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Fractional parts of powers of rational numbers

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Abstract

The author uses Padé approximation techniques and an elementary lemma on primes dividing binomial coefficients to sharpen a theorem of F. Beukers on fractional parts of powers of rationals. In particular, it is proven that $\|((N+1)/N)^k\| > 3^{-k}$ holds for all positive integers N and k satisfying $4 \leqslant N \leqslant k\cdot 3^k$. Other results are described including an effective version of a theorem of K. Mahler for a restricted class of rationals.

1. *Introduction*

The connection between fractional parts of powers of rationals and the number $g(k)$ in Waring's problem is a well known one. In fact, if we define

$$\|x\| = \min_{m\in\mathbb{Z}} (|x-M|)$$

then the inequality
$$\|(3/2)^k\| > (3/4)^k \tag{1}$$
implies that we have
$$g(k) = 2^k+[(3/2)^k]-2. \tag{2}$$

In general, if $p > q \geqslant 2$ are relatively prime integers, then any improvement upon the trivial bound

$$\|(p/q)^k\| \geqslant q^{-k} \tag{3}$$

will provide information about the solutions to certain diophantine equations. Replacing the above by strict inequality, for instance, would imply the truth of Catalan's conjecture.

In 1957, K. Mahler[8] showed that if $\epsilon > 0$ is given, there exists k_0 such that for all $k \geqslant k_0$,

$$\|(p/q)^k\| > e^{-\epsilon k}. \tag{4}$$

The result, however, relies upon Ridout's extension of Roth's theorem and is thus ineffective, i.e. it is not possible to construct the constant k_0 from the proof. It does, though, imply that (1) (and hence (2)) fails for at most finitely many values of k. Efforts to derive effective versions of (4) have either utilized the theory of linear forms in logarithms (A. Baker and J. Coates [1]) or that of Padé approximation to the polynomial $(1-z)^k$ and related functions. From the latter technique, F. Beukers [4] proved that if $N \geqslant 2$ and $k \geqslant 1$ are integers, then

$$\left\|\left(\frac{N+1}{N}\right)^k\right\| > \frac{1}{4}N^{-3/2}(8.4)^{-k}. \tag{5}$$

By improving a technical lemma on the size of common factors of binomial coefficients, D. Easton strengthened (5), showing that for $k \geqslant k_0 = k_0(N)$ (an effective constant), one has

$$\left\|\left(\frac{N+1}{N}\right)^k\right\| > (3.87)^{-k}.$$

In the following, we sharpen both of these bounds, proving:

THEOREM. *If N and k are integers with $4 \leqslant N \leqslant k\cdot 3^k$, then*

$$\left\|\left(\frac{N+1}{N}\right)^k\right\| > 3^{-k}.$$

In a forthcoming paper [2], it is shown that if we let $g_N(k)$ denote the order of $\{1^k, N^k, (N+1)^k, \ldots\}$ as an additive basis for the positive integers (so that $g_2(k) = g(k)$), then the above result implies, for $4 \leqslant N \leqslant 2k/3$, that

$$g_N(k) = N^k + \left[\left(\frac{N+1}{N}\right)^k\right] - 2.$$

Owing to difficulties in obtaining suitably strong effective bounds for $\|(4/3)^k\|$, the case $N = 3$ in this variant of the Ideal Waring problem remains unsettled (as does the case $N = 2$, but see [7]).

2. *Construction of the approximants*

The basic technique involved in our proof is that of (diagonal) Padé or rational approximation to $(1-z)^k$. The approximating and error terms are readily handled asymptotically and the most important factor involved in obtaining a 'good' bound is the ratio between k and the degree n of the Padé approximants. This can be somewhat delicate since this ratio must be fairly small in order to deduce a lower bound, yet large enough that the bound is nontrivial (in the sense of (3)). Our difficulties are at least slightly ameliorated by the presence of large common factors in the coefficients of the approximating numerator and denominator polynomials (as evidenced by Lemma 3).

The following method for producing the Padé approximants to $(1-z)^k$ was suggested to the author by F. Beukers. If we let A, B and C be positive integers and z a real variable, then we can write

$$\int_0^1 t^A(1-t)^B(z-t)^C\,dt = \int_0^z t^A(1-t)^B(z-t)^C\,dt + \int_z^1 t^A(1-t)^B(z-t)^C\,dt. \tag{6}$$

Making the change of variables $t \to zt$ in the first integral on the right hand side of (6) and $t \to 1-t+zt$ in the second yields

$$\int_0^1 t^A(1-t)^B(z-t)^C\,dt = (-1)^C(1-z)^{B+C+1}\int_0^1 t^B(1-t)^C(1-t+zt)^A\,dt + z^{A+C+1}\int_0^1 t^A(1-t)^C(1-zt)^B\,dt$$

and so defining

$$P_A(z) = \frac{(A+B+C+1)!}{A!\,B!\,C!}\int_0^1 t^A(1-t)^B(z-t)^C\,dt,$$

$$Q_A(z) = \frac{(-1)^C(A+B+C+1)!}{A!\,B!\,C!}\int_0^1 t^B(1-t)^C(1-t+zt)^A\,dt \tag{7}$$

and

$$E_A(z) = \frac{(A+B+C+1)!}{A!\,B!\,C!}\int_0^1 t^A(1-t)^C(1-zt)^B\,dt \tag{8}$$

enables us to conclude

$$P_A(z)-(1-z)^{B+C+1}Q_A(z) = z^{A+C+1}E_A(z). \tag{9}$$

One may note, by comparison to e.g. F. Beukers[**4**], that if $A = C$ then $P_A(z)$ and $Q_A(z)$ correspond to the diagonal Padé approximants to $(1-z)^{B+C+1}$, with error term $E_A(z)$. By expanding via the binomial theorem and utilizing

$$\int_0^1 t^p(1-t)^q\,dt = \frac{p!\,q!}{(p+q+1)!} \quad \text{for} \quad p,q \in \mathbb{N}$$

we have:

LEMMA 1. *$Q_A(z)$ and $E_A(z)$ satisfy*:

(a) $$P_A(z) = \sum_{r=0}^{C}\binom{A+B+C+1}{r}\binom{A+C-r}{A}(-z)^r,$$

(b) $$Q_A(z) = (-1)^C\sum_{r=0}^{A}\binom{A+C-r}{C}\binom{B+r}{r}z^r,$$

(c) $$E_A(z) = \sum_{r=0}^{B}\binom{A+r}{r}\binom{A+B+C+1}{A+C+r+1}(-z)^r,$$

so that $P_A(z)$, $Q_A(z)$ and $E_A(z)$ are polynomials in z with integer coefficients. Additionally, we will use

LEMMA 2. *$P_A(z)\,Q_{A+1}(z)-Q_A(z)\,P_{A+1}(z) = cz^{A+C+1}$ where c is a non-zero constant.*

Proof. See e.g. F. Beukers[**4**]. ∎

The next lemma is the principal tool used in this paper to sharpen various estimates. Its form is suggested by the coefficient of the polynomials $Q_A(z)$ as given in Lemma 1(b).

LEMMA 3. *Suppose t is a positive integer satisfying*

$$\left[\frac{At}{A+B+C}\right]+\left[\frac{Bt}{A+B+C}\right]+\left[\frac{Ct}{A+B+C}\right] = t-2 \tag{10}$$

and define

$$M_t = \max\left\{\frac{A}{\left[\frac{At}{A+B+C}\right]+1}, \frac{B}{\left[\frac{Bt}{A+B+C}\right]+1}, \frac{C}{\left[\frac{Ct}{A+B+C}\right]+1}\right\}.$$

If p is a prime such that $M_t < p \leqslant (A+B+C)/t$, then we may conclude that $p \left| \binom{A+C-r}{C}\binom{B+r}{r}\right.$ for all $r = 0, 1, \ldots, A$.

Proof. Since $p > M_t$, we have

$$A/p < \left[\frac{At}{A+B+C}\right]+1,$$

while $p \leqslant (A+B+C)/t$ implies $A/p \geqslant At/(A+B+C)$ and we may conclude that

$$\{A/p\} \geqslant \{At/(A+B+C)\}$$

(where $\{x\}$ denotes the fractional part of x). Similarly,

$$\{B/p\} \geqslant \{Bt/(A+B+C)\}$$

and

$$\{C/p\} \geqslant \{Ct/(A+B+C)\}.$$

Now (10) is readily seen to be equivalent to

$$\left\{\frac{At}{A+B+C}\right\}+\left\{\frac{Bt}{A+B+C}\right\}+\left\{\frac{Ct}{A+B+C}\right\} = 2$$

and hence we have

$$\left\{\frac{A}{p}\right\}+\left\{\frac{B}{p}\right\}+\left\{\frac{C}{p}\right\} \geqslant 2$$

and thus, if r is an integer, $0 \leqslant r \leqslant A$, either

$$\left\{\frac{B}{p}\right\}+\left\{\frac{r}{p}\right\} \geqslant 1 \tag{11}$$

or

$$\left\{\frac{A}{p}\right\}+\left\{\frac{C}{p}\right\} \geqslant 1+\left\{\frac{r}{p}\right\}. \tag{12}$$

In the first instance,

$$\left[\frac{B+r}{p}\right]-\left[\frac{B}{p}\right]-\left[\frac{r}{p}\right] = 1$$

so that

$$p \left| \binom{B+r}{r}\right.$$

$\left(\text{since } \operatorname{ord}_p(n!) = \sum_{i=1}^{\infty}[n/p^i]\right)$. If, however, (12) holds, we must have $\{A/p\} \geqslant \{r/p\}$ and thus

$$\left\{\frac{A-r}{p}\right\} = \left\{\frac{A}{p}\right\}-\left\{\frac{r}{p}\right\}$$

or equivalently
$$\left\{\frac{A-r}{p}\right\}+\left\{\frac{C}{p}\right\}=\left\{\frac{A}{p}\right\}+\left\{\frac{C}{p}\right\}-\left\{\frac{r}{p}\right\}$$
which is $\geqslant 1$ by (12). It follows that
$$\left[\frac{A+C-r}{p}\right]-\left[\frac{A-r}{p}\right]-\left[\frac{C}{p}\right]=1$$
whence $p \,\Big|\, \binom{A+C-r}{C}$. ∎

As a final preliminary lemma, motivated by the integral representations for $Q_A(z)$ and $E_A(z)$ we prove:

LEMMA 4. $\dfrac{(A+B+C)!}{A!\,B!\,C!}<\dfrac{1}{2\pi}\sqrt{\left(\dfrac{A+B+C}{ABC}\right)}\cdot\dfrac{(A+B+C)^{A+B+C}}{A^AB^BC^C}.$

Proof. Using an explicit version of Stirling's formula (see e.g. Stromberg[**12**]), we have the above inequality with the right hand side multiplied by e^X where
$$X=\frac{1}{12(A+B+C)}-\frac{1}{12A+1/4}-\frac{1}{12B+1/4}-\frac{1}{12C+1/4}.$$
Since A, B and C are positive integers, $X<0$ and so $e^X<1$. ∎

3. *Bounding the approximants*

In what follows, we take $c>d\geqslant 1$ and m to be integers, $n=dm$ or $dm-1$ and $s=c/d$. Here and subsequently, let $A=C=n$ and $B=cm-n-1$. If we further define
$$Q(s)=\left(\max_{t\in[0,1]}|F(s,t)|\right)\cdot\alpha(s)$$
and
$$E(s)=\left(\max_{t\in[0,1]}|G(s,t)|\right)\cdot 4,$$
where
$$F(s,t)=t^{s-1}(1-t)\left(1-\left(\frac{N+1}{N}\right)t\right)$$
$$G(s,t)=t(1-t)\left(1+\frac{t}{N}\right)^{s-1}$$
and
$$\alpha(s)=\frac{(s+1)^{s+1}}{4\cdot(s-1)^{s-1}},$$
then the following lemma provides us with upper bounds for the denominator and error terms in our approximation.

LEMMA 5. *We have*
(a) $|Q_n(-1/N)|\ll(4\cdot Q(s))^{dm}$,
(b) $|E_n(-1/N)|\ll(\alpha(s)\cdot E(s))^{dm}$ *where the implied constants are independent of m.*

Proof. (a) From (7) we may write
$$|Q_n(-1/N)|\leqslant\frac{(cm+n)!}{(cm-n-1)!\,n!\,n!}\left|\int_0^1 t^{cm-n-1}(1-t)^n\left(1-\left(\frac{N+1}{N}\right)t\right)^n dt\right|$$

and so, via Lemma 4, we conclude that

$$|Q_n(-1/N)| < C_1 \cdot (4 \cdot \alpha(s))^{dm} \cdot |I|,$$

where I is the above integral and

$$C_1 = \begin{cases} \frac{1}{2\pi}\sqrt{(s^2-1)} & \text{if } n = dm \\ \frac{1}{2\pi}\sqrt{(1/(s^2-1))} & \text{if } n = dm-1. \end{cases}$$

Now from the definition of $F(s,t)$, we have

$$|I| < \left(\max_{t\in[0,1]} |F(s,t)|^d\right)^{m-1} \cdot I_1$$

where

$$I_1 = \begin{cases} \int_0^1 t^{c-d-1}(1-t)^d \left(1-\left(\frac{N+1}{N}\right)t\right)^d dt & \text{if } n = dm, \\ \int_0^1 t^{c-d}(1-t)^{d-1}\left(1-\left(\frac{N+1}{N}\right)t\right)^{d-1} dt & \text{if } n = dm-1, \end{cases}$$

and thus $$|Q_n(-1/N)| < C_2(4 \cdot Q(s))^{dm},$$

where $C_2 = C_1 \cdot I_1/\max_{t\in[0,1]} |F(s,t)|^d$ is independent of m.

(b) Repeating the above argument yields

$$|E_n(-1/N)| < C_3(\alpha(s) \cdot E(s))^{dm}$$

with $C_3 = C_1 \cdot I_2/\max_{t\in[0,1]} |G(s,t)|^d$ and

$$I_2 = \begin{cases} \int_0^1 t^d(1-t)^d\left(1+\frac{t}{N}\right)^{c-d-1} dt & \text{if } n = dm \\ \int_0^1 t^{d-1}(1-t)^{d-1}\left(1+\frac{t}{N}\right)^{c-d} dt & \text{if } n = dm-1. \end{cases} \blacksquare$$

The following lemma motivates the definitions of $Q(s)$ and $E(s)$.

LEMMA 6. *If* $N \geqslant \max\{4, \alpha(s)\}$, *then*

$$Q(s) < 1 < E(s) < 1.07565. \tag{13}$$

Proof. To bound $Q(s)$, we note that $|F(s,t)|$ is maximal in $[0,1]$ either for some

$$t_0 \in \left(0, \frac{N}{N+1}\right) \quad \text{or for} \quad t_1 \in \left(\frac{N}{N+1}, 1\right).$$

Now $t_1 > \frac{N}{N+1}$ implies that $1-t_1 < \frac{1}{N+1}$ and $\left|1-\left(\frac{N+1}{N}\right)t_1\right| < \frac{1}{N}$,

so that $$|F(s,t_1)| < \frac{1}{N(N+1)}.$$

On the other hand, $t_0 < \frac{N}{(N+1)}$ implies $|F(s,t_0)| < (1-t_0)^2 t_0^{s-1} \leqslant \alpha(s)^{-1}$. From

$N \geqslant \alpha(s)$, we have

$$Q(s) = \left(\max_{t\in[0,1]} |F(s,t)|\right)\cdot \alpha(s) < 1.$$

We now consider $E(s)$. That $E(s) > 1$ is immediate. It is also readily observed that $G(s,t)$ is increasing in s and hence to find an upper bound it suffices to suppose (for $N \geqslant 4$) that $\alpha(s) = N$. This implies that

$$E(s) < (1+\alpha(s)^{-1})^{s-1}$$

and hence the inequality in (13) for $N \geqslant 72$. Computing the cases $4 \leqslant N \leqslant 71$ from the definition of $E(s)$ yields (13) (with $E(s)$ attaining the value $1.075644\ldots$ for $N = 6$). ∎

4. *Some remarks on primes dividing binomial coefficients*

To date, in all effective Padé-type lower bounds for $\|(3/2)^k\|$, a crucial factor has been the estimation of common divisors of binomial coefficients (see e.g. F. Beukers [**4**], D. Easton [**6**] and A. K. Dubitskas [**5**]). In fact, the strongest effective bound known is by Dubitskas [**5**], who showed that for $k \geqslant k_0$ (a computable constant), we have

$$\|(3/2)^k\| > (1.734)^{-k}.$$

The proof of this utilizes a special case of Lemma 3. In our situation, we define

$$\mathscr{G}(c,d) = \gcd_{r=0,1,\ldots,n}\left(\binom{2n-r}{n}\binom{cm-n-1+r}{r}\right)$$

and have, by Lemma 3,

$$\mathscr{G}(c,d) \geqslant \prod_t \left(\prod_p p\right), \tag{14}$$

where t satisfies

$$2\left[\frac{nt}{cm+n-1}\right] + \left[\frac{(cm-n-1)\,t}{cm+n-1}\right] = t-2$$

and p is prime with $M_t < p \leqslant (cm+n-1)/t$ (where M_t is as in the statement of the lemma, remembering that we take $A = C = n$, $B = cm-n-1$, $n = dm$ or $dm-1$ and $s = c/d$).

$$L(s) = \exp\left(\sum_t \left(\frac{s+1}{t} - \Theta(s,t)\right)\right)$$

where
$$\Theta(s,t) = \max\left\{\frac{1}{\left[\dfrac{t}{s+1}\right]+1}, \frac{s-1}{\left[\dfrac{(s-1)\,t}{s+1}\right]+1}\right\}$$

and t is such that $\{t/(s+1)\} > 1/2$. By (9), both $\mathscr{G}(c,d)^{-1}\cdot P_n(z)$ and $\mathscr{G}(c,d)^{-1}\cdot Q_n(z)$ have integer coefficients and also:

LEMMA 7. *If $\epsilon > 0$, there is an effective constant $m_0 = m_0(\epsilon, c, d)$ such that if $m \geqslant m_0$, then*

$$\mathscr{G}(c,d) > L(s)^{(1-\epsilon)\,dm}.$$

Proof. The bound follows immediately from the definition of $L(s)$ and the fact that $\sum_{x<p\leqslant y}\log p \sim y-x$ (where p is prime). To quantify this statement, see e.g. Rosser and Schoenfeld [**9**, **10**] and Schoenfeld [**11**]. ∎

One may observe that this lemma represents a sharpening of a result of D. Easton [**6**, Lemma 5·2·8] in that

$$\lim_{s\to\infty} L(s) = \pi/e^{\gamma} \sim 1.76387\ldots, \tag{15}$$

where γ is Euler's constant, while the analogous function of Easton's approaches unity as $s\to\infty$. For a proof of (15), lovers of mysterious constants may refer to Bennett [**3**]. In the work that follows, however, we will not specifically apply Lemma 7, instead computing bounds via Rosser and Schoenfeld [**9**, **10**] and Schoenfeld [**11**], directly from (14) (for certain fixed c and d). We include the lemma here to give an approximate indication of the bounds obtained by this method and note that it is asymptotically best possible (in the sense that $\lim_{m\to\infty}\mathcal{G}(c,d)^{1/(dm)} = L(s)$).

5. *Proof of the theorem*

We separate the proof into two cases. If $N \geqslant 729$, we will prove that the inequality

$$\left\|\left(\frac{N+1}{N}\right)^k\right\| > 3^{-k} \tag{16}$$

holds for all $k \geqslant N/2$. To extend this to $k < N/2$, observe that in this case $((N+1)/N)^k < e^{1/2} \sim 1.6487$ and hence

$$\left(\frac{N+1}{N}\right)^k - 1 > \frac{k}{N} \geqslant 3^{-k}$$

so that (16) holds for these k also. In the second case, if $4 \leqslant N \leqslant 728$, we compute explicit $k_0 = k_0(N)$ such that for $k \geqslant k_0$, we have (16). By checking the remaining values of k, we complete the proof.

To start, we define

$$\Delta = \left(\frac{N+1}{N}\right)^{cm} - MN^{-\delta},$$

where δ is an integer, $0 \leqslant \delta < c$, and M is an arbitrary integer. It follows from (9) that

$$|\Delta|\cdot|Q_n(-1/N)| + N^{-2n-1}|E_n(-1/N)| \geqslant |P_n(-1/N) - MN^{-\delta}Q_n(-1/N)|. \tag{17}$$

By Lemma 2, we can choose $n = dm$ or $dm-1$ such that the right hand side of (17) does not vanish and therefore, by the remarks of the previous section,

$$|\Delta|\cdot|Q_n(-1/N)| + N^{-2n-1}|E_n(-1/N)| \geqslant \mathcal{G}(c,d)\cdot N^{-n-\delta}.$$

Therefore, if we have

$$|E_n(-1/N)| < \tfrac{1}{2}\mathcal{G}(c,d)\cdot N^{n-\delta+1}, \tag{18}$$

we can conclude from Lemma 5(a) that

$$|\Delta| > C_4\,\mathcal{G}(c,d)\cdot N^{-\delta}(4Q(s)\cdot N)^{-dm},$$

where $C_4 = 1/(2C_2)$. Hence, it follows, defining $k = cm-\delta$, that

$$\left|\left(\frac{N+1}{N}\right)^k - M(N+1)^{-\delta}\right| > C_5\,\mathscr{G}(c,d)\,(4Q(s)\cdot N)^{-k/s}, \tag{19}$$

where $C_5 = C_4((N+1)\cdot(4Q(s)\cdot N)^{1/s})^{-\delta}$. The inequality in (16) then obtains if the right hand side of (19) exceeds 3^{-k} (since M was chosen an arbitrary integer).

Let us now suppose that $N \geqslant 729$, $c = [N^{1/3}]$ and $d = 1$. We first show that (18) is satisfied for $k \geqslant N/2$.

Since $\alpha(s) < e^2s^2/4$ by calculus, Lemma 6 yields (in conjunction with Lemma 5)

$$|E_n(-1/N)| < C_3(2s^2)^m.$$

To bound C_3, we suppose $n = m$ and hence that

$$I_2 = \int_0^1 t(1-t)(1+t/n)^{c-2}\,dt.$$

Now
$$G(s,t) = t(1-t)(1+t/N)^{c-1}$$

which implies $I_2/\max_{t\in[0,1]}|G(s,t)| \leqslant 1/(1+t'/N) < 1$ where t' maximizes the function $t(1-t)(1+t/N)^{c-2}$ on $[0,1]$. It follows that $C_3 < C_1 < c/(2\pi)$. The case $n = m-1$ is similar, with a sharper bound for C_3. We may conclude, therefore, that

$$|E_n(-1/N)| < \frac{c}{\pi}(2c^2)^m.$$

Since $\mathscr{G}(c,d) \geqslant 1$, the above implies (18) provided m is such that

$$\left(\frac{N}{2c^2}\right)^m \geqslant \frac{c}{\pi}\cdot N^{c-1}$$

or equivalently,
$$m \geqslant \log\left(\frac{cN^{c-1}}{\pi}\right)\Big/\log\left(\frac{N}{2c^2}\right). \tag{20}$$

Now $\log(cN^{c-1}/\pi) < c\log N$ by our choice of c and N while $\log(N/(2c^2)) \geqslant \log(N^{1/3}/2) \geqslant (2/9)\log N$ (since $N \geqslant 729$). We conclude that the right hand side of (20) is bounded above by $9c/2$ and so if $k \geqslant N/2$, it follows that $m \geqslant N/(2c) \geqslant 9c/2$.

Now by Lemmas 5 and 6

$$|Q_n(-1/N)| < C_2\cdot 4^m$$

and we wish to bound C_2. If $n = m$, we have

$$I_1 = \int_0^1 t^{c-2}(1-t)\left(1+\left(\frac{N+1}{N}\right)t\right)dt = \frac{2+(1-c)/N}{c^3-c} < \frac{2}{c^3-c}$$

and
$$\max_{t\in[0,1]}|F(c,t)| \geqslant F\left(c,\frac{c-1}{c+1}\right) = \left(\frac{2}{c+1}\right)\left(\frac{c-1}{c+1}\right)^{c-1}\left(\frac{2}{c+1}-\frac{c-1}{N(c+1)}\right).$$

This last quantity is $> \frac{181}{182}\alpha(s)^{-1}$ since $c-1 < N/91$, whence

$$\max_{t\in[0,1]}|F(c,t)| > \frac{1}{2c^2}.$$

Table 1

N	c	d	k_0	$\mathscr{G}$	N	c	d	k_0	$\mathscr{G}$
4	25	13	28375	1·73668	16	17	5	16728	1·42391
5	17	8	66045	1·74651	17	7	2	10276	1·40421
6	30	13	162600	1·70934	18	25	7	10325	1·40286
7	37	15	127391	1·68958	19–21	11	3	$\leqslant$ 12749	1·42202
8	60	23	177060	1·64959	22–28	4	1	$\leqslant$ 11288	1·45226
9	52	19	219180	1·61823	29–31	13	3	$\leqslant$ 11466	1·34021
10	20	7	269480	1·58654	32–37	9	2	$\leqslant$ 8703	1·30645
11	86	29	359050	1·55299	38–77	5	1	$\leqslant$ 6175	1·30678
12	46	15	170890	1·52125	78–135	6	1	$\leqslant$ 1359	1
13	79	25	63437	1·48979	136–274	7	1	$\leqslant$ 647	1
14	13	4	36491	1·46801	275–545	8	1	$\leqslant$ 422	1
15	10	3	19900	1·44074	546–728	9	1	$\leqslant$ 382	1

Thus
$$C_2 < \frac{1}{2\pi}\sqrt{(c^2-1)}\cdot\frac{2}{(c^3-c)}\cdot 2c^2 = \frac{2}{\pi}\sqrt{\left(\frac{c^2}{c^2-1}\right)}.$$

Similarly if $n = m-1$, we have
$$C_2 < \frac{1}{2\pi}\sqrt{\left(\frac{1}{c^2-1}\right)}\cdot\frac{1}{c}\cdot 2c^2 = \frac{1}{\pi}\sqrt{\left(\frac{c^2}{c^2-1}\right)}$$

and in either case, we can write
$$|Q_n(-1/N)| < 4^m.$$

The result, then, will follow from (using Lemma 6)
$$(3/(4N)^{1/c})^k \geqslant 8N(N+1)^{c-1}$$
or equivalently
$$k \geqslant \log(8N(N+1)^{c-1})/\log(3/(4N)^{1/c}).$$

Now since $c = [N^{1/3}]$ and $N \geqslant 729$, we have
$$\log(3/(4N)^{1/c}) \geqslant \log(3/(3996)^{1/9}) > 0.17716$$
and
$$\log(8N(N+1)^{c-1}) < \log 8 + c\log(N+1)$$
so that
$$\log(8N(N+1)^{c-1})/\log(3/(4N)^{1/k}) < 11.74 + 5.65N^{1/3}\log(N+1).$$

Since this is less than $N/2$, the result obtains.

For $4 \leqslant N \leqslant 728$, we choose values c and d and use Lemma 3 to find intervals containing primes dividing $\mathscr{G}(c,d)$. To estimate the contribution of these primes, we apply upper and lower bounds on the Chebyshev function $\theta(x) = \sum_{p\leqslant x}\log p$ from theorem 10 of Rosser and Schoenfeld [**9**], the corollary to theorem 6 of Rosser and Schoenfeld [**10**], corollary 2 (9·8) of Schoenfeld [**11**] and the closing remarks to this last paper. We deduce explicit $k_0 = k_0(N)$ for each such N beyond which (16) holds and tabulate the results in Table 1, together with the choices of c and d, and the lower bound derived for $\mathscr{G}(c,d)^{1/(dm)}$ (denoted by $\mathscr{G}$). By checking the required inequality for smaller values of k, we complete the proof. This calculation utilizes Fortran code which computes the N-ary expansion of $(N+1)^k$ and searches for long strings of 0s or $(N-1)$s.

6. *Concluding remarks*

If one desires only to find effective (rather than explicit) bounds, it is possible to sharpen the theorem to

$$\left\|\left(\frac{N+1}{N}\right)^k\right\| > (2.85)^{-k}$$

for all $k \geqslant k_0 = k_0(N)$, a computable constant (where $N \geqslant 4$ as before). The case $N = 3$ appears to be intractable by this method. For larger values of N, however, the lower bound may be improved in the direction of (4). In fact, we can find an effective $k_0 = k_0(N)$ such that $k \geqslant k_0$ implies

$$\left\|\left(\frac{N+1}{N}\right)^k\right\| > (c_1 N)^{-c_2 k/\sqrt{N}}.$$

Here c_1 and c_2 are explicit constants which can be taken to be 4 and $\sqrt{2}$, respectively, without the use of Lemma 7 and decreased somewhat with its application. It follows that, given $\epsilon > 0$, there is an effective N_0 such that if $N \geqslant N_0$, we may find a computable $k_0 = k_0(N)$ with

$$\left\|\left(\frac{N+1}{N}\right)^k\right\| > e^{-\epsilon k}$$

for all $k \geqslant k_0$. In general, though, an effective version of (4) seems unlikely to be forthcoming.

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REFERENCES

[1] A. Baker and J. Coates. Fractional parts of powers of rationals. *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 269–279.

[2] M. Bennett. An ideal Waring problem with restricted summands. In preparation.

[3] M. Bennett. *Fractional parts of powers and related topics*. PhD thesis, University of British Columbia, 1993.

[4] F. Beukers. Fractional parts of powers of rationals. *Math. Proc. Cambridge Philos. Soc.* **90** (1981), 13–20.

[5] A. K. Dubitskas. A lower bound for the quantity $\|(3/2)^k\|$. *Russ. Math. Surveys*, **45** (1990), 163–164.

[6] D. Easton. *Applications of the hypergeometric function in effective Diophantine approximation*. PhD thesis, University of Waterloo, 1986.

[7] J. M. Kubina and M. C. Wunderlich. Extending Waring's conjecture to 471,600,000. *Math. Comp.* **55** (1990), 815–820.

[8] K. Mahler. On the fractional parts of the powers of a rational number: II. *Mathematika*, **4** (1957), 122–124.

[9] J. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.* **6** (1962), 64–94.

[10] J. Rosser and L. Schoenfeld. Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. *Math. Comp.* **29** (1975), 243–269.

[11] L. Schoenfeld. Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$, II. *Math. Comp.* **30** (1976), 337–360.

[12] K. Stromberg. *An Introduction to Classical Real Analysis*. Wadsworth International Mathematical Series, 1981.