1$. *(So it's a polynomial algebra in the generators* $x_{g,m}$*.)*

The multiplication is defined in terms of disjoint unions:

$$1_{\mathcal{Z}(X,\beta)} \cdot 1_{\mathcal{Z}(Y,\gamma)} = 1_{\mathcal{Z}(X \amalg Y,\beta+\gamma)} \,.$$

Corollary

To prove that $GW = PT$*, Pardon only needs to check that*

$$\langle GW, x_{g,m}\rangle = \langle PT, x_{g,m}\rangle \quad \forall g, m\,.$$

In fact, if e *is any theory,* $\langle e, x_{g,m}\rangle$ *is the equivariant version of* e *for the open CY3* $\operatorname{tot} E$*, with* $\mathbb{C}^*$ *action by scalar multiplication on the fibres. The equivariant GW and PT theories were known previously* $\square$*.*

Gromov-Witten theory is a sum over all genera:

$$GW(u)(X/S) = \sum_{h \geq 0} [\overline{M}_h(X/S)]^{\mathrm{vir}} u^{2h-2}\,.$$

Pandharipande-Thomas theory is a sum over all χ:

$$PT(q)(X/S) = \sum_{n} [\mathcal{I}_n(X/S)]^{\mathrm{vir}} q^n\,.$$

For the equivariant invariants, we have the following formulas:

$$\langle GW(u), x_{g,m} \rangle = \sum_{\sum i \cdot k_i = m} \prod_{i=1}^{m} \frac{1}{k_i!} \left(\frac{1}{i} \left(2 \sin \left(\frac{iu}{2} \right) \right)^{2g-2} \right)^{k_i}$$

$$\langle PT(q), x_{g,m} \rangle = \sum_{\sum i \cdot k_i = m} \prod_{i=1}^{m} \frac{1}{k_i! \cdot i^{k_i}} \left((-1)^{g-1} (-q)^{i(1-g)} (1 - (-q)^i)^{2g-2} \right)^{k_i}$$

The PT and GW formulas become equal under the substitution $-q = e^{iu}$.

The result $GW = DT$ uses only that the $x_{g,m}$ generate. We can use the freeness as follows:

Observation

There is a unique CY3 enumerative theory e_0*, such that*

$$\langle e_0, x_{g,m}\rangle = 0 \quad \textit{if } g > 0\textit{, or } m > 1$$

and $\langle e_0, \textit{any polynomial in } x_{g,m} \textit{ of degree other than 1}\rangle = 0$.

Then the $\langle e_0, 1_{\mathcal{Z}(X,\beta)}\rangle$ are the GV invariants $n_0(X, \beta)$.

(To prove this, we need $\langle GW_0, 1_{\mathcal{Z}(X,\beta)}\rangle = \sum_d \frac{1}{d^3}\langle e_0, 1_{\mathcal{Z}(X,\beta/d)}\rangle$
By Pardon's theorem, it is sufficient to prove that
$\langle GW_0, x_{0,m}\rangle = \sum_d \frac{1}{d^3}\langle e_0, x_{0,m/d}\rangle$,
or, $\langle GW_0, x_{0,m}\rangle = \frac{1}{m^3}$,
which is a known result about equivariant Gromov-Witten invariants.)

Sheldon Katz suggested a geometric defn of genus zero GV invariants. Moduli space: $\mathrm{Sh}(X,\beta)$ classifies pure 1-dimensional sheaves $\mathscr{F}$ on X, of holomorphic Euler characteristic $\chi(X,\mathscr{F})=1$, that are Simpson (semi)-stable, and whose associated curve class is β.
Examples: $E\to C\to X$ vector bundle over non-singular curve in X.
$\chi(X,E)=\chi(C,E)=\deg(E)+\mathrm{rk}(E)(1-g(C))=1$.
So: $\mathscr{O}\to\mathbb{P}^1\to X$, or a stable vector bundle over any genus g curve C such that $d+r(1-g)$. We have $\beta=r[C]$, in this case.
The spaces $\mathrm{Sh}(X,\beta)$ are proper, they are schemes (automorphisms of stable sheaves are all scalar $\mathbb{C}^*$, and don't jump). They are -1 shifted symplectic, and so we have integers

$$\mathrm{sh}(X,\beta):=\int_{[\mathrm{Sh}(X,\beta)]^{\mathrm{vir}}}1=\chi(\mathrm{Sh}(X,\beta),\mu)\quad\in\mathbb{Z}\,.$$

Conjecture (Katz)

$\mathrm{sh}(X,\beta)=n_0(X,\beta)$, *the genus 0 GV invariants.*
Equivalently, $\quad GW_0(X,\beta)=\sum_d\frac{1}{d^3}\,\mathrm{sh}(X,\frac{\beta}{d})$

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$\mathrm{sh}(X,\beta) = n_0(X,\beta)$, *the genus 0 GV invariants.*
Equivalently, $GW_0(X,\beta) = \sum_d \frac{1}{d^3}\,\mathrm{sh}(X,\frac{\beta}{d})$

1. implies the integrality of the genus 0 GV invariants,
2. gives a very satisfying geometric meaning to the $g = 0$ GV invariants,
3. satisfyingly locates $g = 0$ GV invariants in the Pardon classification.

We propose the following approach to proving the conjecture. (Joint work in progress with Jim Bryan.) Elevate Katz's invariants to a Pardon enumerative theory sh. Then prove

Conjecture

$$\mathrm{sh} = e_0, \quad \textit{or} \quad \langle \mathrm{sh}, x_{g,m}\rangle = \begin{cases} 0 & \textit{if } g > 0 \\ 0 & \textit{if } m > 1 \\ 1 & \textit{if } g = 0, \textit{ and } m = 1\,, \end{cases}$$

and sh *vanishes on all polynomials in* $x_{g,m}$ *of degree* $\neq 1$.

Conjecture

$g > 0 : \langle \mathrm{sh}, x_{g,m} \rangle = 0 \qquad m > 1 : \langle \mathrm{sh}, x_{0,m} \rangle = 0 \qquad \langle \mathrm{sh}, x_{0,1} \rangle = 1.$
sh *vanishes on all polynomials in* $x_{g,m}$ *of degree* $\neq 1$.

Involves $\mathrm{Sh}(\mathrm{tot}\, E, m)$, where $E \to C$ is vector bundle/curve C of genus g.
Stable sheaves of dimension 1 on $\mathrm{tot}\, E$, of degree m over C, and $\chi = 1$.
(The last claim: cannot have a stable sheaf on a disjoint union.)
The moduli space $\mathrm{Sh}(\mathrm{tot}\, E, m)$ is not compact, but it carries a $\mathbb{C}^*$-action from the vector bundle structure of E. This gives rise to equivariant Katz invariants, which are the numbers $\langle \mathrm{sh}, x_{g,m} \rangle$.
The equivariant invariants are equal to weighted Euler characteristics, which are easier to calculate.
If $g(C) > 0$, then $\mathrm{Pic}^0(C)$ acts on $\mathrm{Sh}(\mathrm{tot}\, E, m)$: $\quad L \cdot \mathscr{F} = L \otimes \mathscr{F}$.
This is a fixed point free action, forcing $\chi(\mathrm{Sh}(\mathrm{tot}\, E, m)) = 0$.
This is still true for the weighted Euler characteristic, because the Milnor number $\pm(1 - \chi(F_P))$ is an intrinsic invariant of the critical scheme.
(The case $g(C) = 0$ was solved by Katz.)

One caveat: equating equivariant invariants with weighted Euler characteristics requires a Calabi-Yau action of $\mathbb{C}^*$. The action has to preserve the determinant: $\det(E) = \det(L \oplus M) = L \otimes M$. This requires to act with opposite weights on L and M. This is a different action than the scalar multiplication action. Have to construct analogues $\widetilde{x}_{g,m}$ of the generators $x_{g,m}$ defined in terms of this Calabi-Yau action, and prove that the $\widetilde{x}_{g,m}$ generate Pardon's algebra of enumeration problems as well. We are currently working on expressing the $x_{g,m}$ in terms of the $\widetilde{x}_{g,m}$, this turns out to be trickier than it looks at first sight.

Thanks!