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PERFECT POWERS WITH THREE DIGITS

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Abstract. We solve the equation $x^a + x^b + 1 = y^q$ in positive integers x, y, a, b and q with $a > b$ and $q \geqslant 2$ coprime to $\phi(x)$. This requires a combination of a variety of techniques from effective Diophantine approximation, including lower bounds for linear forms in complex and p-adic logarithms, the hypergeometric method of Thue and Siegel applied p-adically, local methods, and the algorithmic resolution of Thue equations.

§1. *Introduction.* The problem of digital representations of integers from special sequences is, in many cases, one of considerable subtlety. By way of example, the classification of perfect powers with precisely two binary digits was solved in antiquity, whilst the analogous solution for three digits is of a much more recent vintage (see Szalay [**15**]), that for four such digits is incomplete (but partially understood; see [**2, 10**]), and, for five or more digits, even finiteness results for the problem are unavailable. In a pair of recent papers [**2, 3**], the authors, with Mignotte, have derived a number of results on equations of the shape

$$x_1^a + x_2^b + 1 = y^q \quad \text{and} \quad x_1^a + x_2^b + x_3^c + 1 = y^q,$$

where x_1, x_2 and x_3 are positive integers with the property that $\gcd(x_1, x_2) > 1$ or $\gcd(x_1, x_2, x_3) > 1$, respectively. Such equations are shown to have, effectively, no solutions in exponents a, b and c and integers y and $q \geqslant q_0 = q_0(x_i)$. In the particular case where $x_1 = x_2 = x$, the first equation is proven to have no solutions whatsoever, provided q exceeds some effectively computable absolute constant, at least under the assumption that $\gcd(q, \phi(x)) = 1$. In the paper at hand, we will sharpen this last result, proving the following theorem.

THEOREM 1. *Let x be a positive integer and suppose that there exist non-negative integers a, b, y and $q \geqslant 2$ such that*

$$x^a + x^b + 1 = y^q, \quad a > b > 0 \text{ with } \gcd(q, \phi(x)) = 1. \tag{1}$$

Then

$$(x, a, b, y^q) = (2, 5, 4, 7^2),\ (2, 9, 4, 23^2),\ (3, 7, 2, 13^3) \quad \text{or} \quad (2, 2t, t+1, (2^t + 1)^2),$$

for some integer $t \geqslant 2$.

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The case $x = q = 2$ is the main result (Theorem 1) of Szalay [**15**], whilst the more general situation with $x \in \{2, 3\}$ is a consequence of the following.

THEOREM 2 (Bennett *et al* [**3**, Theorem 1]). *If there exist integers $a > b > 0$ and $q \geqslant 2$ for which*

$$x^a + x^b + 1 = y^q \quad \textit{with } x \in \{2, 3\},$$

then (x, a, b, y^q) is one of $(2, 5, 4, 7^2)$, $(2, 9, 4, 23^2)$ or $(3, 7, 2, 13^3)$, or $(x, a, b, y^q) = (2, 2t, t+1, (2^t + 1)^2)$, for some integer $t \geqslant 2$.

Theorem 1 is, in fact, a special case of our next result, in conjunction with Theorem 2.

THEOREM 3. *Let x be a positive integer and suppose that there exist positive integers a, b, y and an odd prime q such that*

$$x^a + x^b + 1 = y^q, \quad a > b > 0. \tag{2}$$

If we write $x = x_0 \cdot x_1$, where x_0 is comprised solely of prime factors p of x for which $y \equiv 1 \pmod{p}$, then either $(x, a, b, y^q) = (3, 7, 2, 13^3)$ or $x_0 < x_1^3$.

We remark here that we can in fact prove a like result to this theorem with the inequality $x_0 < x_1^3$ replaced by $x_0 < x_1^t$, for any

$$t > \min_{\substack{2 \leqslant m \leqslant q \\ m \in \mathbb{Z}}} \left\{ \frac{m^2 - m + 2q}{2mq - m^2 + m - 2q} \right\},$$

provided we have x^b suitably large (depending, effectively, upon t and q). In particular, as a simple exercise in calculus, we have such a result for any $t > \sqrt{2/q} + 4/q$. To prove this requires appeal to multi-point Padé approximations to $(1 - z)^{i/q}$ for $0 \leqslant i \leqslant m - 1$ and introduces an assortment of technical difficulties. We restrict our attention to the case $t = 3$ for simplicity and to make our conclusions as explicit as possible.

To see how Theorem 1 follows from Theorem 3, in case q has an odd prime factor, observe that the condition $\gcd(q, \phi(x)) = 1$ implies, with equation (2), that $y \equiv 1 \pmod{x}$, and hence that $x_0 = x$ and $x_1 = 1$, whereby $(x, a, b, y^q) = (3, 7, 2, 13^3)$.

It follows from Theorem 1 that every solution (x, a, b, y^q) with $x \geqslant 4$ and q prime to equation (2) satisfies $x \geqslant 2q + 1$.

In case $q = 2$, it is possible to somewhat generalize the aforementioned result of Szalay (classifying solutions to (1) with $x = q = 2$).

THEOREM 4. *Let x be a positive integer and suppose that there exist positive integers a, b and y such that*

$$x^a + x^b + 1 = y^2, \quad a > b > 0. \tag{3}$$

If we write $x = x_0 \cdot x_1$, where x_1 is the largest odd divisor of x, then either

$$(x, a, b, y) = (2, 5, 4, 7), (2, 9, 4, 23) \textit{ or } (2, 2t, t+1, 2^t + 1) \textit{ for } t \geqslant 2,$$

or we have $x_0 < x_1^4$.

Luca [12] (see also Scott [14]) proved that equation (3) has no solutions with $x = p$ an odd prime. In this case, if a solution existed, one would necessarily have either $\gcd(x, y-1) = 1$ or $\gcd(x, y+1) = 1$. One way to generalize this result is thus the following theorem.

THEOREM 5. *Let x be an odd positive integer and suppose that there exist positive integers a, b and y such that*

$$x^a + x^b + 1 = y^2, \quad a > b > 0.$$

Then

$$\min\{\gcd(x, y-1), \gcd(x, y+1)\} > x^{1/6}.$$

Here, the exponent $1/6$ may be replaced by any number smaller than $1/4$, for sufficiently large x; again, our statement is as given to ensure that it is as clean as possible. We observe that our proof of this theorem, being based on techniques from Diophantine approximation, is of an entirely different flavour to that given in [12] (which relies upon the arithmetic of quadratic fields).

It is worth noting that Theorem 1 is directly analogous to a result of Bugeaud *et al* [6], classifying solutions to the Nagell–Ljunggren equation

$$\frac{x^n - 1}{x - 1} = y^q$$

under the constraint that every prime divisor of x divides $y - 1$.

The techniques of this paper may be applied somewhat more generally than just to equations of the shape $x^a + x^b + 1 = y^q$. Indeed, one may complement [1, Theorem 2] (where the first author studied perfect powers with few ternary digits), combining lower bounds for linear forms in two or three Archimedean logarithms with estimates for linear forms in two 3-adic logarithms to compute an integer q_0 with the property that no qth power with $q \geqslant q_0$ has at most three ternary digits. Applying then a result of Corvaja and Zannier [9] leads to the conclusion that, beside the integers of the form $(3^t + 1)^2 = 3^{2t} + 2 \cdot 3^t + 1$, there exist only finitely many perfect powers with at most three ternary digits.

Whilst the techniques of [2, 3] are based almost entirely upon the theory of lower bounds for linear forms in logarithms (p-adic and complex), here we will proceed by combining bounds for non-Archimedean logarithms with the hypergeometric method of Thue and Siegel, applied p-adically. Such arguments have been used previously in, for example, work of Beukers [4] and Corvaja and Zannier [10].

The outline of this paper is as follows. In §2, we begin by appealing to results from linear forms in non-Archimedean logarithms to prove Theorem 3 for suitably large prime exponent q. The quality of our bounds in this section will prove to be of importance later. In §3, we will introduce Padé approximants to the binomial function. Applying these p-adically will enable us to derive upper bounds for x in Theorem 3, essentially reducing the proof to a finite computation. Section 4 continues and sharpens this argument, leaving us with a (feasible) finite computation, the details of which are discussed in §5. Finally, in §6, we treat the cases with exponent $q = 2$ (i.e. Theorems 4 and 5).

§2. *Linear forms in two logarithms.* In this section, we will begin the proof of Theorem 3 by applying estimates for linear forms in two non-Archimedean logarithms to deduce explicit upper bounds for q in equation (2), under the given constraints on x_0.

The assumptions of the theorem we will use (a special case of [**5**, Theorem 3]) appear rather restrictive, but are satisfied in our situation. If an integer $m > 1$ has the factorization $m = p_1^{j_1} \cdots p_k^{j_k}$, where the p_i are distinct primes and $j_i \in \mathbb{N}$, we define, for a non-zero integer x,

$$\nu_m(x) = \min_{1 \leqslant i \leqslant k} \left[\frac{\nu_{p_i}(x)}{j_i} \right],$$

where $\nu_p(x)$ is defined to be the largest integer k such that $p^k | x$.

THEOREM 6. *Let α_1 and α_2 be positive rational numbers with $\alpha_1 \neq 1$, b_1 and b_2 be positive integers and set*

$$\Lambda = \alpha_2^{b_2} - \alpha_1^{b_1}.$$

For any set of distinct primes $p_1, \ldots, p_k$ and positive integers $j_1, \ldots, j_k$, we let $m = p_1^{j_1} \cdots p_k^{j_k}$ and suppose that there exists a positive integer g such that for each i, we have either

$$\nu_{p_i}(\alpha_1^g - 1) \geqslant j_i \quad \textit{and} \quad \nu_{p_i}(\alpha_2^g - 1) \geqslant 1 \quad \textit{if } p_i \geqslant 2,$$

or

$$\nu_{p_i}(\alpha_1^g - 1) \geqslant 2 \quad \textit{and} \quad \nu_{p_i}(\alpha_2^g - 1) \geqslant 2 \quad \textit{if } p_i = 2.$$

Then, if m, b_1 and b_2 are relatively prime, and α_1 and α_2 are multiplicatively independent, we may conclude that

$$\nu_m(\Lambda) \leqslant \frac{53.6g}{(\log m)^4} (\max\{\log b' + \log(\log m) + 0.64, 4 \log m\})^2 \log A_1 \log A_2,$$

where

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1} \quad \textit{and} \quad \log A_i \geqslant \max\{h(\alpha_i), \log m\}.$$

For the remainder of this section, we will suppose that we have a solution to the Diophantine equation (2), where $x = x_0 x_1$ with $x_0 \geqslant x_1^3$. Let p be a prime divisor of x such that p divides $y - 1$ and define u to be the largest integer such that p^u divides x^b. If $p \neq q$, then p^u divides $y - 1$. Otherwise, $\max\{p, p^{u-1}\}$ divides $y - 1$.

We apply Theorem 6 with $\alpha_1 = y$, $\alpha_2 = x^b + 1$, $b_1 = q$, $b_2 = 1$ and $m = x_0^b q^{-\delta}$, where $\delta = 1$, if q divides x_0 (whereby, necessarily, q^2 divides x_0^b), and $\delta = 0$ otherwise. We therefore have $y \equiv 1 \pmod{m}$ and may take $g = 2$ if $b = 1$ and x is even but not divisible by 4. Otherwise, $g = 1$. Clearly, α_1 and α_2 are multiplicatively independent.

Put $A_2 = x^b + 1$ and $A_1 = y$. Since $y \equiv 1 \pmod m$, we deduce that $A_1 \geqslant \max\{y, m\}$. We then have

$$\nu_m(\Lambda) \leqslant \frac{53.6g}{(\log m)^4}(\max\{\log b' + \log(\log m) + 0.64, 4\log m\})^2 \times \log A_1 \ \log A_2, \tag{4}$$

where

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1} \quad \text{and} \quad \log A_i \geqslant \max\{h(\alpha_i), \log m\}.$$

In the other direction, note that

$$\nu_m(\Lambda) \geqslant \lfloor a/b \rfloor \geqslant (a/b) - 1 \geqslant \frac{q \log y}{b \log x} - \frac{\log 2}{b \log x} - 1.$$

Let us assume first that the maximum in inequality (4) is equal to $4\log m$. Then

$$\nu_m(\Lambda) \leqslant \frac{857.6g}{(\log m)^2}(\log(x^b+1))(\log y)$$

and

$$q \leqslant \frac{857.6g}{(\log m)^2}(\log x^b)\log(x^b+1) + \frac{\log 2}{\log y} + \frac{b \log x}{\log y}.$$

From $x_0 \geqslant x_1^3$, we deduce that $x_0^4 \geqslant x_0^3 x_1^3 = x^3$.

When q does not divide x_0, set $\nu = 0$. Otherwise, define ν by $q = (x^b)^\nu$. We therefore have both $0 \leqslant \nu \leqslant 1/2$ and

$$q \leqslant \frac{857.6g}{(3/4-\nu)^2} \cdot \frac{\log(x^b+1)}{\log x^b} + \frac{\log 2}{\log y} + \frac{b \log x}{\log y}.$$

Very roughly,

$$y \geqslant m \geqslant x_0^{b/2} \geqslant x^{3b/8}, \tag{5}$$

and so

$$q \leqslant \frac{857.6g}{(3/4-\nu)^2} \cdot \frac{\log(x^b+1)}{\log x^b} + \frac{11}{3}. \tag{6}$$

From Theorem 2, we may suppose that $x^b \geqslant 5$ (and, more generally, that x^b is not a power of 2 or of 3). If $x^b \leqslant 1500$ and $q > 1500$, then $\nu = 0$ and we infer from (6) that

$$q \leqslant 1524.7 \frac{\log(x^b+1)}{\log x^b} g + \frac{11}{3} < 1702g.$$

If $x^b > 1500$, then (6) implies that

$$q \leqslant \frac{857.7}{(3/4-\nu)^2} g + \frac{11}{3}. \tag{7}$$

All these bounds are under the assumption that the maximum in (4) is equal to $4\log m$. If this condition is not fulfilled, then we obtain

$$\nu_m(\Lambda) \leqslant \frac{53.6g}{(\log m)^4}(\log b' + \log(\log m) + 0.64)^2(\log(x^b+1))(\log y).$$

Since, by (5),

$$\log b' + \log(\log m) \leqslant \log(q + 8/3),$$

we therefore have

$$\frac{q\log y}{b\log x} \leqslant \frac{\log 2}{b\log x} + 1 + \frac{53.6g}{(\log m)^4}(\log(q+8/3)+0.64)^2(\log(x^b+1))(\log y).$$

Consequently,

$$q \leqslant \frac{53.6g}{(3/4-\nu)^4(\log x^b)^2}(\log(q+8/3)+0.64)^2 \cdot \frac{\log(x^b+1)}{\log x^b} + \frac{11}{3}. \tag{8}$$

If $g = 1$, $x^b \leqslant 1500$ and $q > 1500$, then $\nu = 0$ and we infer from (8) that

$$q \leqslant 170\frac{\log(x^b+1)}{\log x^b}\left(\frac{\log(q+8/3)+0.64}{\log x^b}\right)^2 + \frac{11}{3}.$$

For $x^b = 5, 6, 7, 10, 11, 12, 13, 14$ and 15, we obtain the bounds 6500, 4800, 3800, 2400, 2120, 1915, 1755, 1625 and 1520, respectively.

Assume now that $g = 1$, $x^b > 1500$ and $q > 1500$. If $\delta = 1$, then $x^b \geqslant q^2 \geqslant 1500^2$ and, since $\nu \leqslant 1/2$, (8) implies that

$$q \leqslant 7.2g(\log(q+8/3)+0.64)^2 + \tfrac{11}{3}. \tag{9}$$

If $\delta = 0$, then $\nu = 0$ and we get from (8) that

$$q \leqslant 3.2g(\log(q+8/3)+0.64)^2 + \tfrac{11}{3}. \tag{10}$$

To summarize, in the case $g = 1$, we may conclude that $q \leqslant q_0$, where

x^b	q_0	x^b	q_0	x^b	q_0
5	6500	11	2120	$15 \leqslant x^b \leqslant 29$	1577
6	4800	12	1915	$30 \leqslant x^b \leqslant 1500$	1543
7	3800	13	1755	$x^b > 1500$	$\dfrac{857.7}{(3/4-\nu)^2} + \dfrac{11}{3}$
10	2400	14	1625		

(11)

For $g = 2$ (which can occur only if $b = 1$, x is even and not divisible by 4), we essentially have to replace the values of q_0 in (11) by $2q_0$. In the worst case, when $q|x$ and $g = 2$, we therefore have $q \leqslant 27449$, while $q|x$ and $g = 1$ implies $q \leqslant 13723$. If q fails to divide x and $x^b > 1500$, we have $q \leqslant 1523$ or $q \leqslant 3049$, if $g = 1$ or 2, respectively.

§3. *Applications of Padé approximants to hypergeometric functions.* Our goal in the next few sections will be to derive an absolute bound for x satisfying (2) with the additional assumption that $x_0 \geqslant x_1^3$. To do this, we will appeal to the theory of Padé approximants to binomial functions. Such an approach is reasonably common in a variety of number theoretic contexts, see e.g. [**10**, **11**, **13**].

Let us define Padé approximants to $(1+z)^{1/q}$, for $q \geqslant 2$ prime. If n_1 and n_2 are non-negative integers, we set

$$P_{n_1,n_2}(z) = \sum_{k=0}^{n_1} \binom{n_2 + 1/q}{k} \binom{n_1 + n_2 - k}{n_2} z^k$$

and

$$Q_{n_1,n_2}(z) = \sum_{k=0}^{n_2} \binom{n_1 - 1/q}{k} \binom{n_1 + n_2 - k}{n_1} z^k. \tag{12}$$

Then (see e.g. [**7**]), we have that

$$P_{n_1,n_2}(z) - (1+z)^{1/q} Q_{n_1,n_2}(z) = z^{n_1+n_2+1} E_{n_1,n_2}(z), \tag{13}$$

where

$$\begin{aligned} E_{n_1,n_2}(z) = {} & \frac{(-1)^{n_2}\Gamma(n_2 + (q+1)/q)}{\Gamma(-n_1 + 1/q)\Gamma(n_1 + n_2 + 2)} \\ & \times F(n_1 + (q-1)/q, n_2 + 1, n_1 + n_2 + 2, -z), \end{aligned} \tag{14}$$

for F the hypergeometric function given by

$$F(a, b, c, -z) = 1 - \frac{a \cdot b}{1 \cdot c} z + \frac{a \cdot (a+1) \cdot b \cdot (b+1)}{1 \cdot 2 \cdot c \cdot (c+1)} z^2 - \cdots .$$

Appealing twice to (13) and (14) with, in the second instance, n_2 replaced by $n_2 + 1$, and eliminating $(1+z)^{1/q}$, we find that $P_{n_1,n_2+1}(z) Q_{n_1,n_2}(z) - P_{n_1,n_2}(z) Q_{n_1,n_2+1}(z)$ is a polynomial of degree $n_1 + n_2 + 1$ with a zero at $z = 0$ of order $n_1 + n_2 + 1$ (and hence monomial). It follows that we may write

$$P_{n_1,n_2+1}(z) Q_{n_1,n_2}(z) - P_{n_1,n_2}(z) Q_{n_1,n_2+1}(z) = c z^{n_1+n_2+1}, \tag{15}$$

with, as a short calculation reveals, $c \neq 0$.

We will consider (13) with $z = x^b$ and choose non-negative integers n_1 and n_2 such that $n_1 \geqslant n_2$. Let us write $x = x_0 \cdot x_1$, where x_0 is comprised of the primes p dividing x for which $y \equiv 1 \pmod{p}$, and x_1 consists of the largest factor of x coprime to x_0. It is useful for us to observe (see e.g. Lemma 3.1 of Chudnovsky [**8**]) that

$$\binom{n \pm \frac{1}{q}}{k} q^{[qk/(q-1)]} \in \mathbb{Z},$$

so that, in particular, since $n_1 \geqslant n_2$, defining C_{n_1,n_2} by

$$C_{n_1,n_2} = \gcd\left\{ \text{numerator}\left(\binom{n_2 + 1/q}{k} \binom{n_1 + n_2 - k}{n_2} \right), k = 0, \ldots, n_1 \right\},$$

we have

$$q^{\kappa} C_{n_1,n_2}^{-1} P_{n_1,n_2}(x^b) \quad \text{and} \quad q^{\kappa} C_{n_1,n_2}^{-1} Q_{n_1,n_2}(x^b) \in \mathbb{Z},$$

where

$$\kappa = \begin{cases} \left[\dfrac{qn_1}{q-1}\right] & \text{if } \gcd(x,q)=1, \\ 0 & \text{if } q|x \text{ and } \max\{\nu_q(x), b\} > 1. \end{cases}$$

Note that we cannot have $\nu_q(x) = b = 1$, since $y^q \equiv 1 \pmod{q}$ implies that $y^q \equiv 1 \pmod{q^2}$.

We suppose that p is a prime divisor of x. Setting $\eta = (1+x^b)^{1/q}$, since $(1+x^b z)^{1/q}$, $q^{\kappa} P_{n_1,n_2}(z)$ and $q^{\kappa} Q_{n_1,n_2}(z)$ each have p-adic integral coefficients, the same is necessarily true of $q^{\kappa} E_{n_1,n_2}(z)$ and so, via equation (13),

$$|q^{\kappa} C_{n_1,n_2}^{-1} P_{n_1,n_2}(x^b) - \eta q^{\kappa} C_{n_1,n_2}^{-1} Q_{n_1,n_2}(x^b)|_p \leqslant p^{-\nu_p(x) b(n_1+n_2+1)}.$$

On the other hand, since $\eta^q \equiv y^q \pmod{p^{\nu_p(x)a}}$, if we assume that $y \equiv 1 \pmod{p}$ (i.e. that $p|x_0$), we may conclude, if $p|x$ and $(p,q) \neq (2,2)$, that

$$\eta \equiv y \pmod{p^{\nu_p(x)a-\delta}} \quad \text{for } \delta = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases}$$

If x is even and $q = 2$, we have

$$\eta \equiv \pm y \pmod{2^{\nu_2(x)a-1}}$$

for some choice of sign. It follows, in case $p|x$ and $(p,q) \neq (2,2)$, that

$$\begin{aligned} &|q^{\kappa} C_{n_1,n_2}^{-1} P_{n_1,n_2}(x^b) - y q^{\kappa} C_{n_1,n_2}^{-1} Q_{n_1,n_2}(x^b)|_p \\ &\quad \leqslant p^{-\min\{\nu_p(x)a-\delta,\, \nu_p(x) b(n_1+n_2+1)\}}. \end{aligned} \tag{16}$$

Defining

$$\kappa_1 = \begin{cases} \left[\dfrac{qn_1}{q-1}\right] & \text{if } \gcd(x,q)=1, \\ 0 & \text{if } \gcd(x,q) = q, \end{cases}$$

if

$$P_{n_1,n_2}(x^b) \neq y Q_{n_1,n_2}(x^b)$$

we may therefore conclude, assuming $q \geqslant 3$ and $n_1 + n_2 + 1 \geqslant a/b$, that

$$\Lambda_{n_1,n_2} := |q^{\kappa_1} C_{n_1,n_2}^{-1} P_{n_1,n_2}(x^b) - y q^{\kappa_1} C_{n_1,n_2}^{-1} Q_{n_1,n_2}(x^b)| \geqslant x_0^a. \tag{17}$$

We choose

$$n_1 = \left\lceil \frac{(q+1)a}{2qb} \right\rceil \quad \text{and} \quad n_2 = \left\lceil \frac{(q-1)a}{2qb} \right\rceil - \delta$$

for $\delta \in \{0, 1\}$, where $\lceil x \rceil$ denotes the smallest integer $\geqslant x$, so that, in particular, we have the desired inequality $(n_1 + n_2 + 1)b \geqslant a$. Equation (15) readily implies that for at least one of $\delta \in \{0, 1\}$, we must have $P_{n_1,n_2}(x^b) \neq y\, Q_{n_1,n_2}(x^b)$ and hence inequality (17).

Let us assume for the remainder of this section that $x^b \geqslant 10^6$. Before we proceed further, we will make use of a pair of (preliminary) lower bounds upon a/b. Note that $y-1$ is divisible by $q^{-\delta} x_0^b$, where $\delta = 1$ if $q|x_0$ and 0 otherwise. Further, if $\delta = 1$, then necessarily $q^2|x_0^b$. Since we have assumed $x_0 \geqslant x_1^3$, we have $x_0 > x^{3/4}$. Using only that $a \geqslant 3$, $q \geqslant 3$ and $x \geqslant 5$, we find that

$$y < 1.06502 x^{a/q}, \tag{18}$$

and so

$$x^{a/q} > q^{-\delta} 1.06502^{-1} x^{3b/4},$$

whence we have the inequality

$$a > \left(\frac{3q}{4} - \frac{q \log(1.06502q)}{\log(x^b)} \right) b. \tag{19}$$

We next consider $\Lambda_{1,0}$, which divides $|q + x^b - qy|$. We begin by showing that $\Lambda_{1,0} \neq 0$, which is obviously true if q fails to divide x. If $q|x$, say $x = q^{\nu_q(x)} \cdot z$, then if $\Lambda_{1,0} = 0$, necessarily

$$y = 1 + q^{b\nu_q(x)-1} \cdot z^b,$$

and hence the equation $x^a + x^b + 1 = y^q$ becomes

$$q^{a\nu_q(x)} z^a = q^{(b\nu_q(x)-1)q} z^{qb} + \cdots + \binom{q}{2} q^{(b\nu_q(x)-1)2} z^{2b}. \tag{20}$$

If we have $q = 3$ and $b\nu_q(x) = 2$, then

$$3^{a\nu_q(x)-3} z^{a-2b} = z^b + 1,$$

whence

$$3^{a-3} z^{a-4} = a^2 + 1 \quad \text{or } 3^{2a-3} z^{a-2} = z + 1.$$

In either case, we easily obtain a contradiction. Otherwise, from (20), $a\nu_q(x) = 2b\nu_q(x) - 1$, so that $\nu_q(x) = 1$ (whereby $b \geqslant 2$) and $a = 2b - 1$. Comparing terms in (20), we find that

$$q^{2b-1} z^{2b-1} > q^{(b-1)q} z^{qb} \geqslant q^{3b-3} z^{3b},$$

contradicting $b \geqslant 2$. We conclude, as desired, that $\Lambda_{1,0} \neq 0$.

Since $a \geqslant 2b$ (which follows, with care, from (19)), we thus have

$$\min\{\nu_p(x)a,\ \nu_p(x)b(n_1 + n_2 + 1)\} = \min\{\nu_p(x)a,\ 2\nu_p(x)b\} \geqslant 2b,$$

for each prime divisor p of x, whereby, from (16),

$$\Lambda_{1,0} \geqslant x_0^{2b} \geqslant x^{3b/2}.$$

We therefore have $qy > x^{3b/2}$, and so

$$x^{a/q} > (1.06502q)^{-1} x^{3b/2}, \tag{21}$$

i.e.

$$a > \left(\frac{3q}{2} - \frac{q\log(1.06502q)}{\log(x^b)}\right)b. \tag{22}$$

We are now ready to proceed. Inequality (17) provides us with a strong lower bound upon Λ_{n_1,n_2}. On the other hand, following Saradha and Shorey [13] (see the proof of Lemma 18), we have

$$\begin{aligned} |P_{n_1,n_2}(x^b)| &< \left(\frac{n_1+n_2}{2}+1\right)2^{n_1+n_2-1}\left(1+\frac{x^b}{2}\right)^{n_1} \\ &\leqslant \left(\frac{a}{2b}+2\right)2^{a/b+1}\left(1+\frac{x^b}{2}\right)^{(q+1)a/(2qb)+1}. \end{aligned} \tag{23}$$

Let us suppose first that $17 \leqslant q \leqslant 27449$. Combining (19), (23) and the fact that $x^b \geqslant 10^6$, it follows, after a little work, that

$$|P_{n_1,n_2}(x^b)| < 0.66 \cdot 2^{2a/3b} x^{(q+1)a/(2q)+b}.$$

Similarly,

$$|Q_{n_1,n_2}(x^b)| < 2^{n_1+n_2-1}\left(1+\frac{x^b}{2}\right)^{n_2} < 0.13 \cdot 2^{2a/3b} x^{(q-1)a/(2q)+b}$$

and so, from (18),

$$|q^{\kappa_1} P_{n_1,n_2}(x^b) - yq^{\kappa_1} Q_{n_1,n_2}(x^b)| < q^{\kappa_1} 2^{2a/3b} x^{(q+1)a/(2q)+b}.$$

Since

$$q^{\kappa_1} \leqslant q^{(q+1)a/(2(q-1)b)+q/(q-1)}$$

and $C_{n_1,n_2} \geqslant 1$, we thus may conclude from (17) that

$$x_0^b < 2^{2/3} q^{(q+1)/(2(q-1))+bq/a(q-1)} x^{b(q+1)/(2q)+b^2/a},$$

i.e., since $x_0 > x^{3/4}$, that

$$x^b < (2^{2/3} q^{(q+1)/(2(q-1))+bq/a(q-1)})^{(1/4-1/(2q)-b/a)^{-1}}.$$

This, with inequality (22), contradicts the assumption that $x^b \geqslant 10^6$ for $17 \leqslant q \leqslant 317$ and, for $331 \leqslant q \leqslant 27449$, implies, in each case, that $x^b < 1062q^{3/2}$.

§4. *Sharper lower bounds for a/b.* To derive absolute upper bounds upon x^b, for the remaining values of q, i.e. $3 \leqslant q \leqslant 13$, we require rather stronger lower bounds for a/b. We assume, as we may, that x is divisible by a prime exceeding 3. Note that $\Lambda_{2,0}$ divides

$$\left|q^2 + qx^b - \binom{q-1}{2} x^{2b} - q^2 y\right|.$$

Since $y \geqslant 3$ and $x^b \geqslant 1$, this quantity is necessarily non-zero and so, since $a > qb \geqslant 3b$,

$$x^{9b/4} \leqslant x_0^{3b} < \left(\frac{q-1}{2}\right)x^{2b} + q^2 y.$$

For $q \leqslant 13$, this, together with the assumption that $x^b \geqslant 10^6$, implies that $q^2 y > 0.8\, x^{9b/4}$ and so

$$x^{a/q} > (4q^2/3)^{-1} x^{9b/4},$$

i.e.

$$a > \left(\frac{9q}{4} - \frac{q\log(4q^2/3)}{\log(x^b)}\right)b \quad \text{for } 3 \leqslant q \leqslant 13. \tag{24}$$

Inequality (24) is sufficiently strong for what we have in mind, provided $q \in \{11, 13\}$. Indeed, arguing as in the preceding section contradicts $x^b \geqslant 10^6$ in either case. For $3 \leqslant q \leqslant 7$, (24) and $x^b \geqslant 10^6$ implies that

$$\begin{cases} a > 6.21b & \text{if } q = 3, \\ a > 9.98b & \text{if } q = 5, \\ a > 13.63b & \text{if } q = 7. \end{cases}$$

We next observe that $\Lambda_{3,1}$ divides $|81 + 81x^b + 9x^{2b} - x^{3b} - 81y - 54x^b y|$, if $q = 3$, and

$$\begin{aligned} 2^{-\alpha}|2q^3 &+ \tfrac{1}{2}(3q^2(q+1))x^b + \tfrac{1}{2}(q(q+1))x^{2b} \\ &- \tfrac{1}{12}(q^2-1)x^{3b} - 2q^3 y - \tfrac{1}{2}(3q^3 - q^2)x^b y|, \end{aligned}$$

otherwise, where $\alpha = 0$, if $q = 5$, or $\alpha = 1$, if $q = 7$. Again, in each case, the quantity inside the absolute value is negative, whence, appealing to the preceding lower bounds upon a/b, we arrive at the inequalities

$$x^{15b/4} \leqslant x_0^{5b} < x^{3b} + 81y + 54x^b y \quad \text{if } q = 3,$$

or

$$x^{15b/4} \leqslant x_0^{5b} < 2^{-\alpha}\left(\frac{q^2-1}{12}\, x^{3b} + 2q^3 y + \frac{3q^3 - q^2}{2} x^b y\right) \quad \text{if } q > 3.$$

Our assumption that $x^b \geqslant 10^6$ thus implies inequalities of the shape

$$\begin{cases} a > 7.37b & \text{if } q = 3, \\ a > 11.85b & \text{if } q = 5, \\ a > 16.43b & \text{if } q = 7. \end{cases}$$

In case $q = 7$, we consider $n_1 = 5$ and $n_2 = 3$, to find that $\Lambda_{5,3}$ divides $|P - yQ|$, where

$$P = 470596 + 924385\, x^b + 565950\, x^{2b} + 107800\, x^{3b} + 1540x^{4b} - 66x^{5b}$$

and

$$Q = 470596 + 857157\,x^b + 472311\,x^{2b} + 74970\,x^{3b}.$$

Our lower bound upon x^b implies that $P - yQ$ is negative and so, since $a > 16.43b > 9b$, we find that

$$x^{27b/4} \leqslant x_0^{9b} < 66x^{5b} + y(470596 + 857157\,x^b + 472311\,x^{2b} + 74970\,x^{3b}),$$

which, with $x^b \geqslant 10^6$, yields $y > 74971^{-1}x^{15b/4}$. Arguing as previously, we conclude that $a > 20.53b$, if $q = 7$. Feeding this inequality back into the arguments of the preceding section leads to the conclusion that $x^b < 2 \times 10^6$.

In case $q = 5$, we also consider $n_1 = 5$ and $n_2 = 3$, to find that $\Lambda_{5,3}$ divides $|P - yQ|$, where

$$P = 109375 + 218750\,x^b + 137500\,x^{2b} + 27500\,x^{3b} + 550x^{4b} - 22x^{5b}$$

and

$$Q = 109375 + 196875\,x^b + 106875\,x^{2b} + 16625\,x^{3b}.$$

Again, $P - yQ$ is negative and so, since $a > 11.85b > 9b$, we find that

$$x^{27b/4} \leqslant x_0^{9b} < 22x^{5b} + y(109375 + 196875\,x^b + 106875\,x^{2b} + 16625\,x^{3b}),$$

which yields $y > 16626^{-1}x^{15b/4}$ and so $a > 15.20b$. We next consider $\Lambda_{10,4}$. As before, the leading coefficient of $P_{10,4}(z)$ is negative, so that we are led to an inequality of the shape

$$x^{45b/4} \leqslant x_0^{15b} < 456x^{10b} + 5^8 yQ,$$

where

$$Q = 8125 + 22750\,x^b + 23100\,x^{2b} + 10010\,x^{3b} + 1547x^{4b}.$$

After a little work, we conclude that $a > 28.90b$. We finally consider $\Lambda_{14,6}$ (where again the leading coefficient of $P_{14,6}(z)$ is negative), which leads to the inequality

$$x^{63b/4} \leqslant x_0^{21b} < 290377\,x^{14b} + 5^9 yQ,$$

where $Q < 5.3 \times 10^7 x^{6b}$. With $x^b \geqslant 10^6$, we conclude that $a > 37.04b$ and hence, as in the previous section, after some work, that again $x^b < 2 \times 10^6$.

The case $q = 3$ is necessarily more involved, since we require a much larger lower bound upon a/b. In the following table, we use two shorthands to indicate why $\Lambda_{n_1,n_2} \neq 0$. If we write (1), it indicates that the sign of z^{n_1} in $P_{n_1,n_2}(z)$ is negative. If, instead, we write (2), it means that y is known to be suitably larger than $x^{(n_1-n_2)b}$. In either case, appealing to our assumed lower bound for x^b, the term inside the absolute value in the definition of Λ_{n_1,n_2} is negative and hence non-zero.

To implement our arguments, at each stage we require $a \geqslant (n_1 + n_2 + 1)b$ (note that this is the reverse of the inequality we had for our choices of n_1 and n_2

in §3) and suppose throughout that $x^b \geqslant 2 \times 10^7$.

(n_1, n_2)	$\Lambda_{n_1,n_2} \neq 0$	C_{n_1,n_2}	$y \geqslant$	$a/b \geqslant$
(4, 2)	(1)	1	$\frac{1}{1200} x^{13b/4}$	8
(5, 2)	(2)	7	$\frac{1}{892} x^{4b}$	10
(6, 3)	(2)	7	$\frac{1}{15148} x^{9b/2}$	11
(7, 3)	(1)	20	$\frac{1}{28918} x^{21b/4}$	14
(8, 4)	(1)	5	$\frac{1}{665100} x^{23b/4}$	15
(9, 5)	(1)	26	$\frac{1}{5986000} x^{25b/4}$	16
(10, 5)	(2)	13	$\frac{1}{74395200} x^{7b}$	18
(11, 5)	(1)	208	$\frac{1}{26257200} x^{31b/4}$	20
(13, 6)	(2)	532	$(1.03 \times 10^9)^{-1} x^{9b}$	23
(15, 7)	(1)	3344	$(1.65 \times 10^{10})^{-1} x^{41b/4}$	26
(17, 8)	(2)	5225	$(3.57 \times 10^{11})^{-1} x^{23b/2}$	29
(20, 8)	(1)	55	$(4.44 \times 10^{16})^{-1} x^{55b/4}$	35

We carry on in this vein, with results as follows (details are available from the authors upon request); in all cases, we use (1) to conclude that $\Lambda_{n_1,n_2} \neq 0$.

(n_1, n_2)	$a/b \geqslant$	(n_1, n_2)	$a/b \geqslant$	(n_1, n_2)	$a/b \geqslant$	(n_1, n_2)	$a/b \geqslant$	(n_1, n_2)	$a/b \geqslant$
(23, 9)	40	(49, 19)	81	(101, 39)	164	(211, 81)	340	(419, 163)	671
(27, 11)	44	(57, 23)	93	(117, 45)	190	(243, 95)	389	(482, 188)	770
(30, 12)	50	(65, 25)	106	(136, 52)	221	(279, 109)	446	(553, 215)	883
(35, 13)	59	(75, 29)	123	(158, 62)	253	(320, 124)	514	(635, 247)	1016
(41, 15)	69	(88, 34)	142	(182, 70)	293	(367, 143)	586	(730, 284)	1167

In conclusion, we have that

$$a \geqslant 1167b \quad \text{if } q = 3 \text{ and } x^b \geqslant 2 \times 10^7. \tag{25}$$

Inequality (23) now implies that

$$|P_{n_1,n_2}(x^b)| < 0.01 \cdot 2^{0.67a/b} x^{2a/3+b}$$

and we also have that

$$|Q_{n_1,n_2}(x^b)| < 0.07 \cdot 2^{0.67a/b} x^{a/3+b}.$$

From (18),

$$|q^{\kappa_1} P_{n_1,n_2}(x^b) - y q^{\kappa_1} Q_{n_1,n_2}(x^b)| < q^{\kappa_1} 2^{0.67a/b} x^{2a/3+b},$$

whereby, from (17),

$$x_0^b < 2^{0.67} 3^{1+3b/2a} x^{2b/3+b^2/a}, \tag{26}$$

i.e., since $x_0 > x^{3/4}$, that

$$x^b < (2^{0.67} 3^{1+3b/2a})^{(1/12-b/a)^{-1}}. \tag{27}$$

Appealing to (25), we conclude that $x^b < 2 \times 10^8$.

§5. *A finite computation.* Collecting our results from the previous two sections, it remains to treat the values x^b with

$$\begin{cases} 2\times 10^7 \leqslant x^b < 2\times 10^8 & \text{if } q=3 \text{ and (26) is satisfied,}\\ x^b < 2\times 10^7 & \text{if } q=3,\\ x^b < 2\times 10^6 & \text{if } q=5 \text{ or } 7,\\ x^b < 10^6 & \text{if } 11\leqslant q\leqslant 317,\\ x\in\{2q^2, 6q^2\} & \text{if } 331\leqslant q\leqslant 27449,\\ x < 373q^{3/2},\ b=1 & \text{if } 331\leqslant q\leqslant 3049,\ q\nmid x, \quad x\equiv 2 \pmod 4,\\ x^b < 276\,q^{3/2} & \text{if } 331\leqslant q\leqslant 1523,\ q\nmid x, \quad x^b\not\equiv 2 \pmod 4. \end{cases}$$

In the first case, we write $x_0 = x^\theta$, where $3/4 < \theta \leqslant 1$ (so that $x_1 = x^{1-\theta}$). Inequality (26) thus becomes

$$x^b < (2^{0.67}3^{1+3b/2a})^{(\theta-2/3-b/a)^{-1}}. \tag{28}$$

Since $x^b \geqslant 2\times 10^7$ and $a \geqslant 1167b$, it follows that $3/4 < \theta < 0.761$ (so that $x^{0.239} < x_1 < x^{0.25}$). Since the smallest positive integer x which can be factored as $x = x_0\cdot x_1$ with x_1 odd, coprime to x_0 and satisfying $x^{0.239} < x_1 < x^{0.25}$, is $x = 84$, it follows from $x^b < 2\times 10^8$ that $b \leqslant 4$. More precisely, we have, from (26), that either $b = 1$, or $x_1 = 3$, $b = 4$, $x \in \{84, 87, 93, 96\}$.

If $b = 1$, we necessarily have $59 \leqslant x_1 \leqslant 113$, for x_1 odd, $x_1 \not\equiv \pm 3 \pmod 9$. In total, we find that there are precisely 2467984 pairs (x, b) for which $x = x_0\cdot x_1$ with $x_0 > x_1^3$, $\gcd(x_0, x_1) = 1$, $\nu_3(x^b) \neq 1$, x_1 odd, $2\times 10^7 \leqslant x^b < 2\times 10^8$, satisfying (26) with $q = 3$.

For the other cases remaining, we begin by observing that there are 20004842 pairs (x, b) with $x \geqslant 5$ and $x^b < 2\times 10^7$, 2001586 pairs (x, b) with $x \geqslant 5$ and $x^b < 2\times 10^6$, and 1001132 pairs (x, b) with $x \geqslant 5$ and $x^b < 10^6$. There are rather more triples (x, b, q) corresponding to, for instance, the cases with $x^b < 373q^{3/2}$ and $331 \leqslant q \leqslant 3049$, but, all told, we are left with fewer than 10^{10} triples to treat. More careful analysis of our various inequalities reduces this number by roughly half.

To handle the remaining triples (x, b, q), we begin by noting that this is indeed a finite computation since we may consider equation (1) as a special case of the family of Thue equations

$$y^q - x^\delta z^q = x^b + 1, \quad \delta \in \{1, 2, \ldots, q-1\}. \tag{29}$$

We will, in fact, typically solve the remaining equations of the shape (1) by much more elementary methods; our computations, whilst somewhat laborious, took only a few weeks on the first author's laptop.

5.1. *A local sieve.* For each remaining triple (x, b, q), we begin by searching for an integer N such that the congruence

$$x^a + x^b + 1 \equiv y^q \pmod N$$

has no solution in integers a and y. This is rather simpler than the sieve employed in [**3**], where an analogous problem is treated with only the value x

predetermined, though we have many more cases to consider. As in [3], we consider primes $p_i \equiv 1 \pmod q$ for which $\mathrm{ord}_x(p_i) = mq$ with m a "suitably small" integer. By $\mathrm{ord}_l(p_i)$, we mean the smallest positive integer k for which $l^k \equiv 1 \pmod{p_i}$. Fixing $M \in \mathbb{N}$, for each such p_i with $m|M$, we let the exponent a range over integers from 1 to Mq and store the a for which either $x^a + x^b + 1 \equiv 0 \pmod{p_i}$ or

$$(x^a + x^b + 1)^{(p_i-1)/q} \equiv 1 \pmod{p_i}.$$

Denoting by S_i the set of values of a corresponding to a prime p_i, then our goal is to find primes $p_1, p_2, \ldots, p_k$ with $\mathrm{ord}_x(p_i)/q$ dividing M and

$$\bigcap_{i=1}^{k} S_i = \emptyset. \tag{30}$$

Checking that we have such sets of primes (with M reasonably small) for most triples (x, b, q) is a reasonably straightforward, if time-consuming computation. Full details are available from the authors upon request. Indeed, we are able to achieve this, with certain exceptions. These exceptions correspond to two particular families of values of x and b, namely

$$x = t^q - 2, b = 1 \quad \text{where } t \geqslant 2,$$

and

$$x = (t^q - 1)/2, b = 1 \quad \text{where } t \geqslant 2 \text{ is odd},$$

and to the triples

$$(x, b, q) = (18, 1, 3), (18, 2, 3) \quad \text{and} \quad (11, 2, 5).$$

The above local argument fails for the two families here because $x^\delta + x + 1 = t^q$ with $\delta = 0$ or $\delta = 1$, respectively, whilst the other three triples arise from the identities

$$18^2 + 18 + 1 = 7^3 \quad \text{and} \quad 11^2 + 11^2 + 1 = 3^5.$$

We can still rule out a number of these cases locally, however, by considering the corresponding equations

$$(t^q - 2)^a + t^q - 1 = y^q \tag{31}$$

and

$$((t^q - 1)/2)^a + (t^q - 1)/2 + 1 = y^q, \tag{32}$$

modulo q^2, appealing to the fact that $a \geqslant 2$. Indeed, if we suppose that $t \equiv 2 \pmod q$, then equation (31) implies that

$$y^q \equiv (2^q - 2)^a + 2^q - 1 \equiv 2^q - 1 \pmod{q^2}.$$

It follows that $y^q \equiv 1 \pmod q$ and hence $y^q \equiv 1 \pmod{q^2}$, so that

$$2^{q-1} \equiv 1 \pmod{q^2},$$

a contradiction for the primes q under consideration except $q = 1093$ and $q = 3511$ (where our upper bounds upon x^b are exceeded). Similarly, if $t \equiv 3 \pmod{q}$, then (31) implies

$$y^q \equiv (3^q - 2)^a + 3^q - 1 \pmod{q^2}.$$

Since the right-hand side of this congruence is $3 \pmod{q}$, writing $3^q = Aq + 3$, we have

$$Aq + 3 = 3^q \equiv (Aq + 1)^a + Aq + 2 \equiv 3 + (a + 1)Aq \pmod{q^2}$$

and hence $aA \equiv 0 \pmod{q}$, so that either

$$3^{q-1} \equiv 1 \pmod{q^2},$$

a contradiction for $3 \leqslant q < 10^6$, $q \neq 11$, or $q|a$. In the latter case, (31) implies, writing $a = qa_0$,

$$\begin{aligned} t^q - 1 = y^q - (t^q - 2)^{qa_0} &\geqslant ((t^q - 2)^{a_0} + 1)^q - (t^q - 2)^{qa_0} \\ &\geqslant (t^q - 1)^q - (t^q - 2)^q > (2q - 1)t^{q(q-1)}, \end{aligned}$$

contradicting $q \geqslant 3$.

All told, after employing local arguments and appealing to our upper bounds upon x^b, we are left to solve equation (1) for

$$\begin{aligned} (x, b, q) = {} & (797161, 1, 13), (88573, 1, 11), (177145, 1, 11), (1093, 1, 7), \\ & (16382, 1, 7), (39062, 1, 7), (78123, 1, 7), (279934, 1, 7), \\ & (411771, 1, 7), (823541, 1, 7), (11, 2, 5), (121, 1, 5), \\ & (1022, 1, 5), (1562, 1, 5), (3123, 1, 5), (7774, 1, 5), (8403, 1, 5), \\ & (29524, 1, 5), (59047, 1, 5), (80525, 1, 5), (99998, 1, 5), \\ & (161049, 1, 5), (185646, 1, 5), (379687, 1, 5), (537822, 1, 5), \\ & (709928, 1, 5), (759373, 1, 5), (1048574, 1, 5), (1238049, 1, 5), \\ & (18, 2, 3), (18, 1, 3), (27k^3 + 27k^2 + 9k - 1, 1, 3) \quad \text{and} \\ & (4k^3 + 6k^2 + 3k, 1, 3). \end{aligned}$$

In the last two cases, k is an integer with, respectively, $1 \leqslant k \leqslant 90$ and $1 \leqslant k \leqslant 170$. In the first case with $(x, b, q) = (797161, 1, 13)$, we rework the arguments of §3, using the fact that 797161 is prime (so that we may assume $x_0 = 797161$, $x_1 = 1$), to obtain a contradiction. For the remaining triples, we use the computational package Pari to solve the corresponding Thue equations (29) (where now the degrees of the corresponding number fields are small enough to make this computation feasible). In each case only trivial solutions occur. This completes the proof of Theorem 3.

§6. *The case $q = 2$: Theorems 4 and 5.* Let us begin by proving Theorem 5. Suppose that we have $\gcd(x, y + 1) \leqslant x^{1/6}$, say, so that $\gcd(x, y - 1) \geqslant x^{5/6}$. The case with $\gcd(x, y - 1) \leqslant x^{1/6}$ proceeds in a similar fashion, with y replaced by $-y$. Arguing as in §3, we find, if $n_1 + n_2 + 1 \geqslant a/b$, setting

$$\Lambda_{n_1,n_2} := |4^{n_1} C_{n_1,n_2}^{-1} P_{n_1,n_2}(x^b) - y 4^{n_1} C_{n_1,n_2}^{-1} Q_{n_1,n_2}(x^b)|, \tag{33}$$

that either $\Lambda_{n_1,n_2} = 0$ or $\Lambda_{n_1,n_2} \geqslant x^{5a/6}$, where

$$|P_{n_1,n_2}(x^b)| < \left(\frac{n_1+n_2}{2}+1\right)2^{n_1+n_2-1}\left(1+\frac{x^b}{2}\right)^{n_1}$$

and

$$|Q_{n_1,n_2}(x^b)| < 2^{n_1+n_2-1}\left(1+\frac{x^b}{2}\right)^{n_2}.$$

We choose

$$n_1 = \left\lceil \frac{3a}{4b} \right\rceil \quad \text{and} \quad n_2 = \left\lceil \frac{a}{4b} \right\rceil - \delta \tag{34}$$

for $\delta \in \{0, 1\}$, so that $(n_1 + n_2 + 1)b \geqslant a$. Equation (15) again implies that for at least one of $\delta \in \{0, 1\}$, we must have $P_{n_1,n_2}(x^b) \neq yQ_{n_1,n_2}(x^b)$ and so $\Lambda_{n_1,n_2} \neq 0$. We have

$$\Lambda_{n_1,n_2} \leqslant 4^{3a/4b+1}2^{a/b+1}\left((a/2b+2)\left(1+\frac{x^b}{2}\right)^{3a/4b+1} + y\left(1+\frac{x^b}{2}\right)^{a/4b+1}\right).$$

We can solve $x^a + x^b + 1 = y^2$ locally for $x = 5$ and 6. Let us therefore assume that $x \geqslant 7$ and, say, $x^b \geqslant 10^6$. After a little work, if we assume that $a > 156b$ (whereby $y < 1.01x^{a/2}$), we find that

$$x^{5a/6} < 4.8^{a/b}x^{3a/4+b},$$

i.e.

$$x^b < 4.8^{13} < 7.2 \times 10^8. \tag{35}$$

To see that the supposition $a > 156b$ is without loss of generality, let us note first that the assumption that $\gcd(x, y-1) := x_0 > x^{5/6}$ implies that $y > x^{5b/6}$ and so

$$x^a + x^b + 1 > x^{5b/3},$$

whereby $a > 3b/2$. If $a < 2b$, noting that $\Lambda_{1,0} = |2 + x^b - 2y|$ is non-zero (since x is odd), we find that

$$x^{5a/6} < |2 + x^b - 2y| < \max\{x^b, 2y\}.$$

It follows that $y > \frac{1}{2}x^{5a/6}$, contradicting $y < 1.01\, x^{a/2}$. We may thus suppose that $a \geqslant 2b$ and so

$$x_0^{2b} < |2 + x^b - 2y| < \max\{x^b, 2y\},$$

whereby $y > \frac{1}{2}x_0^{2b} > \frac{1}{2}x^{5b/3}$ and so, after a little work, $a \geqslant 3b$.

We proceed in a similar fashion. From $\Lambda_{2,0}$, we find that

$$\min\{x^{5b/2}, x^{5a/6}\} \leqslant \min\{x_0^{3b}, x^{5a/6}\} \leqslant |8 + 4x^b - x^{2b} - 8y| < x^{2b} + 8y$$

and so $y > \frac{1}{8}(x^{5b/2} - x^{2b})$, whereby $a > 4.9b$. From considering $\Lambda_{3,0}$, we find that

$$x^{10b/3} \leqslant x_0^{4b} \leqslant |16 + 8x^b - 2x^{2b} + x^{3b} - 16y| < 16y,$$

whence $a > 6.6b$. Continuing along these lines, with

$$\begin{aligned}(n_1, n_2) = {} & (4, 0), (6, 1), (8, 1), (10, 2), (12, 3), (15, 3), (18, 5), \\ & (21, 5), (26, 6), (30, 9), (35, 10), \\ & (40, 13), (46, 14), (53, 16), (60, 19), (68, 21), \\ & (77, 25), (86, 27), (97, 31), (113, 32),\end{aligned}$$

we eventually conclude that $a > 156b$, as desired.

It remains to handle the pairs (x, b) satisfying (35). As before, our local sieve serves, after lengthy computations to eliminate all pairs except for those corresponding to equations of the shape (31) with $q = 2$ and t odd. Considering the latter equations modulo 4, we find that necessarily a is even, say $a = 2a_0$, whereby

$$\begin{aligned} t^2 - 1 = y^2 - (t^2 - 2)^{2a_0} &\geqslant ((t^2 - 2)^{a_0} + 1)^2 - (t^2 - 2)^{2a_0} \\ &\geqslant (t^2 - 1)^2 - (t^2 - 2)^2 > 2t^2, \end{aligned}$$

a contradiction.

To prove Theorem 4, we suppose that $x = x_0 \cdot x_1$ with $x_0 = 2^{\nu_2(x)}$, $x_1 \equiv 1 \pmod 2$ and $x_1 < x_0^{1/4}$. Then, arguing as before (16), we may choose $\delta_1 \in \{0, 1\}$ such that

$$\begin{aligned} &|C_{n_1,n_2}^{-1} P_{n_1,n_2}(x^b) + (-1)^{\delta_1} y C_{n_1,n_2}^{-1} Q_{n_1,n_2}(x^b)|_2 \\ &\quad \leqslant 2^{-\min\{\nu_2(x)a-1, \nu_2(x)b(n_1+n_2+1)\}}. \end{aligned} \tag{36}$$

For n_1 and n_2 as in (34), and some corresponding choice of $\delta \in \{0, 1\}$, it follows that

$$|C_{n_1,n_2}^{-1} P_{n_1,n_2}(x^b) + (-1)^{\delta_1} y C_{n_1,n_2}^{-1} Q_{n_1,n_2}(x^b)| \geqslant \tfrac{1}{2} x_0^a, \tag{37}$$

and so

$$x^{4a/5} \leqslant x_0^a \leqslant 2^{a/b+2} \left((a/2b + 2) \left(1 + \frac{x^b}{2} \right)^{3a/4b+1} + y \left(1 + \frac{x^b}{2} \right)^{a/4b+1} \right).$$

Assuming that $a \geqslant 40b$, we have, after some work, that

$$x^b < 1.7^{40} < 1.7 \times 10^9.$$

Since we suppose $x_0 > x^{4/5}$, it follows, if $2^{30} < x^b < 1.7 \times 10^9$, then $2^{25} | x^b$, whence $x^b = 2^{25}k$ for $33 \leqslant k \leqslant 50$. Similarly, if $451452826 \leqslant x^b \leqslant 2^{30}$, then $2^{24} | x^b$, so that $x^b = 2^{24}k$ for $27 \leqslant k \leqslant 64$.

§7. *Concluding remarks.* The Diophantine equation we have studied in this paper,

$$x^a + x^b + 1 = y^q, \quad a > b > 0,$$

likely has only the solutions

$$\begin{aligned}(x, a, b, y^q) = {} & (2, 5, 4, 7^2), (2, 9, 4, 23^2), (3, 7, 2, 13^3), \\ & (18, 2, 1, 7^3), (72, 3, 1, 611^2)\end{aligned}$$

or $(2, 2t, t+1, (2^t+1)^2)$, $t \geqslant 2$, in positive integers x, y and $q \geqslant 2$. We are, however, unaware of techniques that would enable one to prove this, without additional assumptions. As a rough indication of the level of difficulty involved, one might observe that for this equation, only with $b = 0$, it is still unknown whether the number of solutions in integers x, y, a, q with $a, q \geqslant 2$ is finite.

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