Canad. J. Math. Vol. **53** (5), 2001 pp. 897–922

On Some Exponential Equations of S. S. Pillai

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Abstract. In this paper, we establish a number of theorems on the classic Diophantine equation of S. S. Pillai, $a^x - b^y = c$, where a, b and c are given nonzero integers with $a, b \geq 2$. In particular, we obtain the sharp result that there are at most two solutions in positive integers x and y and deduce a variety of explicit conditions under which there exists at most a single such solution. These improve or generalize prior work of Le, Leveque, Pillai, Scott and Terai. The main tools used include lower bounds for linear forms in the logarithms of (two) algebraic numbers and various elementary arguments.

1 Introduction

In a series of papers in the 1930's and 1940's, S. S. Pillai [Pi1], [Pi2], [Pi3], [Pi4] studied the Diophantine equation

$$a^x - b^y = c \tag{1.1}$$

in positive integers a, b, x and y, where c is a fixed nonzero integer. Indeed, his famous conjecture that, for each such c, equation (1.1) has at most finitely many solutions in integers a, b, x and y exceeding unity appears for the first time in [Pi2]. This remains an outstanding open problem, though the case $c = 1$ (Catalan's Conjecture) was essentially solved by Tijdeman [Ti] (see Mignotte [Mi2] for an excellent survey of recent developments on this front).

In this paper, we will address the rather more modest problem of equation (1.1) when all three of a, b and c are fixed nonzero integers with $a, b \geq 2$ (this is, in fact, the situation considered by Pillai in [Pi1] and [Pi2]). Here, we can relax the conditions on x and y to include the potential solutions $x = 1$ or $y = 1$. Already in this case, from work of Polya [Po], it was known that equation (1.1) could possess at most finitely many integral solutions. This result was subsequently quantified by Herschfeld [He] (applying arguments of Pillai [Pi1]) who demonstrated that at most nine pairs of positive (x, y) may satisfy (1.1), provided c is sufficiently large relative to a and b and $\gcd(a, b) = 1$. Subsequently, Pillai [Pi2] showed that this equation has, again if c is sufficiently large and $\gcd(a, b) = 1$, at most one such solution. His proof of this result relies upon Siegel's sharpening of Thue's theorem on rational approximation to algebraic numbers and is hence ineffective (in the sense that, *a priori*, there is no way to quantify the term "sufficiently large"). With a modicum of computation, we can, in fact, find a number of examples where there are two

Received by the editors October 5, 2000; revised February 12, 2001.
Supported in part by NSF Grant DMS-9700837.
AMS subject classification: Primary: 11D61, 11D45; secondary: 11J86.

solutions to (1.1) in positive integers x and y, corresponding to the following set of equations:

$$\begin{aligned}
3-2 = 3^2-2^3 &= 1\\
2^3-3 = 2^5-3^3 &= 5\\
2^4-3 = 2^8-3^5 &= 13\\
2^3-5 = 2^7-5^3 &= 3\\
13-3 = 13^3-3^7 &= 10\\
91-2 = 91^2-2^{13} &= 89\\
6-2 = 6^2-2^5 &= 4\\
15-6 = 15^2-6^3 &= 9\\
280-5 = 280^2-5^7 &= 275\\
4930-30 = 4930^2-30^5 &= 4900\\
6^4-3^4 = 6^5-3^8 &= 1215.
\end{aligned} \tag{1.2}$$

There exist no examples of triples (a, b, c) for which equation (1.1) has three positive solutions; this is the content of our first result:

Theorem 1.1 *If a, b and c are nonzero integers with $a, b \geq 2$, then equation (1.1) has at most two solutions in positive integers x and y.*

This theorem sharpens work of Le ([Le, Theorem 2]; see also Shorey [Sh]) who obtained a similar result under the hypotheses $\min\{a, b\} \geq 10^5$, $\min\{x, y\} \geq 2$ and $\gcd(a, b) = 1$ (in case $\gcd(a, b) > 1$, a like result is claimed in [Le], but no proof is provided). We note that the condition $\min\{x, y\} \geq 2$ is actually very restrictive (as is evident from the examples in (1.2)) and appears crucially in the arguments of [Le]. While Theorem 1.1 is essentially sharp, as indicated by (1.2), one might, in light of Pillai's work, believe that something rather stronger is true. We formulate this in the following:

Conjecture 1.2 If a, b and c are positive integers with $a, b \geq 2$, then equation (1.1) has at most one solution in positive integers x and y, except for those triples (a, b, c) corresponding to (1.2).

As evidence for this, we provide a number of results, the first two of which indicate that Conjecture 1.2 is true if c is either "sufficiently large" or "sufficiently small", with respect to a and b. The first of these is an explicit version of the aforementioned theorem of Pillai, valid additionally for pairs (a, b) which fail to be relatively prime (we note that Pillai's treatment of this latter situation in [Pi2] is inadequate).

Theorem 1.3 *If a, b and c are positive integers with $a, b \geq 2$ and*

$$c \geq b^{2a^2 \log a} \quad (\textit{or, if a is prime, } c \geq b^a),$$

then equation (1.1) has at most one solution in positive integers x and y.

We take this opportunity to observe that the exponents above are artifices of our proof and may be somewhat reduced, via more precise application of lower bounds for linear forms in logarithms of algebraic numbers.

If, instead, we suppose that c is suitably small, relative to a and b, elaborating an argument of Terai [Te], we may derive a complementary result to Theorem 1.3. To state our result concisely, we require some notation. Let us define, given a and b integers exceeding unity, a_0 to be the largest positive integral divisor of a satisfying $\gcd(a_0, b) = 1$ and write

$$\delta(a,b) = \frac{\log a_0}{\log a} \quad \text{and} \quad \delta^*(a,b) = \max\{\delta(a,b), 1-\delta(a,b)\}.$$

Theorem 1.4 *If a, b and c are positive integers with $a, b \geq 2$, then equation (1.1) has at most one solution in positive integers x and y with*

$$b^y \geq 6000 \quad c^{1/\delta^*(a,b)}.$$

Terai [Te, Theorem 3] obtained a result of this shape, under the additional assumptions that $(x, y) = (1, 1)$ is a solution of (1.1) and that $\gcd(a, b) = 1$. His stated constant is 1697 rather than 6000, which reflects both the further constraints imposed and discrepancies between the lower bounds for linear forms in two logarithms used in [Te] and in the paper at hand. We note that the constant 6000 may be readily reduced by arguing somewhat more carefully.

In case $c = 1$, Conjecture 1.2 is a well known theorem of Leveque [Lev] (proved, independently, by Cassels [Ca]). Terai (Theorem 4 of [Te]) considered the case $c = 2$ under the restrictive (and, as it transpires, unnecessary) condition that $(x, y) = (1, 2)$ is a solution to (1.1). In fact, one may derive an efficient procedure for testing the validity of this conjecture for any fixed c, thereby generalizing Leveque's theorem; for small values, we have:

Theorem 1.5 *If a, b and c are integers with $a, b \geq 2$ and $1 \leq c \leq 100$, then equation (1.1) has at most one solution in positive integers x and y, except for triples (a, b, c) satisfying*

$$(a,b,c) \in \{(3,2,1), (2,3,5), (2,3,13), (4,3,13), (16,3,13), \\ (2,5,3), (13,3,10), (91,2,89), (6,2,4), (15,6,9)\}.$$

In each of these cases, (1.1) has precisely two positive solutions.

Finally, if we restrict our attention to prime values of a (where we assume that c is positive), we may verify Conjecture 1.2 for a number of fixed values of a. The first result of this nature was obtained by Scott [Sc] in the case $a = 2$ (we will discuss this in more detail in Section 2). We prove:

Theorem 1.6 *If $a, b \geq 2$ and c are positive integers, with a prime and $b \equiv \pm 1 \pmod{a}$, then (1.1) has at most one positive solution (x, y) unless*

$$(a, b, c) \in \{(3, 2, 1), (2, 3, 5), (2, 3, 13)\}.$$

In each of these cases, there are precisely two such solutions.

An (almost) immediate corollary of this, which proves Conjecture 1.2 for $a = 2^n + 1$ prime (*i.e.*, for the Fermat primes; a presumably finite set), is the following:

Corollary 1.7 *If $a \in \{3, 5, 17, 257, 65537\}$ and $b \geq 2$, then (1.1) has at most one positive solution (x, y) unless $(a, b, c) = (3, 2, 1)$, in which case there are two solutions $(x, y) = (1, 1)$ and $(x, y) = (2, 3)$.*

It appears to be difficult to prove Conjecture 1.2 for an infinite family of values of a or, for that matter, for even a single fixed b. We note that Conjecture 1.2, in the special case where (1.1) possesses a minimal solution $(x, y) = (1, 1)$, has been considered from a rather different viewpoint by Mignotte and Pethő [MP], motivated by computations of Fielder and Alford [FA]. Additionally, results of Mordell [Mo] and Pintér [Pin] on elliptic Diophantine equations may be recast as cases of Conjecture 1.2, where we specify values of positive solutions (x_1, y_1) and (x_2, y_2) as $(x_1, y_1, x_2, y_2) = (1, 1, 2, 3)$ and $(2, 1, 3, 2)$, respectively. Further, (1.1) is a simple example of an S-unit equation. Though general bounds for the number of solutions to such equations have reached an admirable state of refinement (see *e.g.* Beukers and Schlickewei [BS]) or Shorey and Tijdeman [ShTi]), we feel there is still some merit in careful examination of a restricted situation.

2 Elementary Results

Before we proceed with the proofs of our Theorems, we will mention a related result due to Scott [Sc]. By applying elementary properties of integers in quadratic fields, Scott proved the following (an immediate consequence of Theorems 3 and 4 of [Sc]):

Proposition 2.1 *If $b > 1$ and c are positive integers and a is a positive rational prime, then equation (1.1) has at most one solution in positive integers x and y unless either $(a, b, c) = (3, 2, 1), (2, 3, 5), (2, 3, 13)$ or $(2, 5, 3)$, or $a > 2$, $\gcd(a, b) = 1$ and the smallest $t \in \mathbb{N}$ such that $b^t \equiv 1 \pmod{a}$ satisfies $t \equiv 1 \pmod{2}$. In these situations, the given equation has at most two such solutions. If equation (1.1), with the above hypotheses, has distinct positive solutions (x_1, y_1) and (x_2, y_2), then $y_2 - y_1 \equiv 1 \pmod{2}$, unless $(a, b, c) = (3, 2, 1), (2, 3, 5), (2, 3, 13), (2, 5, 3)$ or $(13, 3, 10)$.*

This result establishes Conjecture 1.2 in the case $a = 2$ and includes the pairs $(a, b) = (3, 2)$ and $(2, 3)$ as special cases (Conjecture 1.2 for these pairs was an old question of Pillai [Pi2], resolved via the theory of linear forms in logarithms of algebraic numbers by Stroeker and Tijdeman [StTi]; see also Chein [Ch] and Herschfeld [He]). From Proposition 2.1, we can, in the proof of Theorem 1.1, restrict attention to those a that possess at least two distinct prime factors (so that $a \geq 6$). For Theorems 1.3, 1.4 and 1.5, we will also suppose that $a \geq 6$, an assumption we will not justify until Section 7 (the proof of Theorem 1.6 will not rely upon any prior results). In all cases, we will henceforth assume, without loss of generality, that a and b are not perfect powers and that c is positive.

3 Proof of Theorem 1.1

In this section, we will prove that equation (1.1) has at most two positive solutions (x, y), provided a, b and c are positive integers with $a, b \geq 2$. Let us suppose that, in fact, there are three such solutions (x_i, y_i) in positive integers, where

$$x_1 < x_2 < x_3 \quad \text{and} \quad y_1 < y_2 < y_3.$$

We begin by noting that, for $i = 1, 2$, we have

$$y_{i+1}x_i - x_{i+1}y_i > 0. \tag{3.1}$$

To see this, observe that the function $A^x - B^x$ is monotone increasing for $x \geq 1$, provided $A > B > 1$, and so

$$a^{x_{i+1}} - b^{y_i \frac{x_{i+1}}{x_i}} > c = a^{x_{i+1}} - b^{y_{i+1}}.$$

It follows that $y_{i+1}x_i > y_i x_{i+1}$, as desired. Inequality (3.1), though extremely simple, will prove to be of crucial importance in establishing a "gap principle" for the solutions (x_i, y_i); *i.e.*, a result which guarantees that these solutions do not lie too close together. In the context of equation (1.1), this inequality occurs first in work of Terai [Te].

We first suppose that $\gcd(a, b) > 1$. There thus exists a prime p dividing a and b, say with $\operatorname{ord}_p a = \alpha \geq 1$ and $\operatorname{ord}_p b = \beta \geq 1$. Since

$$a^{x_i}(a^{x_{i+1}-x_i} - 1) = b^{y_i}(b^{y_{i+1}-y_i} - 1),$$

it follows that $\alpha x_i = \beta y_i$ for $i = 1, 2$, whereby

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{\beta}{\alpha},$$

contradicting (3.1). We will therefore assume, for the remainder of this section, that $\gcd(a, b) = 1$.

Let us write

$$\Lambda_i = x_i \log a - y_i \log b,$$

whereby

$$e^{\Lambda_i} - 1 = \frac{c}{b^{y_i}}$$

and so

$$\log |\Lambda_i| < \log \left(\frac{c}{b^{y_i}} \right).$$

We will use this inequality to show, at least for $i \geq 2$, that $|\Lambda_i|$ is "small". From this, we will (eventually) derive a contradiction. Arguing crudely, since $x_3 > x_2 > x_1$, we have

$$a^{x_3} \geq a^{x_1+2} > a^2 c \quad \text{and} \quad a^{x_2} \geq a^{x_1+1} > ac,$$

whence

$$\frac{a^{x_i}}{b^{y_i}} = \frac{a^{x_i}}{a^{x_i} - c} < \frac{a^{i-1}}{a^{i-1} - 1} \quad \text{for } 2 \leq i \leq 3.$$

It follows that

$$b^{y_i} < a^{x_i} < \frac{a^{i-1}}{a^{i-1} - 1} b^{y_i}$$

and so

$$\log|\Lambda_i| < \log\left(\min\left\{\frac{a^{i-1}c}{(a^{i-1} - 1)a^{x_i}}, \frac{c}{b^{y_i}}\right\}\right) \tag{3.2}$$

for $2 \leq i \leq 3$. Let us also note that

$$y_{i+1}\Lambda_i - y_i\Lambda_{i+1} = (x_i y_{i+1} - x_{i+1} y_i) \log a \geq \log a,$$

where the inequality follows from (3.1). Since $\Lambda_{i+1} > 0$, we thus have

$$\frac{x_{i+1}}{\log b} > \frac{y_{i+1}}{\log a} > \frac{1}{\Lambda_i}. \tag{3.3}$$

The following is the Corollary to Theorem 2 of Mignotte [Mi]; here, $h(\alpha)$ denotes the absolute logarithmic Weil height of α, defined, for an algebraic integer α, by

$$h(\alpha) = \frac{1}{[\mathbb{Q}(\alpha) : \mathbb{Q}]} \log \prod_\sigma \max\{1, |\sigma(\alpha)|\},$$

where σ runs over the embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$.

Lemma 3.1 *Consider the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_2$$

where b_1 and b_2 are positive integers and α_1, α_2 are nonzero, multiplicatively independent algebraic numbers. Set

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$$

and let ρ, λ, a_1 and a_2 be positive real numbers with $\rho \geq 4$, $\lambda = \log \rho$,

$$a_i \geq \max\{1, \rho|\log \alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i)\} \quad (1 \leq i \leq 2)$$

and

$$a_1 a_2 \geq \max\{20, 4\lambda^2\}.$$

Further suppose h is a real number with

$$h \geq \max\left\{3.5, 1.5\lambda, D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+1.377\right)+0.023\right\},$$

$\chi = h/\lambda$ *and* $\upsilon = 4\chi + 4 + 1/\chi$. *We may conclude, then, that*

$$\log|\Lambda| \geq -(C_0+0.06)(\lambda+h)^2 a_1 a_2,$$

where

$$C_0 = \frac{1}{\lambda^3}\left\{\left(2+\frac{1}{2\chi(\chi+1)}\right)\left(\frac{1}{3}+\sqrt{\frac{1}{9}+\frac{4\lambda}{3\upsilon}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\frac{32\sqrt{2}(1+\chi)^{3/2}}{3\upsilon^2\sqrt{a_1a_2}}}\right)\right\}^2.$$

We apply this lemma to $|\Lambda_3|$ where, in the notation of Lemma 3.1, we have

$$D=1, \quad \alpha_1 = b, \quad \alpha_2 = a, \quad b_1 = y_3, \quad b_2 = x_3$$

and, since we assume $b \geq 2$ and $a \geq 6$, may take

$$a_1 = (\rho+1)\log b, \quad a_2 = (\rho+1)\log a.$$

Choosing $\rho = 4.74$, it follows that $a_1a_2 \geq \max\{20, 4\lambda^2\}$. Let

$$h = \max\left\{9.365, \log\left(\frac{x_3}{\log b}\right)+0.788\right\}.$$

That this is a valid choice for h follows from the inequality

$$\frac{x_3}{\log b} > \frac{y_3}{\log a}.$$

We will treat the two possible choices for h in turn. Suppose first that

$$h = \log\left(\frac{x_3}{\log b}\right)+0.788$$

whereby we have

$$\frac{x_3}{\log b} > 5308. \tag{3.4}$$

If $b=2$, from Proposition 2.1, we may assume that $a \geq 15$, while, for $b \geq 3$, we may suppose that $a \geq 6$. It follows that $\frac{1}{a_1}+\frac{1}{a_2}$ and $\frac{1}{a_1a_2}$ are both maximal for $(a,b) = (15,2)$ and hence, in Lemma 3.1, we have $C_0 < 0.615$. Applying this lemma, we conclude that

$$\log|\Lambda_3| > -22.24\left(\log\left(\frac{x_3}{\log b}\right)+2.345\right)^2 \log a \log b.$$

Combining this with (3.2), we find, since $a \geq 6$, that

$$\frac{x_3}{\log b} < \frac{\log c}{\log a \log b} + \frac{\log(36/35)}{\log a \log b} + 22.24 \left(\log \left(\frac{x_3}{\log b} \right) + 2.345 \right)^2 .$$

Since (x_1, y_1) is a solution to equation (1.1), it follows that $c < a^{x_1}$ and so, in conjunction with $\log a \log b \geq \log 2 \log 15$, we have

$$\frac{x_3 - x_1}{\log b} < 0.01 + 22.24 \left(\log \left(\frac{x_3}{\log b} \right) + 2.345 \right)^2 .$$

From (1.1), we obtain

$$a^{x_{i+1}-x_i} \equiv 1 \pmod{b^{y_i}} \quad \text{and} \quad b^{y_{i+1}-y_i} \equiv 1 \pmod{a^{x_i}} \tag{3.5}$$

and, consequently,

$$a^{x_3 - x_2} > b^{y_2} > b^{y_1} a^{x_1}.$$

It follows that $x_3 - x_1 > x_1$ and so

$$\frac{x_3}{\log b} < 0.02 + 44.48 \left(\log \left(\frac{x_3}{\log b} \right) + 2.345 \right)^2 ,$$

contradicting (3.4).

We therefore have that $h = 9.365$, whereby

$$\frac{x_3}{\log b} < 5309. \tag{3.6}$$

Since (3.2) and (3.3) yield

$$\frac{x_3}{\log b} > \frac{1}{\Lambda_2} > \frac{b^{y_2}}{c} > \frac{a^{x_2} - a^{x_1}}{c} > a^{x_2 - x_1} - 1,$$

where the last two inequalities follow from $a^{x_2} - a^{x_1} < b^{y_2} < a^{x_2}$ and $a^{x_1} > c$, we may thus conclude that

$$a^{x_2 - x_1} \leq 5309.$$

Since $a \geq 6$ and (via Proposition 2.1) $\omega(a) \geq 2$ (*i.e.*, a possesses at least two distinct prime factors), we are left to consider

$$\begin{array}{ll} x_2 - x_1 = 1 & 6 \leq a \leq 5308 \\ x_2 - x_1 = 2 & 6 \leq a \leq 72 \\ x_2 - x_1 = 3 & 6 \leq a \leq 15 \\ x_2 - x_1 = 4 & a = 6. \end{array} \tag{3.7}$$

To deal with the remaining cases, we first note that, from (3.2), we have

$$\left| \frac{\log b}{\log a} - \frac{x_i}{y_i} \right| < \frac{c}{y_i b^{y_i} \log a}. \tag{3.8}$$

We may thus conclude that $\frac{x_i}{y_i}$ is a convergent in the simple continued fraction expansion to $\frac{\log b}{\log a}$, provided

$$\frac{c}{y_i b^{y_i} \log a} < \frac{1}{2y_i^2}$$

i.e., if

$$\frac{b^{y_i} \log a}{c y_i} > 2.$$

In particular, since (3.5) yields

$$b^{y_{i+1}-y_i} > a^{x_i} > b^{y_i},$$

we have

$$\frac{b^{y_3} \log a}{c y_3} > \frac{b^{y_3-y_2+y_1}}{y_3} \log a \geq \frac{b^{\frac{1}{2}y_3+\frac{1}{2}+y_1}}{y_3} \log a > 2,$$

where the last inequality follows from $y_{i+1} \geq 2y_i + 1$ (whereby $y_3 \geq 7$) and $b \geq 2$. Thus $\frac{x_3}{y_3}$ is a convergent in the simple continued fraction expansion to $\frac{\log b}{\log a}$. On the other hand, if p_r/q_r is the r-th such convergent, then

$$\left| \frac{\log b}{\log a} - \frac{p_r}{q_r} \right| > \frac{1}{(a_{r+1} + 2)q_r^2}$$

where a_{r+1} is the $(r+1)$-st partial quotient to $\frac{\log b}{\log a}$ (see *e.g.* [Kh]). It follows, then, if $\frac{x_3}{y_3} = \frac{p_r}{q_r}$, that

$$a_{r+1} > \frac{b^{y_3} \log a}{c y_3} - 2 > \frac{b^{\frac{1}{2}y_3+\frac{1}{2}+y_1}}{y_3} \log a - 2. \tag{3.9}$$

For each $1 \leq x_2 - x_1 \leq 4$ and each a in the ranges given in (3.7) we compute, for each b dividing $a^{x_2-x_1} - 1$, the initial terms in the infinite simple continued fraction expansion to $\frac{\log b}{\log a}$. To carry out this calculation, we utilize Maple V and find, in all cases except $(a, b) = (3257, 148), (4551, 25)$ and $(5261, 526)$, that the denominator of the 19-th convergent to $\frac{\log b}{\log a}$ satisfies $q_{19} \geq 5309 \log a$. Since 3257 and 5261 are prime and $25 = 5^2$, these cases are excluded by hypothesis. It follows from

$$\frac{y_3}{\log a} < \frac{x_3}{\log b}$$

and (3.6) that $y_3 < 5309 \log a$ and so we necessarily have $\frac{x_3}{y_3} = \frac{p_r}{q_r}$ with $1 \leq r \leq 18$.

The only a and b under consideration for which we find a partial quotient a_k with $k \leq 19$ and $a_k \geq 100000$ are given in the following table

(3.10)

a	b	a_k
1029	257	$a_4 = 146318$
1837	204	$a_{16} = 1859087$
2105	526	$a_{14} = 149863$
2179	33	$a_8 = 169118$
2194	731	$a_4 = 251316$
3741	5	$a_{14} = 197241$
4348	621	$a_{15} = 132488$.

On the other hand, (3.9) implies, since $b \geq 2$ (and $a \geq 15$ if $b = 2$), that $a_{r+1} > 100000$ provided $y_3 \geq 38$. It follows that $y_3 \leq 37$ in all cases (since a much stronger result is a consequence of (3.9) for those (a, b) listed in (3.10)). Since

$$\frac{y_3}{\log a} > a^{x_2 - x_1} - 1,$$

we have $(a^{x_2-x_1} - 1)\log a < 37$, whereby $6 \leq a \leq 14$ and $x_2 - x_1 = 1$ (whence $(a, b) \in \{(6, 5), (10, 3), (14, 13)\}$). For these three cases, we find that $q_k \geq 5309 \log a$ with $k = 12, 9$ and 9, respectively and the largest partial quotient under consideration is $a_3 = 34$ to $\frac{\log 13}{\log 14}$. Together with (3.9), this contradicts

$$y_3 \geq 2y_2 + 1 \geq 4y_1 + 3 \geq 7,$$

completing the proof of Theorem 1.1.

4 Effective Pillai

One deficiency in the main theorem of Pillai [Pi2] is the ineffectivity stemming from the application of Siegel's Theorem. In this section, we will derive an effective (indeed, explicit) version, valid, additionally, for pairs (a, b) which fail to be relatively prime.

We will have use of the following result (see Ribenboim [Ri, (C6.5), pp. 276–278] for a proof); to state it, we require some notation. If $\gcd(a, b) = 1$, define $m(a, b)$ and $n(a, b)$ to be positive integers such that

$$b^{n(a,b)} = 1 + la^{m(a,b)}$$

with l an integer, $\gcd(l, a) = 1$, $m(a, b) \geq 2$ and $n(a, b)$ minimal. That such $m(a, b)$ and $n(a, b)$ exist follows from *e.g.* Ribenboim [Ri, (C6.5)]. We have

Lemma 4.1 *Suppose that a and b are relatively prime integers with $a, b \geq 2$. If $N, M \geq 2$ are positive integers with $M \geq m(a, b)$ and $b^N \equiv 1 \pmod{a^M}$, then N is divisible by $n(a, b)a^{M-m(a,b)}$.*

In essence, this follows from the well known fact that, if x and y are non-zero, relatively prime integers and $n > 1$, then

$$\gcd\left(x - y, \frac{x^n - y^n}{x - y}\right) = \gcd(x - y, n).$$

To apply this lemma, we require an upper bound for $m(a, b)$.

Lemma 4.2 *If $a, b \geq 2$ are relatively prime integers, then*

$$m(a, b) < \phi(a^2)\frac{\log b}{\log 2},$$

where ϕ denotes Euler's totient function.

Proof We follow work of Pillai [Pi2, see the erratum on p. 215]. Let us begin by writing

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

where $p_1, \ldots, p_r$ are distinct primes and $\alpha_i \in \mathbb{N}$, and choosing $t_1 \in \mathbb{N}$ minimal such that

$$b^{t_1} \equiv 1 \ (\mathrm{mod}\ a^2).$$

We thus have

$$b^{t_1} = 1 + M_1 p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r} a^{s_1}$$

where $s_1 \geq 2$, $M_1 \in \mathbb{N}$, $\gcd(M_1, a) = 1$ and $\beta_i \leq \alpha_i - 1$ for at least one value of $1 \leq i \leq r$. If $r = 1$ and $a \geq 3$, it follows that

$$b^{t_1 p_1^{\alpha_1 - \beta_1}} = 1 + M_2 a^{s_1 + 1}$$

where $\gcd(M_2, a) = 1$, and so $m(a, b) \leq s_1 + 1$. By the definition of t_1, we have $t_1 \leq \phi(a^2)$ and so, since $a^{s_1} < b^{t_1}$,

$$m(a, b) < \phi(a^2)\frac{\log b}{\log a} + 1 < \phi(a^2)\frac{\log b}{\log 2}.$$

On the other hand, if $a = 2$, then necessarily $\beta_1 = 0$ and so

$$m(a, b) = s_1 < \phi(a^2)\frac{\log b}{\log 2}.$$

If $r \geq 2$, then arguing as in (C6.5) of Ribenboim [Ri, see pp. 275–276], if $k \in \mathbb{N}$ is minimal such that $\beta_i < k\alpha_i$ for $1 \leq i \leq r$, we have

$$m(a, b) \leq s_1 + k + 1.$$

Suppose, without loss of generality, that $(k - 1)\alpha_1 \leq \beta_1 < k\alpha_1$, so that

$$a^{s_1} < (M_1 p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r})^{-1} b^{t_1} < p_1^{-(k-1)\alpha_1} b^{t_1}.$$

From $t_1 \leq \phi(a^2)$, we have
$$a^{s_1} < p_1^{-(k-1)\alpha_1} b^{\phi(a^2)}$$
whereby
$$s_1 \log a + (k-1)\alpha_1 \log p_1 < \phi(a^2) \log b.$$
Since we assume that $r \geq 2$, we thus have $a \geq 6$, whence
$$s_1 \log 6 + (k-1) \log 2 < \phi(a^2) \log b.$$
Using that $s_1 \geq 2$, we conclude that
$$m(a,b) \leq s_1 + k + 1 < \phi(a^2) \frac{\log b}{\log 2},$$
as desired. ■

We will first prove Theorem 1.3 in the situation where $\delta(a,b) = 0$. In this case, something stronger is true.

Lemma 4.3 *If a, b and c are positive integers with $a, b \geq 2$ and $\delta(a,b) = 0$, then equation (1.1) has at most a single solution in positive integers (x, y).*

Proof To prove this, note first that $\delta(a,b) = 0$ implies $\gcd(a,b) > 1$. If (1.1) has two distinct positive solutions (say (x_1, y_1) and (x_2, y_2), with $x_2 > x_1$), then from
$$a^{x_2} - b^{y_2} = a^{x_1} - b^{y_1} = c > 0,$$
if p is a prime dividing $\gcd(a,b)$, with $\operatorname{ord}_p a = \alpha$ and $\operatorname{ord}_p b = \beta$, we have
$$x_1 \alpha = y_1 \beta \tag{4.1}$$
and, by (3.1),
$$x_2 \alpha < y_2 \beta.$$
It follows that
$$\operatorname{ord}_p c = x_2 \alpha. \tag{4.2}$$
Since we have assumed that $\delta(a,b) = 0$, every prime dividing a also divides $\gcd(a,b)$ and thus a^{x_2} divides $c = a^{x_1} - b^{y_1}$, contradicting $x_2 > x_1$. ■

If $\delta(a,b) > 0$, Theorem 1.3 is a consequence of the following result.

Proposition 4.4 *If a, b and c are positive integers with $a, b \geq 2$ and $\delta(a,b) > 0$, then equation (1.1) has at most a single solution in positive integers (x, y) with*
$$x \geq \frac{m(a_0, b) + 5}{\delta(a,b)}.$$

Proof The constant 5 on the right hand side of the above inequality may likely be replaced by 0, with a certain amount of effort; we will not undertake this here. Since $\delta(a,b) > 0$, we have $2 \leq a_0 \leq a$. Let us suppose that we have two solutions to (1.1) in positive integers, say (x_1, y_1) and (x_2, y_2), with

$$x_2 > x_1 = \frac{m(a_0,b)+k}{\delta(a,b)},$$

where $k \geq 5$. From the equation

$$a^{x_1}(a^{x_2-x_1}-1) = b^{y_1}(b^{y_2-y_1}-1),$$

it follows that

$$b^{y_2-y_1} \equiv 1 \pmod{a_0^{x_1}}$$

and so Lemma 4.1 implies that $a_0^{x_1-m(a_0,b)}$ divides $y_2 - y_1$. Thus

$$y_2 > a_0^{\frac{m(a_0,b)+k}{\delta(a,b)}-m(a_0,b)} = (a/a_0)^{m(a_0,b)}a^k \geq a^5. \tag{4.3}$$

On the other hand, $c < a^{x_1}$, so

$$\log c < x_1 \log a = \frac{\big(m(a_0,b)+k\big)\log^2 a}{\log a_0}.$$

The first inequality in (4.3) thus implies that

$$\frac{y_2 \log b}{\log c} > \frac{(a/a_0)^{m(a_0,b)}a^k \log a_0 \log b}{\big(m(a_0,b)+k\big)\log^2 a} \geq \frac{a^k \log b}{\big(m(a_0,b)+k\big)\log a}.$$

From Lemma 4.2, we have

$$m(a_0,b) < \frac{\phi(a_0^2)\log b}{\log 2} < \frac{a_0^2 \log b}{\log 2} \leq \frac{a^2 \log b}{\log 2}$$

and so

$$\frac{y_2 \log b}{\log c} > \frac{a^k}{\left(\frac{a^2}{\log 2}+\frac{k}{\log b}\right)\log a} > 73, \tag{4.4}$$

where the second inequality follows from $k \geq 5$, $a \geq 6$ and $b \geq 2$. We will use (4.3) and (4.4) to deduce absolute upper bounds upon a and y_2, in conjunction with Lemma 3.1.

Let us write

$$\Lambda_2 = x_2 \log a - y_2 \log b,$$

where, in the notation of Lemma 3.1, we have

$$D = 1,\ \alpha_1 = b,\ \alpha_2 = a,\ b_1 = y_2,\ b_2 = x_2,\ a_1 = (\rho+1)\log b,\ a_2 = (\rho+1)\log a.$$

Further, take $\rho = 4.1$ and

$$h = \max \left\{ 9, \log \left(\frac{x_2}{\log b} \right) + 0.9 \right\}.$$

As before, these are valid choices in Lemma 3.1. Suppose first that

$$h = \log \left(\frac{x_2}{\log b} \right) + 0.9,$$

whereby we have

$$\frac{x_2}{\log b} > 3294. \tag{4.5}$$

By Proposition 2.1 and our assumption that $a \geq 6$, it follows that

$$\frac{1}{a_1} + \frac{1}{a_2} \quad \text{and} \quad \frac{1}{a_1 a_2}$$

are both maximal for $(a, b) = (6, 2)$ and hence, in Lemma 3.1, we have $C_0 < 0.87$. Applying this lemma, we conclude that

$$\log |\Lambda_2| > -24.2 \left(\log \left(\frac{x_2}{\log b} \right) + 2.4 \right)^2 \log a \log b.$$

Combining this with (3.2), we find, since $a \geq 6$, that

$$\frac{x_2}{\log b} < \frac{\log(6c/5)}{\log a \log b} + 24.2 \left(\log \left(\frac{x_2}{\log b} \right) + 2.4 \right)^2. \tag{4.6}$$

Since

$$\frac{x_2}{\log b} > \frac{y_2}{\log a} > \frac{73 \log(c)}{\log a \log b},$$

where the latter inequality follows from (4.4), (4.6) thus implies (with $a \geq 6$ and $b \geq 2$) that

$$\frac{x_2}{\log b} < 0.2 + 24.6 \left(\log \left(\frac{x_2}{\log b} \right) + 2.4 \right)^2$$

which contradicts (4.5). We therefore have

$$\frac{y_2}{\log a} < \frac{x_2}{\log b} < 3295.$$

From inequality (4.3), it follows that

$$\frac{a^5}{\log a} < 3295.$$

Since $a \geq 6$, this contradiction completes the proof of Proposition 4.4. ∎

We will now prove Theorem 1.3. As previously mentioned, the cases $a = 3$ and $a = 5$ will be treated in Section 7. From Proposition 2.1, we therefore assume $a \geq 6$. If $\delta(a, b) = 0$, then the desired conclusion is immediate from Lemma 4.3. Let us suppose that a, b and c are positive integers with $a, b \geq 2$, $\delta(a, b) > 0$ and $c \geq b^{2a^2 \log a}$, for which equation (1.1) possesses distinct positive solutions (x_1, y_1) and (x_2, y_2) (with $x_2 > x_1$). Since $a^{x_1} > c$, we thus have $x_1 > 2a^2 \log b$. On the other hand, Lemma 4.2 gives

$$\frac{m(a_0, b)}{\delta(a, b)} = \frac{m(a_0, b) \log a}{\log a_0} < \frac{a_0^2 \log b \log a}{\log 2 \log a_0} \leq \frac{a^2 \log b}{\log 2}$$

and so

$$x_1 - \frac{m(a_0, b)}{\delta(a, b)} > \left(2 - \frac{1}{\log 2}\right) a^2 \log b.$$

Since $a \geq 6$, $b \geq 2$ and $a_0 \geq 2$, this last quantity exceeds $\frac{5 \log a}{\log a_0}$, completing the proof of Theorem 1.3, in case a is composite.

Let us now suppose that $a \geq 7$ is prime, $b \geq 2$ and $c \geq b^a$. Since $\delta(a, b) > 0$, it follows that $\gcd(a, b) = 1$. We begin by calculating $m(a, b)$ more precisely in this situation. Choose n to be the smallest positive integer such that $b^n \equiv 1 \pmod{a}$ and write $b^n = 1 + ja$. If $\gcd(j, a) = 1$, then

$$b^{an} \equiv 1 + ja^2 + \binom{a}{2} j^2 a^2 \pmod{a^3}.$$

Since $a > 2$ is prime, it follows that $b^{an} \equiv 1 + ja^2 \pmod{a^3}$, whence

$$\gcd\left(\frac{b^{an} - 1}{a^2}, a\right) = 1$$

and therefore $m(a, b) = 2$. If, on the other hand, $\gcd(j, a) > 1$ (so that a divides j), then we can write $b^n = 1 + la^{m(a,b)}$. Since n divides $\phi(a) = a - 1$ and, via Proposition 2.1, we may assume that n is odd, we have $n \leq \frac{a-1}{2}$ and so

$$a^{m(a,b)} < b^n \leq b^{\frac{a-1}{2}}.$$

It follows that

$$m(a, b) < \max\left\{2, \frac{a - 1}{2} \frac{\log b}{\log a}\right\}. \tag{4.7}$$

Now

$$\frac{a - 1}{2} \frac{\log b}{\log a} \geq 2$$

for $a \geq 7$ prime, unless $b = 2$ and $7 \leq a \leq 17$ or $(a, b) = (7, 3)$. From Proposition 2.1, if $(a, b) \neq (7, 2)$, since $a^{x_1} > c \geq b^a$, (4.7) implies that $x_1 > 2m(a, b)$ and thus Lemma 4.1 yields

$$y_2 > a^{x_1 - m(a,b)} \geq a^{\frac{x_1+1}{2}}.$$

Applying the arguments immediately preceding and following (4.6), we obtain $\frac{y_2}{\log a} < 3295$. If $a \geq 47$, this implies that $x_1 \leq 4$, contradicting $x_1 > 2m(a,b) \geq 4$. Similarly, we have $x_1 \leq 9$ (if $a = 7$), $x_1 \leq 7$ (if $11 \leq a \leq 13$), $x_1 \leq 6$ (if $17 \leq a \leq 19$) and $x_1 \leq 5$ (if $23 \leq a \leq 43$). Since $a^{x_1} > c \geq b^a$, we derive the inequalities $b \leq 12$ (if $a = 7$), $b \leq 4$ (if $a = 11$), $b \leq 3$ (if $a = 13$), $b = 2$ (if $a = 17$ or 19) and a contradiction for larger values of a. After applying Proposition 2.1, we are left to consider only the pairs $(a,b) = (7,11), (11,3)$ and $(13,3)$. In each case, we have $m(a,b) = 2$ and so the inequalities

$$a^{x_1 - m(a,b)} < 3295 \log a, \quad a^{x_1} > c \geq b^a \quad \text{and} \quad x_1 \geq 2m(a,b) + 1 = 5$$

lead to immediate contradictions. Finally, if we suppose that $(a,b) = (7,2)$, then

$$(-1)^{y_1} \equiv (-1)^{y_2} \pmod 3,$$

which implies that $y_1 \equiv y_2 \pmod 2$, contradicting Proposition 2.1.

5 Generalizing Terai

In [Te], Terai obtains a result which implies Conjecture 1.2 in case equation (1.1) has the solution $(x,y) = (1,1)$ and (a,b,c) are relatively prime, positive integers with $a \geq 2$ and $b \geq 1697c$. In this section, we will generalize this to include the possibility that $\gcd(a,b) > 1$ and eliminate any suppositions about the size of the smallest solution (x,y).

Suppose that a, b and c are positive integers, with $a, b \geq 2$, for which we have two positive solutions (x_1, y_1) and (x_2, y_2) to (1.1), with $x_1 < x_2$ and

$$b^{y_2} > b^{y_1} \geq 6000 c^{1/\delta^*(a,b)},$$

where

$$\delta^*(a,b) = \max\{\delta(a,b), 1 - \delta(a,b)\}.$$

Let us define, as in Section 1, a_0 and b_0 to be the largest positive integral divisors of a and b, respectively, relatively prime to b and a, respectively. From (4.2) and the arguments preceding it, we have that $(a/a_0)^{x_2}$ divides c and so

$$c \geq a^{(1-\delta(a,b))x_2}.$$

Since

$$a^{x_1} > b^{y_1} > 6000 c^{1/\delta^*(a,b)}, \tag{5.1}$$

it follows that

$$x_1 > \frac{\left(1 - \delta(a,b)\right)}{\delta^*(a,b)} x_2.$$

From $x_2 > x_1$ and the definition of $\delta^*(a,b)$, we thus have $\delta(a,b) > 1/2$ and so $\delta^*(a,b) = \delta(a,b)$. From (4.1),

$$(a/a_0)^{x_1} = (b/b_0)^{y_1}$$

and so $b^{y_1} > 6000c$, $a^{x_1} = b^{y_1} + c$ and $a^{x_2-x_1} \equiv 1 \pmod{b_0^{y_1}}$ together imply that

$$a^{x_2-x_1} > b_0^{y_1} > \frac{6000}{6001} a_0^{x_1} = \frac{6000}{6001} a^{\delta(a,b)x_1}.$$

We may therefore conclude that

$$x_2 > \big(1 + \delta(a,b)\big)\, x_1 - \frac{\log(6001/6000)}{\log a}. \tag{5.2}$$

Since $\delta(a,b) > 1/2$, a and b are necessarily multiplicatively independent and we may again apply Lemma 3.1 to $\Lambda_2 = x_2 \log a - y_2 \log b$, where we take

$$D = 1,\ \alpha_1 = b,\ \alpha_2 = a,\ b_1 = y_2,\ b_2 = x_2,\ a_1 = (\rho+1)\log b,\ a_2 = (\rho+1)\log a.$$

Choosing $\rho = 4.7$ and

$$h = \max\left\{9.45, \log\left(\frac{x_2}{\log b}\right) + 0.79\right\}.$$

we argue as in Section 3 (with $\frac{1}{a_1} + \frac{1}{a_2}$ and $\frac{1}{a_1 a_2}$ maximal for $(a,b) = (6,2)$, whence $C_0 < 0.65$). Our conclusion is that, if

$$h = \log\left(\frac{x_2}{\log b}\right) + 0.79,$$

whence

$$\frac{x_2}{\log b} > 5767, \tag{5.3}$$

we have

$$\frac{x_2}{\log b} < \frac{\log(6c/5)}{\log a \log b} + 23.1 \left(\log\left(\frac{x_2}{\log b}\right) + 2.4\right)^2.$$

From (5.1),

$$c < 6000^{\delta(a,b)} a^{\delta(a,b)x_1}$$

and so, combining this with (5.2) and using that $\delta(a,b) > 1/2$, we find that

$$\frac{x_2}{\log b} - \frac{\log(6c/5)}{\log a \log b} > \frac{1}{1+\delta(a,b)} \frac{x_2}{\log b},$$

whereby

$$\frac{x_2}{\log b} < 23.1\big(1+\delta(a,b)\big) \left(\log\left(\frac{x_2}{\log b}\right) + 2.4\right)^2.$$

Since $\delta(a,b) \le 1$, this contradicts (5.3). It follows that

$$\log\left(\frac{x_2}{\log b}\right) + 0.79 < 9.45,$$

or

$$\frac{x_2}{\log b} < 5768.$$

On the other hand, (3.2) and (3.3) imply that

$$\frac{x_2}{\log b} > \frac{1}{\Lambda_1} > \frac{b^{y_1}}{c} > 6000,$$

which yields the desired contradiction.

6 Small Values of c

In this section, we will prove Conjecture 1.2 for all $1 \leq c \leq 100$, including cases with $\gcd(a, b) > 1$. Our proof will, in contrast to those of Leveque [Lev] and Cassels [Ca] for $c = 1$, rely upon lower bounds for linear forms in logarithms. It does not appear to be a routine matter to extend their arguments to larger values of c.

Suppose first that $\gcd(a, b) = 1$ and that we have two positive solutions (x_1, y_1) and (x_2, y_2) to (1.1), with $x_1 < x_2$ and $y_1 < y_2$. Once again, applying Lemma 3.1 to

$$\Lambda_2 = x_2 \log a - y_2 \log b,$$

we may choose $\rho = 5.11$ and

$$h = \max \left\{ 8.56, \log \left(\frac{x_2}{\log b} \right) + 0.773 \right\}.$$

If we have

$$h = \log \left(\frac{x_2}{\log b} \right) + 0.773,$$

then

$$\frac{x_2}{\log b} > 2409 \tag{6.1}$$

and thus, since $\frac{1}{a_1} + \frac{1}{a_2}$ and $\frac{1}{a_1 a_2}$ are maximal for $(a, b) = (7, 2)$, $C_0 < 0.556$. Applying Lemma 3.1, we conclude that

$$\log |\Lambda_2| > -22.997 \left(\log \left(\frac{x_2}{\log b} \right) + 2.405 \right)^2 \log a \log b.$$

Combining this with (3.2), we find, since $a \geq 6$, that

$$\frac{x_2}{\log b} < \frac{\log(6c/5)}{\log a \log b} + 22.997 \left(\log \left(\frac{x_2}{\log b} \right) + 2.405 \right)^2.$$

From $1 \leq c \leq 100$ and $\log a \log b \geq \log 7 \log 2$, we thus have

$$\frac{x_2}{\log b} < 3.715 + 22.997 \left(\log \left(\frac{x_2}{\log b} \right) + 2.405 \right)^2,$$

contradicting (6.1). It follows that

$$\frac{b^{y_1}}{c} < \frac{x_2}{\log b} < 2410. \tag{6.2}$$

For each value of $1 \leq c \leq 100$, this provides an upper bound upon b^{y_1} and, via $a^{x_1} = b^{y_1} + c$, upon a^{x_1}. To complete the proof of Theorem 1.5 for relatively prime a and b, we will argue as in Section 3. Let us first suppose that

$$\frac{b^{y_2} \log a}{c y_2} > 2,$$

so that $\frac{x_2}{y_2}$ is a convergent in the simple continued fraction expansion to $\frac{\log b}{\log a}$, say $\frac{x_2}{y_2} = \frac{p_r}{q_r}$. In fact, we must have $x_2 = p_r$ and $y_2 = q_r$. If not, then $\gcd(x_2, y_2) = d > 1$ and so, writing $x_2 = dx$ and $y_2 = dy$,

$$a^{x_2} - b^{y_2} = (a^x - b^y) \cdot \sum_{i=0}^{d-1} a^{ix} b^{(d-i-1)y} = c.$$

It follows that

$$\sum_{i=0}^{d-1} a^{ix} b^{(d-i-1)y} \leq c. \tag{6.3}$$

If $x_1 = 1$, this is an immediate contradiction, since $a > a - b^{y_1} = c$. Similarly, if $x_1 = 2$, we have $x_2 \geq 3$ and so $a^{(d-1)x} > a^2 - b^{y_1}$. We may thus assume that $x_1 \geq 3$ (so that $x_2 \geq 4$). If $d = 2$ and $x_2 = 4$, we have $y_2 \geq 6$, whereby inequality (6.3) implies that $a^2 + b^3 \leq c \leq 100$. Since we assume that a and b are not perfect powers, with $\gcd(a, b) = 1$, Proposition 2.1 implies $(a, b) = (7, 2)$, contradicting $0 < a^4 - b^{y_2} \leq 100$. If $d = 2$ and $x_2 \geq 6$, then $y_2 \geq 4$ and so $a^3 + b^2 \leq 100$, contradicting $a \geq 6$. Finally, if $d \geq 3$ and $x_2 \geq 4$, then $(d-1)x \geq 3$ and so $a^3 < 100$, again contradicting $a \geq 6$.

We thus have

$$a_{r+1} > \frac{b^{y_2} \log a}{c y_2} - 2 = \frac{b^{q_r} \log a}{c q_r} - 2. \tag{6.4}$$

For each pair (a, b) under consideration, we compute the initial terms in the continued fraction expansions to $\frac{\log b}{\log a}$ via Maple V and check to see if there exists a convergent p_r/q_r with $p_r < 2410 \log b$, $p_r \geq 2$, $q_r \geq 3$ and related partial quotient a_{r+1} satisfying (6.4). This is a relatively substantial calculation, as there are roughly seven million pairs (a, b) to treat. We find that the numerators p_r satisfy $p_{16} > 2410 \log b$, with precisely three exceptions corresponding to $(a, b) = (98, 17), (108, 53)$ and $(165, 91)$. In the first of these, we have $p_{18} > 2410 \log b$, while in the second and third, we have $p_{17} > 2410 \log b$. The largest partial quotient we encounter, associated with a convergent for which $p_r < 2410 \log b$, is $a_8 = 15741332$, corresponding to $(a, b) = (1968, 1937)$ (this contradicts (6.4), however). In fact, the only (a, b) not excluded by Proposition 2.1 for which we find convergents and partial quotients satisfying all the desired properties have either $(p_r, q_r) = (2, 3)$ or are as given in the following table:

(6.5)

a	b	r	a_r	p_r	q_r	c
23	2	4	10	2	9	15, 17, 19, 21
45	2	3	30	2	11	29, 37, 41, 43
91	2	4	31	2	13	87, 89
13	3	4	79	3	7	10, 88
47	3	4	54	2	7	22, 38, 44
421	3	4	1034	2	11	94
56	5	4	228	2	5	11, 31, 51
130	7	4	175	2	5	93
6	11	3	21	4	3	95
3	13	3	79	7	3	14, 68, 74.

Since a theorem of Mordell [Mo] ensures that the Diophantine equation

$$a - b = a^2 - b^3$$

has precisely the solutions

$$(a, b) \in \{(-14, 6), (-2, 2), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (1, 1), (3, 2), (15, 6)\},$$

we may restrict attention to (a, b, c) in (6.5). It is easily checked that amongst these, there exist positive integers $x_1 < p_r$ and $y_1 < q_r$ with

$$a^{p_r} - b^{q_r} = a^{x_1} - b^{y_1} > 0$$

only for $(a, b, c) = (91, 2, 89)$ and $(13, 3, 10)$.

Next suppose that

$$\frac{b^{y_2} \log a}{c y_2} \leq 2. \tag{6.6}$$

Since $a \geq 6$, $y_2 \geq 3$ and $1 \leq c \leq 100$, we thus have $2 \leq b \leq 7$. More precisely, if $b = 7$, it follows that $y_2 = 3$ and $y_1 = 1$, whereby a^{x_1} divides 48. This implies that $c \leq 41$, contradicting (6.6). Similarly, if $b = 6$, $y_2 = 3$, $y_1 = 1$, 35 is divisible by a^{x_1} and so $c \leq 29$, again contrary to (6.6). If $b = 5$, $y_1 = 1$, $y_2 = 3$ or 4 and a^{x_1} divides 24 or 124, respectively. We thus have $a^{x_1} \in \{6, 12, 24\}$ if $y_2 = 3$ or $a^{x_1} \in \{31, 62, 124\}$, if $y_2 = 4$, in each case contradicting (6.6). If $b = 3$, then Proposition 2.1 implies that we may assume $a \geq 10$, so (6.6) and $c \leq 100$ yield $y_2 \leq 5$, whence a^{x_1} divides 8 (if $(y_1, y_2) = (1, 3)$), 26 (if $(y_1, y_2) = (1, 4)$ or $(2, 5)$) or 80 (if $(y_1, y_2) = (1, 5)$). The first of these is excluded by Proposition 2.1, the second and third by inequality (6.6).

If $b = 2$, $a \geq 7$ and so $y_2 \leq 10$. It follows that a^{x_1} divides $2^{y_2-y_1} - 1$, where $2 \leq y_2 - y_1 \leq 9$. From $a \geq 6$ and

$$a^2 \leq a^{x_2} \leq 2^{y_2} + 100 \leq 1124,$$

whereby $a \leq 33$, we have that a^{x_1} is equal to one of 7 (with $y_2 - y_1 \in \{3, 6, 9\}$), 15 ($y_2 - y_1 \in \{4, 8\}$), 17 ($y_2 - y_1 = 8$), 21 ($y_2 - y_1 = 6$) or 31 ($y_2 - y_1 = 5$). The cases $a^{x_1} = 17$ and 21 immediately contradict (6.6). If $a^{x_1} = 7$ then (6.6) implies $(y_1, y_2) = (1, 4)$ so that $7^{x_2} = 21$. Similarly, $a^{x_1} = 15$ leads to $15^{x_2} = 45$ and $a^{x_1} = 31$ implies $31^{x_2} = 93$. These contradictions complete the proof of Theorem 1.5 for pairs (a, b) with $\gcd(a, b) = 1$.

Finally, we turn our attention to triples (a, b, c) with $\gcd(a, b) > 1$ and $1 \leq c \leq 100$. If the equation at hand has two positive solutions, then, from

$$a^{x_1}(a^{x_2-x_1} - 1) = b^{y_1}(b^{y_2-y_1} - 1),$$

if $\operatorname{ord}_p a = \alpha$ and $\operatorname{ord}_p b = \beta$, equations (4.1) and (4.2) are necessarily satisfied. For fixed c, this yields bounds upon α and x_2, and hence upon x_1, y_1 and β. Since

$$y_2\beta \geq x_2\alpha + 1, \tag{6.7}$$

the equation

$$(b^{y_1} + c)^{\frac{x_2}{x_1}} - b^{y_2} = c \tag{6.8}$$

provides explicit bounds upon b and, via $a^{x_1} = b^{y_1} + c$, upon a.

By way of example, we will give our arguments in detail for $c = 4, 8$ and 9, noting that we need not consider squarefree values of c. If $c = 4$, then $\gcd(a, b) > 1$ implies $\gcd(a, b) = 2$ and so, from (4.2), $x_2 = 2$ and $\alpha = 1$. Equation (4.1) thus implies that $x_1 = y_1 = \beta = 1$ and so, from (6.7) and (6.8),

$$(b + 4)^2 - b^3 \geq 4.$$

This implies that $b \leq 3$. Since 2 divides b, it follows that $b = 2$. We thus have $2^{y_2} = 32$ and so $y_2 = 5$, corresponding to $6 - 2 = 6^2 - 2^5 = 4$.

If $c = 8$, we have, if $\gcd(a, b) > 1$, that $\gcd(a, b) = 2$, $x_2\alpha = 3$ (so that $x_2 = 3$ and $\alpha = 1$) and $x_1 \in \{1, 2\}$. In the first case (where we have $y_1 = \beta = 1$), we are led to

$$(b + 8)^3 - b^4 \geq 8,$$

whereby $b \leq 7$. Since $\operatorname{ord}_2 b = \beta = 1$, in this situation, we thus have $b = 2$ or $b = 6$, whence $2^{y_2} = 992$ or $6^{y_2} = 2736$, both contradictions. If, instead, $x_1 = 2$, then (4.1) implies that $y_1\beta = 2$. In case $y_1 = 1$, we have, from $y_2 \geq 2y_1 + 1$,

$$(b + 8)^{3/2} - b^3 \geq 8,$$

and so $b \leq 3$, contradicting $\beta = 2$. If $y_1 = 2$, $y_2 \geq 5$ and

$$(b^2 + 8)^{3/2} - b^5 \geq 8,$$

whence $b \leq 2$ (so that $b = 2$). Since 12 is not a square, we conclude that equation (1.1) has at most one positive solution (x, y) provided $c = 8$ and $\gcd(a, b) > 1$.

Similarly, if we consider $c = 9$ with $\gcd(a, b) > 1$, then necessarily $x_1 = y_1 = \alpha = \beta = 1$ and $x_2 = 2$, so that

$$(b + 9)^2 - b^3 \geq 9.$$

This implies that $b \leq 6$. Since 3 divides b, we are thus left with the cases $b = 3$ and $b = 6$. In the former, (6.8) yields $3^{y_2} = 135$, a contradiction, while the latter leads to $6^{y_2} = 216$; *i.e.*, to the known example $15 - 6 = 15^2 - 6^3 = 9$. Arguing similarly for the remaining 36 non-squarefree values of $c \leq 100$, we find no other additional triples (a, b, c) for which (1.1) has two positive solutions and $\gcd(a, b) > 1$. This completes the proof of Theorem 1.5.

7 Prime Values of a

In the previous section, we studied the problem of deducing Conjecture 1.2 for fixed values of c. Essentially, we used the fact that Theorem 1.4 enables one to bound a and b explicitly in terms of c. If instead, we suppose that a is fixed (where c is positive), we cannot usually obtain such bounds upon c, solely in terms of a. In the special case where $a = 2$, however, Conjecture 1.2 is a consequence of Proposition 2.1. We will now extend this to include all primes of the form $a = 2^n+1$, for $n \in \mathbb{N}$. Unfortunately, this is, in all likelihood, just the set

$$a \in \{3, 5, 17, 257, 65537\}$$

(*i.e.*, the known Fermat primes). From Proposition 2.1, if a is prime and s is the smallest positive integer such that $b^s \equiv 1 \pmod{a}$, then Conjecture 1.2 obtains provided s is even. Theorem 1.6 will therefore follow from showing that a like conclusion is valid for $s = 1$. We suppose, then, that $a \geq 3$ is prime and $b \equiv 1 \pmod{a}$. Further, assume, as usual, that we have distinct positive solutions (x_1, y_1) and (x_2, y_2) to (1.1), with $x_2 > x_1$. From Section 4, either $b = 1 + ja$ with $\gcd(j, a) = 1$ (whereby $n(a, b) = a$ and $m(a, b) = 2$) or $b = 1 + la^{m(a,b)}$ for some positive integer l with $\gcd(l, a) = 1$. In the first case, $a^{x_1} > b^{y_1} \geq b > a$ and so $x_1 \geq 2$. Lemma 4.1 thus implies that a^{x_1-1} divides $y_2 - y_1$. In the second, since $a^{x_1} > b > a^{m(a,b)}$, we have $x_1 \geq m(a, b) + 1$ and $y_2 - y_1$ divisible by $a^{x_1-m(a,b)}$.

Arguing as in previous sections, we find, if $x_2 \geq 2410 \log b$, that

$$\frac{x_2}{\log b} < \frac{\log\big(ac/(a-1)\big)}{\log a \log b} + 22.997 \left(\log\left(\frac{x_2}{\log b}\right) + 2.405\right)^2. \tag{7.1}$$

Since $c < a^{x_1}$,

$$\frac{\log\big(ac/(a-1)\big)}{\log a \log b} < \frac{\log(3/2)}{\log 3 \log 7} + \frac{x_1}{\log b} < 0.19 + \frac{x_1}{\log b}. \tag{7.2}$$

Let us first suppose that $b = 1 + ja$ with $\gcd(j, a) = 1$ (so that we may write $y_2 - y_1 = ta^{x_1-1}$ for t a positive integer and $x_1 \geq 2$). If $x_1 = 2$, then $b \geq 7$, with (7.1) and (7.2), contradicts $x_2 \geq 2410 \log b$ and hence

$$\frac{y_2}{\log a} < \frac{x_2}{\log b} < 2410 \quad \text{or} \quad y_2 < 2410 \log a. \tag{7.3}$$

From $b > a$, we have $y_1 = 1$, $1 \leq j \leq a - 1$ and, via Proposition 2.1, may assume that y_2 is even (so that t is odd). It follows that x_2 is odd, since otherwise, writing $x_2 = 2x$ and $y_2 = 2y$,

$$a^2 - b = a^{2x} - b^{2y} \geq a^x + b^y > a^2.$$

We may thus restrict attention to those pairs (a, b) for which the smallest positive integer s with $a^s \equiv 1 \pmod{b}$ is odd. In particular, this enables us to suppose that $1 < j < a - 1$, since $a^s \equiv 1 \pmod{a^2 - a + 1}$ implies $s \equiv 0 \pmod 6$, while $a^s \equiv 1 \pmod{a + 1}$ implies $s \equiv 0 \pmod 2$. Further, considering the equation

$$a^2 - (1 + ja) = a^{x_2} - (1 + ja)^{y_2} \tag{7.4}$$

modulo 3 and modulo 8 implies that $a \not\equiv 2 \pmod 3$, $b \not\equiv 2 \pmod 3$ and either $a - j \equiv 1 \pmod 8$ or $j \equiv -1 \pmod 8$. Similarly, working modulo a^4, we find that

$$\frac{(1 + ta)}{2} t j^2 a + 1 + t j \equiv 0 \pmod{a^2}. \tag{7.5}$$

In particular, a divides $1 + tj$ and thus, since $j < a - 1$, it follows that $t \geq 3$. Inequality (7.3) thus yields $ta < 2410 \log a$, whence $3 \leq a \leq 7121$. For each prime a between 3 and 7121, we search, via Maple V, for integers j and t (*i.e.*, for quadruples (a, b, c, y_2)) which satisfy the above elementary constraints. We find none.

We argue similarly for larger values of x_1, where we again deduce

$$a^{x_1 - 1} < y_2 < 2410 \log a, \tag{7.6}$$

so that $3 \leq x_1 \leq 8$ and $3 \leq a \leq 103$. We search for quadruples (a, b, c, y_2) satisfying $b = 1 + ja$, $y_2 = y_1 + ta^{x_1 - 1}$, $c = a^{x_1} - b^{y_1}$ and, analogous to (7.5),

$$\frac{(ta^{x_1-1} + 2y_1 - 1)}{2} t j^2 a + 1 + t j \equiv 0 \pmod{a^2}.$$

Here, j and t are integers with $\gcd(j, a) = 1$, t odd and

$$1 \leq j \leq a^{\frac{x_1 - y_1}{y_1}}, \quad 1 \leq t < \frac{2410 \log a}{a^{x_1 - 1}}.$$

For each of these quadruples, we obtain congruence conditions upon x_2 such that $a^{x_2} = b^{y_2} + c$, by considering the equation modulo m for $m \in \{3, 4, 5, 7, 11, 13\}$. In conjunction with the fact that $x_2 \equiv x_1 \pmod{s}$ where s is the smallest positive integer such that $a^s \equiv 1 \pmod b$, these conditions lead to contradictions in all cases except $(a, b, c, y_2) = (79, 243321, 249718, 6242)$. For this quadruple (a, b, c, y_2), we have $b^{y_2} + c \not\equiv 0 \pmod{a^6}$, and hence conclude as desired.

Now, let us turn our attention to those $b \equiv 1 \pmod a$ of the form $b = 1 + la^{m(a,b)}$, for $l \in \mathbb{N}$ with $\gcd(l, a) = 1$. As mentioned previously, we may write $x_1 = m(a, b) + k$ and $y_2 - y_1 = ta^k$, where k and t are positive integers and $m(a, b) < \frac{\log b}{\log a}$. Again, if $x_2 \geq 2410 \log b$, we have (7.1) and, from (7.2),

$$\frac{\log\big(ac/(a-1)\big)}{\log a \log b} < 0.19 + \frac{x_1}{\log b} < 0.19 + \frac{1}{\log a} + \frac{k}{\log b}. \tag{7.7}$$

Since $y_2 > a^k$, $b > a^{m(a,b)}$ and $\frac{x_2}{\log b} > \frac{y_2}{\log a}$, we find that

$$x_2 > m(a, b)a^k. \tag{7.8}$$

Combining (7.1), (7.7) and (7.8), from $a \geq 3$, we find that $x_2 < 2410 \log b$, contrary to our assumptions. It follows that necessarily $y_2 < 2410 \log a$.

Consider now equation (1.1), or, in our case,

$$a^{m(a,b)+k} - (1 + la^{m(a,b)})^{y_1} = a^{x_2} - (1 + la^{m(a,b)})^{y_2}. \tag{7.9}$$

Since (3.5) implies $a^{x_2 - x_1} > b^{y_1}$, we have

$$x_2 > (y_1 + 1)m(a, b) + k \geq 2m(a, b) + k,$$

and so, expanding (7.9) by the binomial theorem, we find that

$$a^{m(a,b)+k} + \sum_{r=1}^{y_2} \left(\binom{y_2}{r} - \binom{y_1}{r} \right) l^r a^{rm(a,b)} \equiv 0 \pmod{a^{2m(a,b)+k}}.$$

Since $y_2 - y_1 = ta^k$, if α is the largest nonnegative integer such that a^α divides $r!$, we find that a^β divides $\left(\binom{y_2}{r} - \binom{y_1}{r} \right) a^{rm(a,b)}$, for $r \geq 3$, where

$$\beta \geq rm(a, b) + k - \alpha > rm(a, b) + k - \frac{r}{a-1} > 2m(a, b) + k.$$

From

$$\binom{y_2}{2} - \binom{y_1}{2} = \frac{(y_1 + y_2 - 1)}{2} ta^k,$$

we conclude that

$$1 + tl \equiv 0 \pmod{a^{m(a,b)}}. \tag{7.10}$$

Further, $b = 1 + la^{m(a,b)} < a^{x_1} = a^{m(a,b)+k}$, whereby $l < a^k$ and so the above congruence implies the inequality $y_2 - y_1 + 1 > a^{m(a,b)}$. We conclude, therefore, that

$$a^{\max\{m(a,b),k\}} < y_2 < 2410 \log a, \tag{7.11}$$

whence $3 \le a \le 103$, $1 \le k \le 7$ and $2 \le m(a,b) \le 7$.

To eliminate the remaining possibilities, we begin by supposing that $23 \le a \le 103$, whence, from (7.11), $1 \le k \le 2$, $m(a,b) = 2$, and necessarily $y_1 = 1$. Again, since we may assume that y_2 is even, necessarily x_2 is odd. There are precisely 69 triples (a, t, l) for the primes a under consideration with t odd and satisfying (7.10) and (7.11) (30 with $k = 1$ and 39 with $k = 2$). For the examples corresponding to $k = 1$, we deduce a contradiction to the parity of x_2 by considering the equation $a^{x_2} = b^{y_2} + c$ modulo one of 3, 4, 5 or 7, unless $(a, b, c, y_2) = (59, 83545, 121834, 8556)$ or $(83, 385785, 186002, 10210)$, where a like contradiction is obtained modulo 13 and 11, respectively. For the examples with $k = 2$, we have that the smallest $s \in \mathbb{N}$ with $a^s \equiv 1 \pmod{b}$ is even (again, contrary to the fact that x_2 is odd) for all but the case $(a, b, c, y_2) = (23, 25393, 254448, 5820)$, which leads to a contradiction modulo 3.

We may thus suppose that $3 \le a \le 19$. It is easy to see that we have $y_1 = 1$ unless $\big(a, m(a,b), k\big)$ is in the set

$$\{(3,2,3),(3,3,4),(5,2,3),(3,2,4),(3,3,5),(5,2,4),(3,2,5),(3,2,6),(3,2,7)\}$$

in which case we can have $y_1 = 2$ (or $y_1 = 3$ if $\big(a, m(a,b), k\big) = (3,2,5)$). Again, we use (7.10) and (7.11) to reduce the set of possible quadruples to a manageable level and then eliminate those remaining with local arguments (though we could just as easily check to see if, in any case, $b^{y_2} + c$ is a perfect power). Considerations modulo 3, 4, 5, 7, 8, 11 and 13 suffice to deal with all quadruples (a, b, c, y_2) other than $(a, b, c, y_2) = (5, 74376, 3749, 2626)$ and $(17, 751401, 668456, 4914)$. For these we obtain contradictions modulo 19 and 16, respectively. This completes the proof of Theorem 1.6. Corollary 1.7 is now immediate upon noting, if $a = 2^k + 1$ is prime with $k \in \mathbb{N}$, that the desired result follows from Proposition 2.1, unless $b \equiv 1 \pmod{a}$. In this latter case, we apply Theorem 1.6 to obtain the stated conclusion.

8 Concluding Remarks

Arguments similar to those in this paper may be applied to sharpen results of Le [Le] and Shorey [Sh] on the somewhat more general Diophantine equation

$$ra^x - sb^y = c,$$

where a, b, c, r and s are given positive integers (again, with $a, b \ge 2$). It is also worth noting that the finiteness of the list of exceptions to Conjecture 1.2 may be shown to follow in somewhat nontrivial fashion from the *abc*-conjecture of Masser-Oesterlé.

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