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COUNTING ZEROS OF DIRICHLET L-FUNCTIONS

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Abstract. We give explicit upper and lower bounds for $N(T,\chi)$, the number of zeros of a Dirichlet L-function with character χ and height at most T. Suppose that χ has conductor $q>1$, and that $T\ge 5/7$. If $\ell=\log\frac{q(T+2)}{2\pi}>1.567$, then

$$\left|N(T,\chi)-\left(\frac{T}{\pi}\log\frac{qT}{2\pi e}-\frac{\chi(-1)}{4}\right)\right|\le 0.22737\ell+2\log(1+\ell)-0.5.$$

We give slightly stronger results for small q and T. Along the way, we prove a new bound on $|L(s,\chi)|$ for $\sigma<-1/2$.

1. Statement of results

For any Dirichlet character χ, the Dirichlet L-function is defined by

$$L(s,\chi)\coloneqq\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \tag{1.1}$$

when $\Re s>1$, and by analytic continuation for other complex numbers s. We adopt the usual convention of letting $\rho=\beta+i\gamma$ denote a zero of $L(s,\chi)$, so that $\beta=\Re\rho$ and $\gamma=\Im\rho$ by definition. We let

$$\mathcal{Z}(\chi)\coloneqq\{\rho\in\mathbb{C}\colon 0<\beta<1,\ L(\rho,\chi)=0\} \tag{1.2}$$

be the set of zeros of $L(s,\chi)$ inside the critical strip (technically a multiset, since multiple zeros, if any, are included according to their multiplicity). Notice in particular that the set $\mathcal{Z}(\chi)$ does not include any zeros on the imaginary axis, even when χ is an imprimitive character; consequently, if χ is induced by another character χ^*, then $\mathcal{Z}(\chi)=\mathcal{Z}(\chi^*)$. If $\bar\chi$ is the conjugate character to χ, then $\mathcal{Z}(\chi)=\overline{\mathcal{Z}(\bar\chi)}$.

We write $N(T,\chi)$ for the standard counting function for zeros of $L(s,\chi)$ with $0<\beta<1$ and $|\gamma|\le T$. In other words,

$$N(T,\chi)\coloneqq\#\{\rho\in\mathcal{Z}(\chi)\colon|\gamma|\le T\},$$

counted with multiplicity if there are any multiple zeros. The primary aim of this work is to provide explicit upper and lower bounds on $N(T,\chi)$, when χ is a primitive character, in terms of $\chi(-1)$, the conductor q and the height T.

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Theorem 1.1. *Let χ be a primitive character with conductor $q > 1$ and let $T \geq 5/7$. Set $\ell := \log \frac{q(T+2)}{2\pi}$. If $\ell \leq 1.567$, then $N(T,\chi) = 0$. If $\ell > 1.567$, then*

$$\left| N(T,\chi) - \left(\frac{T}{\pi} \log \frac{qT}{2\pi e} - \frac{\chi(-1)}{4} \right) \right| \leq 0.22737\ell + 2\log(1+\ell) - 0.5.$$

There have been two earlier papers dedicated to finding explicit bounds for the quantity $N(T,\chi)$, by McCurley [6] in 1984 and by Trudgian [13] in 2015. Both authors gave bounds of the shape

$$\left| N(T,\chi) - \frac{T}{\pi} \log \frac{qT}{2\pi e} \right| \leq C_1 \log qT + C_2 \tag{1.3}$$

for positive constants C_1 and C_2. In McCurley [6], which assumes $T \geq 1$, these constants $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ are functions of a parameter $\eta \in (0, 1/2]$; for all such values of η, one finds that necessarily $C_1(\eta) > 1/\pi \log 2 > 0.45$. Trudgian pushed McCurley's techniques further, giving [13, Theorem 1] a table of ten pairs of values (C_1, C_2) under the assumption $T \geq 1$ and ten further pairs under the assumption $T \geq 10$. All of his pairs have $C_1 \geq 0.247$, and in his proof it is asserted that C_1 could be made as small as $(\pi \log 4)^{-1} \doteq 0.229612$.

Regrettably, Trudgian's paper contains an error that renders his proof incomplete. In short, the various parameters introduced in his proofs need to satisfy certain inequalities, and he incorrectly argued that one of the inequalities was redundant. The same difficulty unfortunately recurs in [12] (where bounds are derived for zeros of the Riemann zeta-function) and in [13, Theorem 2] (devoted to analogous results for Dedekind zeta-functions). On a certain level, the main purpose of the paper at hand is to repair these problems for Dirichlet L-functions, motivated by the fact that the authors appealed to [13, Theorem 1] in the course of proving the main results of [2].

Our bound in Theorem 1.1 has a slightly more complicated shape (and uses the offset of $\chi(-1)$) to make the bound as small as possible; however, for any $C_1 > 0.22737$, it is a simple calculus exercise to calculate a constant C_2 such that Theorem 1.1 implies the bound (1.3). We can therefore deduce the following corollary of Theorem 1.1 in a straightforward way:

Corollary 1.2. *Let χ be a character with conductor $q > 1$. If $T \geq 5/7$, then*

$$\left| N(T,\chi) - \frac{T}{\pi} \log \frac{qT}{2\pi e} \right| \leq \min\{0.247 \log qT + 6.894, 0.298 \log qT + 4.358\}.$$

Corollary 1.2 improves upon all twenty of Trudgian's claimed pairs as well as upon McCurley's parametric bound. Figure 1 shows the (C_1, C_2) pairs implied by McCurley, and the twenty pairs claimed by Trudgian, as well as the (C_1, C_2) pairs implied for $T \geq 1$ by Theorem 1.1; the two marked points are the two (C_1, C_2) pairs from Corollary 1.2.

As noted earlier, the current work is focused on fixing the aforementioned error in [13], while at the same time introducing a number of further improvements. Most notably, in Theorem 5.7 we extend a bound of Rademacher [9] on $|L(s,\chi)|$ from $-\frac{1}{2} \leq \sigma \leq \frac{3}{2}$ to all real σ, allowing us to set our parameters more liberally. Also, we make a choice for η in terms of q and T that is nearly optimal, allowing us to deduce a rather simpler bound. Thirdly, we computed all 806,544 zeros of primitive L-functions, corresponding to 80,818 characters, with $\ell \leq 6$ and $1 < q < 935$, to sufficient precision to verify the bounds in Theorem 1.1 in this range, allowing us

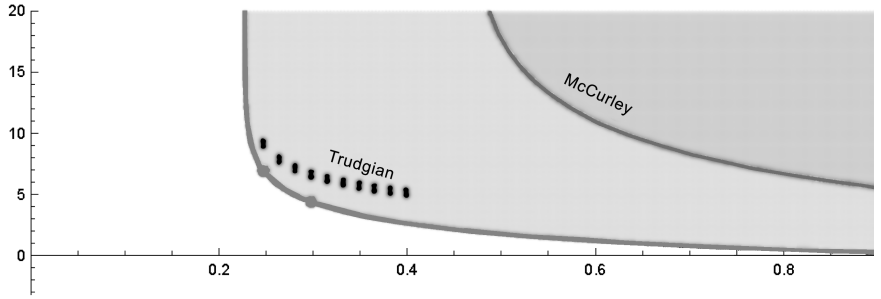

FIGURE 1. The valid (C_1, C_2) pairs proved by McCurley (upper curve), claimed by Trudgian (twenty points), and implied by Theorem 1.1 (lower curve); the two points on the lower curve represent Corollary 1.2.

to assume greater lower bounds on T in our proofs. Finally, we are also interested in lower bounds on $N(T,\chi)$ when T is small, as in Conjecture 1.3 below, and so we state the inequality in Theorem 1.1 in a form that is more useful towards that end.

For any fixed q, we should note that Theorem 1.1 is not particularly of practical interest. The conductor q will either be so large that "explicit" is not helpful, or small enough that one can compute the low height zeros to great precision. For large T, the main term $\frac{T}{\pi}\log\frac{qT}{2\pi e}$ so greatly exceeds the error term (even in McCurley's form), that any improvement is truly minor. Also, the requirement that $T \geq 5/7$ makes our bound unhelpful for those studying zeros of extremely low height.

Using $T \geq 1$ (or $T \geq 10$) would markedly simplify the computational work, and the lower order terms would be improved, but the applicability of the *lower* bound on $N(T,\chi)$ would no longer be optimized. Namely, Theorem 1.1 implies that

$$\lim_{q\to\infty} \max_{\mathrm{cond}(\chi)=q} N(T,\chi) = \infty$$

for $T \geq 5/7 + 10^{-5}$, but not for $T = 5/7$. This is, ultimately, why we chose to work with $5/7$ and not 1. The restriction that $T \geq 5/7$ may appear peculiar, but it is not entirely arbitrary. It would be somewhat challenging to replace $5/7$ with a smaller number, as the functions defined in Definition 3.3 oscillate badly for small T, and that oscillation drives both the size of the lower-order terms in the bound in Theorem 1.1 and the computational effort needed to verify those bounds rigorously. It is also worth noting that we could work with $T \geq T_0 \geq 5/7$, introducing a a term like C/T_0 or $(C_1T+C_2)/(T^2+C_3T+C_4)$ in the bound. However, the added difficulty (an extra variable makes the interval analysis algorithms we use significantly slower) does not seem justified by the slight improvement in the bound that could result.

Where this result is useful is when T is small, but a large range of values of q are to be worked with, and the need for an explicit bound arises not from the large number of zeros that come with large T for one character but from the large number of characters under consideration. For example, in [2], the authors needed to treat all moduli up to $q \leq 10^5$, a total of 1,847,865,075 primitive characters. McCurley's bound implies that there are at most 32,456,205,589 corresponding zeros of height at most 1 and conductor at most 10^5, while Trudgian's claims (one of which we used in [2]) would cut this down to 21,880,443,454. Theorem 1.1 reduces this number to just 16,461,465,486. Some computations for low height zeros and the proof of

Theorem 1.1, tailored specifically for $T = 1$ and shown in Table 1, lower the number still further to just 14,431,705,483.

It is disappointing that, for *fixed* T, the main term and the error term in Theorem 1.1 are of comparable size. We are thus motivated to conjecture, as we are unable to prove, that the error term should be an actual error term, that is, genuinely smaller than the main term. We state this conjecture in a more qualitative form:

Conjecture 1.3. *For every real $T > 0$ and every integer $M \geq 1$, there is an integer q_0 such that every character χ with conductor at least q_0 satisfies $N(T, \chi) \geq M$.*

Assuming the generalized Riemann hypothesis for Dirichlet L-functions, Selberg [10] proved that the error term in the counting function for $N(T, \chi)$ is $O\big(\frac{\log q(T+1)}{\log\log q(T+3)}\big)$ uniformly in q and T; in particular, Conjecture 1.3 follows from GRH. McCurley's bound implies that this conjecture holds for $T > \frac{1}{\log 2} \doteq 1.443$, and Theorem 1.1 implies this conjecture for $T \geq 5/7 + 10^{-5}$. By way of example, we know of characters with conductor 840 for which $N(1, \chi) = 0$; Theorem 1.1 implies that $N(1, \chi) \geq 1$ when $q \geq 1.3 \times 10^{47}$. The largest conductor of a character χ in our dataset with $N(2, \chi) = 0$ is 241; Theorem 1.1 implies that $N(2, \chi) \geq 1$ when $q \geq 1.2 \times 10^{7}$.

Motivated by Selberg's bound, as well as our somewhat substantial computation of zeros, we make a rather speculative conjecture.

Conjecture 1.4. *Let χ be a character with conductor $q > 1$. Recall that $\ell = \log \frac{q(T+2)}{2\pi}$. If $T \geq 5/7$, then*

$$\left| N(T, \chi) - \left(\frac{T}{\pi} \log \frac{qT}{2\pi e} - \frac{\chi(-1)}{4} \right) \right| \leq \frac{\ell}{\log(2 + \ell)}.$$

The outline of this paper is as follows. In Section 2, following the approach of McCurley, we derive our first estimates for $N(T, \chi)$, from which our main results will follow. Section 3 is devoted to sharp inequalities for the Gamma function. In Section 4, we begin the task of bounding the argument of $L(s, \chi)$, by constructing a function whose zeros measure changes in the argument. In Section 5, we complete this process through application of Backlund's trick and Jensen's formula. Finally, in Section 6, we complete the proof of Theorem 1.1.

The technical details of our computations can be found in data files at:

`http://www.nt.math.ubc.ca/BeMaObRe2/`

2. The main term

Assuming that χ is a primitive character with conductor $q > 1$, the completed L-function, an entire function, is defined as

$$\Lambda(s, \chi) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s + a_\chi}{2}\right) L(s, \chi);$$

we note that the zeros of $\Lambda(s, \chi)$ are precisely those of $L(s, \chi)$. The *functional equation* is

$$\Lambda(s, \chi) = \varepsilon(\chi) \Lambda(1 - s, \bar{\chi}), \tag{2.1}$$

where $\varepsilon(\chi)$ is independent of s and has absolute value 1.

Fix $\sigma_1 > 1$. By integrating $\frac{\Lambda'}{\Lambda}(s,\chi)$ around the rectangle with corners at $\sigma_1 \pm iT$ and $1-\sigma_1 \pm iT$ (where T is not the height of a zero of $L(s,\chi)$), and appealing to equation (2.1) on the left half of the contour, we arrive at the identity

$$N(T,\chi) = \frac{T}{\pi}\log\frac{q}{\pi} + \frac{2}{\pi}\Im \ln\Gamma(\tfrac{1}{4} + \tfrac{a_\chi}{2} + i\,\tfrac{T}{2}) + \frac{1}{\pi}\arg L(s,\chi)\Big|_{s=1/2-iT}^{1/2+iT}, \tag{2.2}$$

where

$$a_\chi \coloneqq \begin{cases} 0, & \text{if } \chi(-1)=1, \\ 1, & \text{if } \chi(-1)=-1, \end{cases}$$

is the sign of the character. Define

$$g(a,T) \coloneqq \frac{2}{\pi}\Im \ln\Gamma(\tfrac{1}{4} + \tfrac{a}{2} + i\tfrac{T}{2}) - \frac{T}{\pi}\log\frac{T}{2e} - \frac{2a-1}{4}, \tag{2.3}$$

so that

$$\frac{T}{\pi}\log\frac{q}{\pi} + \frac{2}{\pi}\Im \ln\Gamma(\tfrac{1}{4} + \tfrac{a_\chi}{2} + i\,\tfrac{T}{2}) = \frac{T}{\pi}\log\frac{qT}{2\pi e} - \frac{\chi(-1)}{4} + g(a_\chi,T). \tag{2.4}$$

We have that

$$\begin{aligned} &\arg L(s,\chi)\Big|_{s=1/2-iT}^{1/2+iT} \\ &\quad = \arg L(s,\chi)\Big|_{s=1/2-iT}^{\sigma_1-iT} + \arg L(s,\chi)\Big|_{s=\sigma_1-iT}^{\sigma_1+iT} + \arg L(s,\chi)\Big|_{s=\sigma_1+iT}^{1/2+iT} \\ &\quad = \arg L(\sigma-iT,\chi)\Big|_{\sigma=1/2}^{\sigma_1} + \arg L(\sigma_1+it,\chi)\Big|_{t=-T}^{T} + \arg L(\sigma+iT,\chi)\Big|_{\sigma=\sigma_1}^{1/2}. \end{aligned}$$

In particular,

$$\begin{aligned} \left|\arg L(s,\chi)\Big|_{s=1/2-iT}^{1/2+iT}\right| \le{}& \left|\arg L(\sigma-iT,\chi)\Big|_{\sigma=1/2}^{\sigma_1}\right| + \left|\arg L(\sigma_1+it,\chi)\Big|_{t=-T}^{T}\right| \\ &+ \left|\arg L(\sigma+iT,\chi)\Big|_{\sigma=\sigma_1}^{1/2}\right|. \end{aligned} \tag{2.5}$$

These three terms sometimes all have the same sign in practice, suggesting that there is no possibility of finding cancellation in general. Since

$$\overline{L(\sigma-iT,\chi)} = L(\sigma+iT,\bar\chi),$$

we have

$$\left|\arg L(\sigma-iT,\chi)\Big|_{\sigma=1/2}^{\sigma_1}\right| \le \max_{\tau\in\{\chi,\bar\chi\}}\left|\arg L(\sigma+iT,\tau)\Big|_{\sigma=\sigma_1}^{1/2}\right|.$$

If χ is a real character then we have equality in this statement, so again there is no recoverable loss in general.

Trivial bounds on $|L(s,\tau)|$ come from comparing the Euler products of $L(s,\tau)$ and $\zeta(s)$, leading immediately to the following.

Lemma 2.1. *If $s=\sigma+it$ and $\sigma>1$, then $\frac{\zeta(2\sigma)}{\zeta(\sigma)} \le |L(s,\tau)| \le \zeta(\sigma)$.*

Proposition 2.2. *For $\sigma_1>1$, $\left|\arg L(\sigma_1+it,\tau)\Big|_{t=-T}^{T}\right| \le 2\log\zeta(\sigma_1)$.*

Proof. For t between $-T$ and T, Lemma 2.1 implies that

$$|\arg L(\sigma_1 + it, \tau)| \leq |\log L(\sigma_1 + it, \tau)| \leq \log \zeta(\sigma_1).$$

The proposition thus follows from the fact that $\arg L$ is trapped between $-\log \zeta(\sigma_1)$ and $\log \zeta(\sigma_1)$, whereby its net change is at most $2 \log \zeta(\sigma_1)$. □

We have thus arrived at the inequality

$$\begin{aligned} &\left| N(T, \chi) - \left(\frac{T}{\pi} \log \frac{qT}{2\pi e} - \frac{\chi(-1)}{4} + g(a_\chi, T) \right) \right| \\ &\qquad \leq \frac{2}{\pi} \log \zeta(\sigma_1) + \frac{2}{\pi} \max_{\tau \in \{\chi, \bar{\chi}\}} \left| \arg L(\sigma + iT, \tau) \Big|_{\sigma=\sigma_1}^{1/2} \right|. \end{aligned} \tag{2.6}$$

We will use Stirling's approximation to estimate $g(a_\chi, T)$ in the next section, and the remainder of the work is spent on bounding the change of $\arg L(\sigma + iT, \tau)$ on the segment $\sigma \in [1/2, \sigma_1]$. Up to this point, we have followed McCurley's approach to the problem verbatim.

3. The gamma function

We require bounds for the Gamma function in two contexts. The first of these is in equation (2.2) where the real part of the argument is either $1/4$ or $3/4$, while the second is in the situation where we have a fixed imaginary part $T/2$ and varying real part. Both usages are nicely handled by a suitable shifted version of Stirling's approximation.

Lemma 3.1 (Stirling's approximation)**.** *Let x and y be positive real numbers. Then $\Im \ln \Gamma(x + iy)$ is within*

$$\frac{(4 + 3\pi)/1440}{((x+2)^2 + y^2)^{3/2}}$$

of the expression

$$\begin{aligned} y \log \frac{y}{e} + \frac{\pi}{2} \left(x - \frac{1}{2} \right) - \left(x + \frac{3}{2} \right) \arctan \frac{x+2}{y} - \frac{y/12}{(x+2)^2 + y^2} \\ + \frac{y}{2} \log \left(1 + \frac{(x+2)^2}{y^2} \right) + \arctan \frac{x}{y} + \arctan \frac{x+1}{y}. \end{aligned}$$

Proof. By [5, Proposition 2.1], we have the identity

$$\ln \Gamma(z) = \ln \Gamma(z+1) - \log z = \ln \Gamma(z+2) - \log z - \log(z+1).$$

Thus,

$$\Im \ln \Gamma(z) = \Im \ln \Gamma(z+2) - \operatorname{Arg}(z) - \operatorname{Arg}(z+1). \tag{3.1}$$

We will use the version of Stirling's series and corresponding error bounds given in [3]: for $\Re(z) > 0$, there is a complex function R_2 with $|R_2(z)| \leq \frac{4+3\pi}{1440|z|^3}$ and

$$\ln \Gamma(z) = \left(z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log 2\pi + \frac{1}{12z} + R_2(z).$$

(The upper bound for $|R_2(z)|$ is a factor of $\frac{3\pi}{4} + 1$ larger than the asymptotically true size, but the overapproximation is unimportant for our purposes.) For x and

y positive real numbers, we have $\operatorname{Arg}(x+iy) = \frac{\pi}{2} - \arctan(x/y)$. Equation (3.1) now becomes

$$\begin{aligned}\Im \ln \Gamma(x+iy) = y\log\frac{y}{e} + \frac{\pi}{2}\left(x-\frac{1}{2}\right) - \left(x+\frac{3}{2}\right)\arctan\frac{x+2}{y} \\ - \frac{y/12}{(x+2)^2+y^2} + \frac{y}{2}\log\left(1+\frac{(x+2)^2}{y^2}\right) \\ + \arctan\frac{x}{y} + \arctan\frac{x+1}{y} + \Im R_2(x+2+iy),\end{aligned}$$

and the lemma follows from $|\Im R_2(z+2)| \le |R_2(z+2)|$. □

Proposition 3.2. *For $a \in \{0,1\}$, $T \ge 5/7$ and $g(a,T)$ defined as in (2.3), we have*

$$|g(a,T)| \le \frac{2-a}{50T}.$$

Proof. We need only apply Lemma 3.1 with $x = \frac{a}{2} + \frac{1}{4}$ and $y = T/2$, finding that $g(a,T)$ is within $\frac{(8+6\pi)/45}{(81+40a+4T^2)^{3/2}}$ of

$$\begin{aligned}(3.2)\quad &\frac{16+12\pi-60T\sqrt{40a+4T^2+81}}{45\pi\left(81+40a+4T^2\right)^{3/2}} + \frac{T}{2\pi}\log\left(1+\frac{40a+81}{4T^2}\right) \\ &+ \frac{2}{\pi}\left(\arctan\frac{2a+1}{2T} + \arctan\frac{2a+5}{2T} - \left(\frac{a}{2}+\frac{7}{4}\right)\arctan\frac{2a+9}{2T}\right).\end{aligned}$$

Proving the four inequalities (upper and lower, $a = 0$ and $a = 1$) is a typical problem for interval analysis. □

A number of times in this work we will assert that some inequality is true "by interval analysis". Full details are available in Mathematica notebooks on the website

`http://www.nt.math.ubc.ca/BeMaObRe2/`

but we wish to indicate the idea behind this under-utilized technique here. One extends the domain of some primitive real functions (like addition, multiplication, arctangents, logarithms, etc.) to include intervals, and so that

$$f(X_1,\dots,X_n) = \{f(x_1,\dots,x_n)\colon x_i \in X_i\}.$$

The fundamental theorem of interval analysis says that if h is defined by a composition of primitive functions and $x_i \in X_i$, then

$$h(x_1,\dots,x_n) \in h(X_1,\dots,X_n).$$

For instance, (3.2), multiplied by T and with $a = 0$ and $T = [1, \frac{129}{128}]$ becomes

$$\left[\frac{12\pi+16-\frac{5805\sqrt{38713}}{2048}}{3825\sqrt{85}\pi} + \frac{\log\left(\frac{38713}{1849}\right)}{2\pi} + \frac{2\arctan\frac{25026}{8321} - \frac{7}{2}\arctan\frac{9}{2}}{\pi},\right.$$
$$\left.\frac{129}{128}\left(\frac{262144\left(12\pi+16-60\sqrt{85}\right)}{47036295\sqrt{38713}\pi} + \frac{129\log\frac{85}{4}}{256\pi} + 2 - \frac{2\arctan 12 + \frac{7}{2}\arctan\frac{192}{43}}{\pi}\right)\right],$$

a subset of $[0.022, 0.035]$. Also, $T \cdot \frac{(8+6\pi)/45}{(81+40a+4T^2)^{3/2}}$ becomes

$$\left[\frac{262144(8+6\pi)}{47036295\sqrt{38713}}, \frac{43(8+6\pi)}{163200\sqrt{85}}\right] \subseteq [0.0007, 0.0008].$$

This computation then constitutes a proof that $0.0213 \le Tg(0,T) \le 0.0358$ for $1 \le T \le \frac{129}{128}$. It should be noted that this proof works without floating point arithmetic, except at moments when one needs to decide which of two expressions represents a smaller number.

One can then proceed to a proof for all $T \ge 5/7$ by breaking the interval $[5/7, \infty)$ into sufficiently small intervals. By the definition of uniform continuity, if the domain is broken into sufficiently small pieces, then interval arithmetic will yield a sufficiently tight bound on the range of the function. There is a theoretical and a practical difficulty with this paradigm for generating proofs of inequalities. The theoretical problem is that we need not only the function to be uniformly continuous, but for every sub-computation involved to be uniformly continuous. This may require cleverly rewriting the expression or by introducing more primitive functions, each such introduction requiring some (usually easy) calculus proof.

The practical difficulty that arises is that "sufficiently small pieces" can quickly become too numerous to be useful. This can be partially addressed by rewriting the expression, but also by introducing a simple expression between the target function and the planned bound. For example,

$$\frac{T}{2\pi}\log\left(1+\frac{81}{4T^2}\right) \le \frac{T}{2\pi}\times\frac{81}{4T^2} = \frac{81}{8\pi T},$$

and for $T = [100, 200]$ this improves the naive interval arithmetic upper bound of $\frac{100}{\pi}\log\left(\frac{40081}{40000}\right) \approx 0.064$ to $\frac{81}{800\pi} \approx 0.032$. That is, a theoretically tighter bound in real arithmetic may be theoretically worse in interval arithmetic. The best expression to use may even depend on the specific interval under consideration.

In the course of our interval analysis bounds in this paper, we use Alirezaei's uncommonly sharp bounds for $\arctan x$ [1] and Topsøe's Padé-inspired bounds for $\log(1+x)$ [11].

Definition 3.3. For $a \in \{0,1\}$, $d \ge 0$ and $T \ge 5/7$, we define

$$\mathcal{E}(a,d,T) := \left|\Im \ln\Gamma\left(\tfrac{\sigma+a+iT}{2}\right)\Big|_{\sigma=1/2}^{1/2+d} + \Im \ln\Gamma\left(\tfrac{\sigma+a+iT}{2}\right)\Big|_{\sigma=1/2}^{1/2-d}\right|.$$

We set $E(a,d,T)$ to be the expression given in Figure 2, so that Lemma 3.1 applied to the definition of $\mathcal{E}(a,d,T)$ gives

$$\mathcal{E}(a,d,T) \le E(a,d,T)$$

for $0 \le d < \frac{9}{2}$ and $T > 0$. While E contains many terms, they are each easy to work with computationally. Figure 3 shows E for typical arguments.

Lemma 3.4. *Suppose that* $0 \le \delta_1 \le d < \frac{9}{2}$, *with* $a \in \{0,1\}$ *and* $T \ge \frac{5}{7}$. *Then*

$$0 < E(a,\delta_1,T) \le E(a,d,T).$$

For $a \in \{0,1\}$, $\frac{1}{4} \le d \le \frac{5}{8}$ *and* $T \ge \frac{5}{7}$,

$$\frac{E(a,d,T)}{\pi} \le \frac{(640+216a)d - 112 - 39a}{1536(3T+3a-1)} + \frac{1}{2^{10}}.$$

Proof. This is proved using interval analysis. For fixed a and T, a degree 3 Taylor model with center 0 is used to show that the derivative of $T\cdot E(a,d,T)$ with respect to d is positive for small d (using some algebra and the Moore–Skelboe algorithm to bound the 4th derivative of E with respect to d), and, simply, the Moore–Skelboe

$$
\begin{aligned}
E(a,d,T) :=& \frac{2T/3}{(2a+2d+17)^2+4T^2} + \frac{2T/3}{(2a-2d+17)^2+4T^2} \\
&- \frac{4T/3}{(2a+17)^2+4T^2} + \frac{T}{2}\log\left(1+\frac{(2a+17)^2}{4T^2}\right) \\
&- \frac{T}{4}\log\left(1+\frac{(2a+2d+17)^2}{4T^2}\right) - \frac{T}{4}\log\left(1+\frac{(2a-2d+17)^2}{4T^2}\right) \\
&+ \frac{(8+6\pi)/45}{((2a+2d+17)^2+4T^2)^{3/2}} \\
&+ \frac{(8+6\pi)/45}{((2a-2d+17)^2+4T^2)^{3/2}} + \frac{2(8+6\pi)/45}{((2a+17)^2+4T^2)^{3/2}} \\
&+ 2\arctan\frac{2a+1}{2T} - \arctan\frac{2a+2d+1}{2T} - \arctan\frac{2a-2d+1}{2T} \\
&+ 2\arctan\frac{2a+5}{2T} - \arctan\frac{2a+2d+5}{2T} - \arctan\frac{2a-2d+5}{2T} \\
&+ 2\arctan\frac{2a+9}{2T} - \arctan\frac{2a+2d+9}{2T} - \arctan\frac{2a-2d+9}{2T} \\
&+ 2\arctan\frac{2a+13}{2T} - \arctan\frac{2a+2d+13}{2T} - \arctan\frac{2a-2d+13}{2T} \\
&+ \frac{2a+2d+15}{4}\arctan\frac{2a+2d+17}{2T} \\
&+ \frac{2a-2d+15}{4}\arctan\frac{2a-2d+17}{2T} - \frac{2a+15}{2}\arctan\frac{2a+17}{2T}
\end{aligned}
$$

FIGURE 2. Definition of $E(a,d,T)$, for $a \in \{0,1\}$, $0 \le d < 9/2$, $T \ge 5/7$.

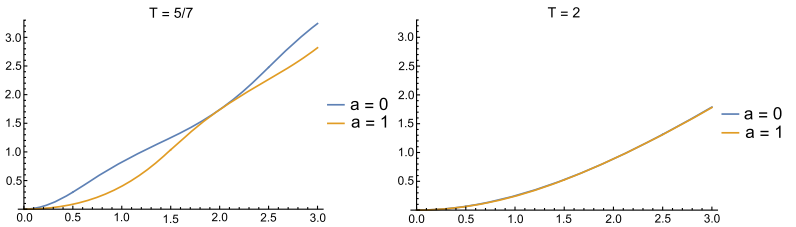

FIGURE 3. Graphs of $\mathcal{E}(a,d,T)$ for $T = 5/7$ and $T = 2$.

algorithm for larger d. As $E(a,0,T) > 0$, this shows that E is positive. Consult the website for details. □

4. Bounds on $L(s,\tau)$, and bounds on $f_m(s)$

4.1. **Introduction of the auxiliary function f_m.** We will construct a function that has many zeros if $\arg L$ changes substantially on the interval on $[\frac{1}{2}+iT, \sigma_1+iT]$. To wit, let m be a large integer, and define, for s a complex number,

$$f_m(s) := \frac{1}{2}\big(L(s+iT,\tau)^m + L(s-iT,\bar{\tau})^m\big).$$

Notice the use of $s \pm iT$, rather than $\sigma \pm iT$; this is done so that f_m is holomorphic. Note further that, for real arguments, f_m simplifies nicely:

$$\begin{aligned} f_m(\sigma) &= \frac{1}{2}\big(L(\sigma + iT, \tau)^m + L(\sigma - iT, \bar{\tau})^m\big) \\ &= \frac{1}{2}\big(L(\sigma + iT, \tau)^m + \overline{L(\sigma + iT, \tau)^m}\big) \\ &= \Re L(\sigma + iT, \tau)^m. \end{aligned}$$

The tactic we will employ follows McCurley [6]. If $\arg L$ changes, then $\arg L^m$ changes m times more, and this causes $L(\sigma + iT, \tau)^m$ to be purely imaginary many times, whereby f_m will have many real zeros. We use Jensen's formula to bound the number of zeros in terms of an integral of $\log |f_m|$, and then bound the integral using a variety of estimates, trivial and non-trivial. As $m \to \infty$, this tactic captures the total variation of $\arg L$, which is sometimes as small as the net change.

Definition 4.1. We define n_m to be that integer (depending on m) for which

$$n_m \le \frac{1}{\pi}\left|\arg L(\sigma + iT, \tau)^m\Big|_{\sigma=\sigma_1}^{1/2}\right| < n_m + 1. \tag{4.1}$$

Since we will take $m \to \infty$, the reader will be well-served to think of n_m as being very large. We will find upper and lower bounds for n_m/m.

Lemma 4.2. *The function f_m has as at least n_m zeros on the real segment $[\frac{1}{2}, \sigma_1]$, and*

$$\frac{n_m}{m} \le \frac{1}{\pi}\left|\arg L(\sigma + iT, \tau)\Big|_{\sigma=\sigma_1}^{1/2}\right| < \frac{n_m + 1}{m}.$$

Proof. By Definition 4.1, we know that the expression $\frac{1}{2} + \frac{1}{\pi}\arg L(\sigma + iT, \tau)^m$ is an integer for at least n_m different values of σ in the interval $[1/2, \sigma_1]$. In other words,

$$f_m(\sigma) = \Re L(\sigma + iT, \tau)^m = 0$$

for at least n_m different values of σ.

As m is an integer, we have that $\arg L(s, \tau)^m = m \arg L(s, \tau)$. Thus, dividing the inequalities in line (4.1) by m leads to the desired conclusion. □

If we had defined f_m with subtraction instead of addition, thereby picking out the imaginary part instead of the real part of $L(s, \tau)$, the analogue of the proof of Lemma 4.3 would not be valid.

Lemma 4.3. *For any real $c > 1$, there is an infinite sequence of integers m with $f_m(c) \neq 0$, and moreover, along that sequence*

$$\lim_m \left(-\frac{1}{m}\log |f_m(c)|\right) \le \log \frac{\zeta(c)}{\zeta(2c)}.$$

Proof. Define K and ψ by $L(c + iT, \tau) = Ke^{\psi i}$. Since $L(s, \tau) \neq 0$ for $\sigma > 1$ and $c > 1$, we know that $K > 0$, and also $L(c - iT, \bar{\tau}) = \overline{L(c + iT, \tau)} = Ke^{-\psi i}$. We have

$$\frac{f_m(c)}{L(c + iT, \tau)^m} = \frac{1}{2}\left(1 + \frac{L(c - iT, \bar{\tau})^m}{L(c + iT, \tau)^m}\right) = \frac{1}{2}\left(1 + e^{-2m\psi i}\right).$$

Whatever the value of ψ, there is a sequence of values of m with the property that $-2m\psi \to 0 \pmod{2\pi}$, whence $\frac{f_m(c)}{L(c+iT,\tau)^m} \to 1$.

For $\sigma > 1$, we have the Euler product bound from Lemma 2.1

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq |L(\sigma + iT, \tau)|.$$

This translates, for m a sequence of integers with the above property, into the bound

$$1 = \lim_m \frac{f_m(c)}{L(c+iT,\tau)^m} \leq \lim_m \left| \frac{f_m(c)}{\left(\zeta(2c)/\zeta(c)\right)^m} \right|,$$

which becomes

$$0 \leq \lim_m \log|f_m(c)| - m \log \frac{\zeta(2c)}{\zeta(c)},$$

completing the proof. □

5. Jensen's formula

We apply Jensen's formula to the sequence of functions $f_m(s)$ and the open disk $D(c,r)$ with center c and radius r. Here, m ranges through the sequence of positive integers defined in Lemma 4.3. As τ and $\bar{\tau}$ are nonprincipal, each f_m is entire, and in particular holomorphic on $D(c,r)$. Let $\mathcal{Z}_m(X)$ be the multiset of zeros of f_m in the set $X \subseteq \mathbb{C}$. Let

$$S_m(c,r) := \frac{1}{m} \sum_{z \in \mathcal{Z}_m(D(c,r))} \log \frac{r}{|z-c|}. \tag{5.1}$$

In our setting and notation, Jensen's formula is as follows.

Theorem 5.1 (Jensen's formula)**.** *Let $c \in \mathbb{C}$, and let $r > 0$ be real. If $f_m(c) \neq 0$, then*

$$S_m(c,r) = -\frac{1}{m}\log|f_m(c)| + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{m}\log|f_m(c+re^{i\theta})|\,d\theta.$$

We apply this to derive an upper bound upon $S_m(c,r)$.

Proposition 5.2. *Let c, r and σ_1 be real numbers with*

$$c - r < \frac{1}{2} < 1 < c < \sigma_1 < c + r,$$

and $F_{c,r} : [-\pi, \pi] \to \mathbb{R}$ an even function with $F_{c,r}(\theta) \geq \frac{1}{m}\log|f_m(c+re^{i\theta})|$. Then

$$\lim_{m\to\infty} S_m(c,r) \leq \log\frac{\zeta(c)}{\zeta(2c)} + \frac{1}{\pi}\int_0^{\pi} F_{c,r}(\theta)\,d\theta.$$

We will give a lower bound on the sum that involves $\frac{n_m}{m}$ in Section 5.1, and an upper bound on the integral, via an explicit $F_{c,r}$, using classical and new bounds on L-functions in Section 5.2. What will then remain is the work of choosing good values for c, r and σ_1, which we do in Section 6.

5.1. Backlund's trick and the Jensen sum.

Lemma 5.3. *Let d and T be positive real numbers, and $\mathcal{E}(a,d,T)$ be as in Definition 3.3. Then*

$$\left|\arg L(\sigma+iT,\tau)^m\Big|_{\sigma=1/2}^{1/2+d}\right| \leq \left|\arg L(\sigma+iT,\tau)^m\Big|_{\sigma=1/2}^{1/2-d}\right| + m\mathcal{E}(a_\tau,d,T).$$

Proof. By the functional equation (2.1),

$$\arg \Lambda(\sigma + iT, \tau)\Big|_{\sigma=1/2}^{1/2+d} + \arg \Lambda(\sigma + iT, \tau)\Big|_{\sigma=1/2}^{1/2-d} = 0.$$

Since

$$\begin{aligned}\arg \Lambda(\sigma + iT, \tau) &= \arg \left(\frac{q}{\pi}\right)^{(s+a_\tau)/2} + \arg \Gamma(\tfrac{s+a_\tau}{2}) + \arg L(s, \tau)\\ &= \frac{T}{2} \log \frac{q}{\pi} + \Im \ln \Gamma(\tfrac{s+a_\tau}{2}) + \arg L(s, \tau),\end{aligned}$$

we see that the terms

$$\arg L(\sigma + iT, \tau)\Big|_{\sigma=1/2}^{1/2+d} + \arg L(\sigma + iT, \tau)\Big|_{\sigma=1/2}^{1/2-d}$$

and

$$\Im \ln \Gamma(\tfrac{\sigma+a_\tau+iT}{2})\Big|_{\sigma=1/2}^{1/2+d} + \Im \ln \Gamma(\tfrac{\sigma+a_\tau+iT}{2})\Big|_{\sigma=1/2}^{1/2-d}$$

add to 0, and so have the same absolute value. This last displayed equation has the same absolute value as $\mathcal{E}(a_\tau, d, T)$. As $\arg L(\sigma + iT, \tau)^m = m \arg L(\sigma + iT, \tau)$, we have established this lemma. □

We will appeal to the following proposition with rather weak constraints on c and r; if r is much larger than c, then we can in fact do slightly better. The source of the error in [13] is in not tracking the constraints on c and r and how they impact the applicability of "Backlund's trick".

Proposition 5.4 (Backlund's trick)**.** *Let c and r be real numbers , and set*

$$\sigma_1 := c + \frac{(c - 1/2)^2}{r} \quad \text{and} \quad \delta := 2c - \sigma_1 - \frac{1}{2}.$$

Further, let $E_\delta := E(a_\tau, \delta, T)$. If $1 < c < r$ and $0 \le \delta < \frac{9}{2}$, then

$$\left|\arg L(\sigma + iT, \tau)\Big|_{\sigma=\sigma_1}^{1/2}\right| \le \frac{\pi\, S_m(c, r)}{2 \log r/(c - 1/2)} + \frac{E_\delta}{2} + \frac{\pi}{m}.$$

Proof. The conditions on c and r imply the inequalities

$$c - r < \frac{1}{2} - \delta \le \frac{1}{2} \le \frac{1}{2} + \delta = 2c - \sigma_1 \le c \le \sigma_1 < c + r.$$

For $z \in D(c, r)$, we see that $\log \frac{r}{|z-c|} > 0$, so that

$$S_m(c, r) := \frac{1}{m} \sum_{z \in \mathcal{Z}_m(D(c,r))} \log \frac{r}{|z - c|} \ge \frac{1}{m} \sum_{z \in \mathcal{Z}_m((c-r,\sigma_1])} \log \frac{r}{|z - c|}.$$

We will further only consider particular zeros in the real interval $(c - r, \sigma_1]$, noting that omitting zeros from the computation weakens rather than invalidates the claimed bound.

For real σ, such as those in the interval $(c - r, c + r)$, we have that

$$f_m(\sigma) = \Re L(\sigma + iT, \tau)^m,$$

and so for $\sigma \in \mathcal{Z}_m((c - r, c + r))$, we have

$$0 = f_m(\sigma) = \Re L(\sigma + iT, \tau)^m,$$

whence $\arg f_m(\sigma) = \frac{\pi}{2} + j\pi$ for some integer j. By the definition of n_m, we are then guaranteed at least n_m values of σ in the interval $[1/2, \sigma_1]$ with $f_m(\sigma) = 0$.

For $1 \le k \le n_m$, let δ_k be the smallest nonnegative real number with

$$f_m(1/2+\delta_k) = 0 \qquad \text{and} \qquad k-1 \le \frac{1}{\pi}\left|\arg L(\sigma+iT,\tau)^m\Big|_{\sigma=1/2}^{1/2+\delta_k}\right|.$$

We set $z_k := \frac{1}{2}+\delta_k$. Define x_1 to be the number of z_k's that lie in the interval $[1/2, 1/2+\delta) = [1/2, 2c-\sigma_1)$, and let $x_2 = n_m - x_1$ be the number of z_k's in $[2c-\sigma_1, \sigma_1]$. We have

$$0 \le \delta_1 < \delta_2 < \cdots < \delta_{x_1} < \delta \le \delta_{x_1+1} < \cdots < \delta_{n_m} \le \sigma_1 - 1/2.$$

Using Lemma 5.3,

$$\begin{aligned} k-1 &\le \frac{1}{\pi}\left|\arg L(\sigma+iT,\tau)^m\Big|_{\sigma=1/2}^{1/2+\delta_k}\right| \\ &< \frac{1}{\pi}\left|\arg L(\sigma+iT,\tau)^m\Big|_{\sigma=1/2}^{1/2-\delta_k}\right| + \frac{m}{\pi}\,\mathcal{E}(a_\tau,\delta_k,T) \\ &\le \frac{1}{\pi}\left|\arg L(\sigma+iT,\tau)^m\Big|_{\sigma=1/2}^{1/2-\delta_k}\right| + \frac{m}{\pi}\,E(a_\tau,\delta_k,T). \end{aligned}$$

For each $j \ge 1$, if k is minimal with

$$\frac{1}{\pi}\left|\arg L(\sigma+iT,\tau)^m\Big|_{\sigma=1/2}^{1/2-\delta_k}\right| > k-1-\frac{m}{\pi}\,E(a_\tau,\delta_k,T) \ge j,$$

then f_m has at least j zeros in $[\frac{1}{2}-\delta_k, \frac{1}{2})$. We define δ_{-k} so that $\frac{1}{2}-\delta_{-k}$ is the smallest (furthest from $1/2$) of the "at least j" zeros. We say that the zero $z_k = \frac{1}{2}+\delta_k$ has a pair, namely $z_{-k} = \frac{1}{2}-\delta_{-k}$. By construction, $\delta_{-k} \le \delta_k$.

If $z_k \in [\frac{1}{2}, \frac{1}{2}+\delta]$ is unpaired, then it contributes (using $\frac{1}{2}+\delta \le c$)

$$\frac{1}{m}\log\frac{r}{|c-z_k|} = \frac{1}{m}\log\frac{r}{c-(\frac{1}{2}+\delta_k)} \ge \frac{1}{m}\log\frac{r}{c-1/2}$$

to $S_m(c,r)$. If $z_k \in [\frac{1}{2}, \frac{1}{2}+\delta]$ is paired, then it (together with its paired zero, which is at least $\frac{1}{2}-\delta$ and so in $(c-r, c+r)$) contributes

$$\begin{aligned} \frac{1}{m}\log\frac{r}{|c-z_k|} + \frac{1}{m}\log\frac{r}{|c-z_{-k}|} &= \frac{1}{m}\log\frac{r^2}{|c-(1/2+\delta_k)|\cdot|c-(1/2-\delta_{-k})|} \\ &\ge \frac{1}{m}\log\frac{r^2}{|(c-1/2)^2-\delta_k^2|} \ge \frac{1}{m}\log\frac{r^2}{(c-1/2)^2} \end{aligned}$$

to $S_m(c,r)$. If $z_k \in [\frac{1}{2}+\delta, \sigma_1]$, then it contributes

$$\begin{aligned} \frac{1}{m}\log\frac{r}{|c-z_k|} &\ge \min\left\{\frac{1}{m}\log\frac{r}{c-(\frac{1}{2}+\delta)}, \frac{1}{m}\log\frac{r}{\sigma_1-c}\right\} \\ &= \frac{1}{m}\log\frac{r}{\max\{c-\frac{1}{2}-\delta, \sigma_1-c\}} \\ &= \frac{1}{m}\log\frac{r}{\sigma_1-c} = \frac{1}{m}\log\frac{r^2}{(c-1/2)^2} \end{aligned}$$

to $S_m(c,r)$, revealing the wisdom in setting $\delta = 2c-\sigma_1-\frac{1}{2}$ and $\sigma_1 = c+\frac{(c-1/2)^2}{r}$.

Suppose there are x zeros in $[\frac{1}{2}, \frac{1}{2}+\delta]$, and x' of them are unpaired, and there are $n_m - x$ zeros in $(\frac{1}{2}+\delta, \sigma_1]$. We then have

$$\begin{aligned} S_m(c,r) &\geq \frac{x'}{m}\log\left(\frac{r}{c-1/2}\right) + \frac{x-x'}{m}\log\left(\frac{r^2}{(c-1/2)^2}\right) \\ &\quad + \frac{n_m - x}{m}\log\left(\frac{r^2}{(c-1/2)^2}\right) \\ &= \frac{x' + 2(x-x') + 2(n_m - x)}{m}\log\left(\frac{r}{c-1/2}\right) \\ &= \frac{2n_m - x'}{m}\log\left(\frac{r}{c-1/2}\right). \end{aligned}$$

If all of the zeros were unpaired, then $2n_m - x' = n_m$, and this argument would reduce to McCurley's. Fortunately, by construction $x' \leq mE_\delta/\pi$, and so

$$S_m(c,r) \geq \frac{2n_m - mE_\delta/\pi}{m}\log\left(\frac{r}{c-1/2}\right),$$

whence

$$\frac{n_m}{m} \leq \frac{S_m(c,r)}{2\log\left(r/(c-1/2)\right)} + \frac{E_\delta}{2\pi}.$$

Lemma 4.2 completes this proof. □

No effort was made to use the pairs of zeros in $(\frac{1}{2}+\delta, \sigma_1]$. This is because the pairs of such zeros may lie outside $(c-r, c+r)$ and so may not contribute to S. With a stronger assumption about r, we can guarantee that the pair should get counted and obtain a slightly stronger but more involved bound. In practice, the paired zero is very close to the edge of $D(c,r)$, and so the improvement is very slight except for tiny q and T, which we may handle by direct computation anyway.

Proposition 5.5 (Backlund's trick, inelegant version)**.** *Let c and r be real numbers, and set $\sigma_1 := \frac{1}{2} + \sqrt{2}(c - \frac{1}{2})$ and $\delta := 2c - \sigma_1 - \frac{1}{2}$. Further, let*

$$S_m(c,r) := \frac{1}{m}\sum_{z \in \mathcal{Z}_m(D(c,r))} \log\frac{r}{|z-c|},$$

$E_\delta := E(a_\tau, \delta, T)$, and $E_{\sigma_1} := E(a_\tau, \sigma_1 - \frac{1}{2}, T)$. If $r > (1+\sqrt{2})(c-\frac{1}{2})$, $c > 1$ and $\frac{1}{4} \leq \delta < \sigma_1 < \frac{9}{2}$, then

$$\left|\arg L(\sigma + iT, \tau)\Big|_{\sigma=\sigma_1}^{1/2}\right| \leq \frac{\pi\, S_m(c,r)}{2\log r/(c-1/2)} + \frac{E_\delta}{2} + \frac{\pi}{m} + \frac{E_{\sigma_1} - E_\delta}{2}\left(1 - \frac{\log(1+\sqrt{2})}{\log r/(c-1/2)}\right).$$

Proof. The proof is essentially identical to that of the preceding proposition. We arrive at the inequality

$$\begin{aligned} S_m(c,r) &\geq \frac{x_1'}{m}\log\left(\frac{r}{c-1/2}\right) + \frac{x_1 - x_1'}{m}\log\left(\frac{r^2}{(c-1/2)^2}\right) + \frac{x_2'}{m}\log\left(\frac{r}{\sigma_1 - c}\right) \\ &= \frac{2n_m}{m}\log\left(\frac{r}{c-1/2}\right) - \frac{x_1'}{m}\log\left(\frac{r}{c-1/2}\right), \end{aligned}$$

from which this proposition follows, again upon invoking Lemma 4.2. □

5.2. **The Jensen integral.** To use Proposition 5.2, we require an explicit function $F_{c,r}(\theta)$ that will bound the quantity $\frac{1}{m}\log|f_m(c+re^{i\theta})|$ and that is even as a function of θ.

We begin by quoting some useful bounds on $L(s,\tau)$. The first bound (5.2) is straightforward from the Euler products for $L(s,\tau)$ and $\zeta(s)$. The second bound (5.3) is Rademacher's convexity bound [9]. The third bound (5.4) follows from the second with $\eta=-\sigma$, although it is actually a primary ingredient in Rademacher's proof of equation (5.3).

Lemma 5.6. *Let τ be a primitive character with modulus $q>1$. Fix a parameter $\eta\in(0,\frac{1}{2}]$ and let $s=\sigma+it$. If $\sigma\geq 1+\eta$, then*

$$|L(s,\tau)|\leq\zeta(\sigma). \tag{5.2}$$

If $-\eta\leq\sigma\leq 1+\eta$, then

$$|L(s,\tau)|\leq\zeta(1+\eta)\left(\frac{q}{2\pi}\cdot|s+1|\right)^{(1+\eta-\sigma)/2}. \tag{5.3}$$

If $-\frac{1}{2}\leq\sigma\leq-\eta$, then

$$|L(s,\tau)|\leq\zeta(1-\sigma)\left(\frac{q}{2\pi}\cdot|s+1|\right)^{\frac{1}{2}-\sigma}. \tag{5.4}$$

We can leverage Rademacher's argument to also provide bounds to the left of $\sigma=-1/2$. For a real number x, let $[x]$ be the integer closest to x, choosing the one closer to 0 if there are two integers equally close to x. We note that $[-x]=-[x]$.

For $-\frac{1}{2}\leq\sigma<0$, the following result reduces to equation (5.4).

Theorem 5.7. *Let τ be a primitive character with modulus $q>1$. Suppose $s=\sigma+it$, with $\sigma<0$. Then*

$$|L(s,\tau)|\leq\zeta(1-\sigma)\left(\frac{q}{2\pi}\right)^{\frac{1}{2}-\sigma}\cdot\left|s-[\sigma]+1\right|^{\frac{1}{2}-\sigma+[\sigma]}\cdot\prod_{j=1}^{-[\sigma]}|s+j-1|. \tag{5.5}$$

Proof. Let $a=a_\tau$ be the sign and q the modulus (and conductor) of τ. From the functional equation (2.1), it follows that

$$|L(s,\tau)|=\left(\frac{q}{\pi}\right)^{-\sigma+1/2}\cdot|L(1-s,\bar{\tau})|\cdot\left|\frac{\Gamma(\frac{a}{2}+\frac{1-s}{2})}{\Gamma(\frac{a}{2}+\frac{s}{2})}\right|.$$

As $\sigma<0$, we may apply Lemma 2.1 to conclude that

$$|L(1-s,\bar{\tau})|\leq\zeta(1-\sigma).$$

We are therefore left with a ratio of Gamma functions to bound. To do this, we appeal to Euler's reflection formula

$$\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)}$$

and Legendre's duplication formula

$$\Gamma(z)\Gamma(z+\tfrac{1}{2})=2^{1-2z}\sqrt{\pi}\cdot\Gamma(2z).$$

It follows, as $a \in \{0,1\}$, that

$$\begin{aligned}\frac{\Gamma(\frac{a}{2}+\frac{1-s}{2})}{\Gamma(\frac{a}{2}+\frac{s}{2})} &= \frac{\Gamma(1-\frac{s+1-a}{2})}{\Gamma(\frac{a}{2}+\frac{s}{2})}\\ &= \frac{\pi}{\Gamma(\frac{a}{2}+\frac{s}{2})\Gamma(\frac{s+1-a}{2})\sin(\frac{\pi}{2}(s+1-a))}\\ &= \frac{\sqrt{\pi}}{2^{1-s}\Gamma(s)\sin(\frac{\pi}{2}(s+1-a))}.\end{aligned}$$

Replacing s by $s+k$ and a by $b \in \{0,1\}$, we obtain

$$\frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})} = \frac{\sqrt{\pi}}{2^{1-(s+k)}\Gamma(s+k)\sin(\frac{\pi}{2}(s+k+1-b))}.$$

Comparing the last two equations reveals that

$$\begin{aligned}\frac{\Gamma(\frac{a}{2}+\frac{1-s}{2})}{\Gamma(\frac{a}{2}+\frac{s}{2})} &= \frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})}\frac{2^{1-(s+k)}}{2^{1-s}}\frac{\Gamma(s+k)}{\Gamma(s)}\frac{\sin(\frac{\pi}{2}(s+k+1-b))}{\sin(\frac{\pi}{2}(s+1-a))}\\ &= \frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})}2^{-k}\Big(\prod_{j=1}^{k}(s+j-1)\Big)\frac{\sin(\frac{\pi}{2}(s+k+1-b))}{\sin(\frac{\pi}{2}(s+1-a))}.\end{aligned}$$

If we now choose $b \equiv a+k \pmod 2$, then the last factor becomes simply ± 1 and hence the desired inequality follows upon choosing $k = -[\sigma]$, bounding $\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})/\Gamma(\frac{b}{2}+\frac{s+k}{2})$ trivially and taking absolute values. □

Lemma 5.8. *Fix a parameter $\eta \in (0, \frac{1}{2}]$ and $T > 0$. Write $s = \sigma + it$. If $\sigma \geq 1+\eta$, then*

$$\frac{1}{m}\log|f_m(s)| \leq \log\zeta(\sigma). \tag{5.6}$$

If $-\eta \leq \sigma \leq 1+\eta$, then

$$\frac{1}{m}\log|f_m(s)| \leq \log\zeta(1+\eta)+\frac{1+\eta-\sigma}{2}\log\frac{q}{2\pi}+\frac{1+\eta-\sigma}{4}\log\big((\sigma+1)^2+(|t|+T)^2\big). \tag{5.7}$$

If $\sigma \leq -\eta$, then

$$\begin{aligned}\frac{1}{m}\log|f_m(s)| \leq{}& \log\zeta(1-\sigma)\\ &+\frac{1-2\sigma}{2}\log\frac{q}{2\pi}+\frac{1-2\sigma+2[\sigma]}{4}\log\big((1+\sigma-[\sigma])^2+(|t|+T)^2\big)\\ &+\frac{1}{2}\sum_{k=1}^{-[\sigma]}\log\big((\sigma+k-1)^2+(|t|+T)^2\big).\end{aligned} \tag{5.8}$$

Proof. For $\sigma \geq 1+\eta > 1$, the real parts of $s+iT$ and $s-iT$ are both at least $1+\eta$, and so Lemma 2.1 gives

$$\begin{aligned}|f_m(s)| &:= \left|\frac{1}{2}\left(L(s+iT,\tau)^m + L(s-iT,\bar{\tau})^m\right)\right| \\ &\leq \frac{1}{2}\left(|L(s+iT,\tau)|^m + |L(s-iT,\bar{\tau})|^m\right) \\ &\leq \frac{1}{2}\left(\zeta(\sigma)^m + \zeta(\sigma)^m\right) = \zeta(\sigma)^m.\end{aligned}$$

Taking real logarithms yields the inequality (5.6).

For the second claim, with $-\eta \leq \sigma \leq 1+\eta$, we have

$$\begin{aligned}|f_m(s)| &:= \left|\frac{1}{2}\left(L(s+iT,\tau)^m + L(s-iT,\bar{\tau})^m\right)\right| \\ &\leq \frac{1}{2}\left(|L(s+iT,\tau)|^m + |L(s-iT,\bar{\tau})|^m\right) \\ &\leq \frac{1}{2}\left(\left(\zeta(1+\eta)\left(\frac{q}{2\pi}|s+iT+1|\right)^{(1+\eta-\sigma)/2}\right)^m \right. \\ &\qquad \left. + \left(\zeta(1+\eta)\left(\frac{q}{2\pi}|s-iT+1|\right)^{(1+\eta-\sigma)/2}\right)^m\right).\end{aligned}$$

Writing $s = \sigma + it$, with $T > 0$, we have[1]

$$|s \pm iT + 1|^2 = (\sigma+1)^2 + (t \pm T)^2 \leq (\sigma+1)^2 + (|t|+T)^2.$$

Thus,

$$|f_m(s)| \leq \left(\zeta(1+\eta)\left(\frac{q}{2\pi}\sqrt{(\sigma+1)^2 + (|t|+T)^2}\right)^{(1+\eta-\sigma)/2}\right)^m,$$

and routine manipulation of logarithms yields the inequality (5.7).

Finally, for $-\frac{1}{2} \leq \sigma < -\eta$, we have $-[\sigma] = 0$, and so line (5.8) follows from (5.7) upon setting $\eta = -\sigma$ and noting that the summation over k is empty. Assume now that $\sigma < -\frac{1}{2}$, so that $[\sigma] \leq -1$. As the exponent $[\sigma] - \sigma + 1/2$ is nonnegative, we see that the bound from Theorem 5.7 is monotone increasing for $\Im(s) > 0$ (with $\Re(s)$ fixed). We apply Theorem 5.7 to s with imaginary parts $t+T$ and $t-T$, and $|t \pm T| \leq |t| + T$. Thus,

$$\begin{aligned}|f_m(s)| &= \left|\frac{1}{2}\left(L(s+iT,\tau)^m + L(s-iT,\bar{\tau})^m\right)\right| \\ &\leq \zeta(1-\sigma)^m \left(\frac{q}{2\pi}\right)^{m(\frac{1}{2}-\sigma)} \left((\sigma-[\sigma]+1)^2 + (|t|+T)^2\right)^{m\cdot\frac{1}{2}\cdot(\frac{1}{2}-\sigma)} \\ &\qquad \cdot \left(\prod_{k=1}^{-[\sigma]}\left((\sigma+k-1)^2 + (|t|+T)^2\right)\right)^{1/2}.\end{aligned}$$

Taking real logarithms completes the proof. □

[1] As we are ultimately concerned with $m \to \infty$, we could care only about whichever one of $|s \pm iT + 1|$ is larger, saving a factor of 2. But also, as $m \to \infty$ and we ultimately care about $\frac{1}{m}\log|f_m(s)|$, a factor of 2 is irrelevant.

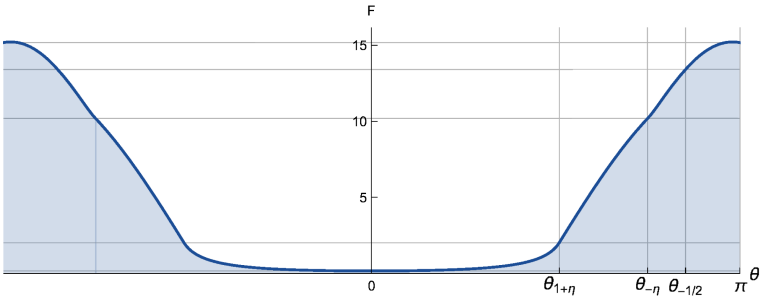

FIGURE 4. The function $F_{1.2,1.9}(\theta)$ for $T = 1$, $q = 10^6$, and $\eta = 0.141$. With these parameters, $\int_0^\pi F_{1.2,1.9}(\theta)\,d\theta \doteq 16.37$.

Definition 5.9. We set

$$L_j(\theta) := \log \frac{(j + c + r\cos\theta)^2 + (|r\sin\theta| + T)^2}{(T+2)^2}.$$

We note that $L_j(\theta)$ depends on c, r and T. If we suppose that $0 \le \theta \le \pi$ and $T \ge 5/7$, then, from the inequality $\log x \le x-1$, we find that $L_j(\theta) \le L_j^\star(\theta)/(T+2)$, where

$$L_j^\star(\theta) := 2r\sin\theta - 4 + \tfrac{7}{19}\left((j + c + r\cos\theta)^2 + (r\sin\theta - 2)^2\right).$$

Definition 5.10. Let $\sigma = c + r\cos\theta$ and $t = r\sin\theta$, where $-\pi \le \theta \le \pi$. Define

$$F_{c,r}(\theta) := \begin{cases} \log\zeta(\sigma), & \text{if } \sigma \ge 1+\eta, \\ \log\zeta(1+\eta) + \frac{1+\eta-\sigma}{2}\ell + \frac{1+\eta-\sigma}{4}L_1(\theta), & \text{if } -\eta \le \sigma \le 1+\eta, \\ \log\zeta(1-\sigma) + \dfrac{1-2\sigma}{2}\ell + \dfrac{1-2\sigma+2[\sigma]}{4}L_{1-[\sigma]}(\theta) & \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{k=1}^{-[\sigma]} L_{k-1}(\theta), & \text{if } \sigma < -\eta. \end{cases}$$

We note that $F_{c,r}(\theta)$ depends on q, T and η implicitly. Usefully, $F_{c,r}$ is a continuous, even function of θ. Figure 4 shows the function $F_{1.2,1.9}(\theta)$ for $T = 1$ and $q = 10^6$ and $\eta = 0.141$.

Definition 5.11. If c and r are real numbers, with r positive, define θ_σ as

$$\theta_\sigma := \begin{cases} 0, & \text{if } c + r \le \sigma, \\ \arccos\frac{\sigma - c}{r}, & \text{if } c - r \le \sigma \le c + r, \\ \pi, & \text{if } \sigma \le c - r. \end{cases}$$

We remark that for $c - r \le \sigma \le c + r$, we have $c + r\cos\theta_\sigma = \sigma$.

Definition 5.12. We also define

$$\kappa_1 := (\theta_{-\eta} - \theta_{1+\eta})\frac{1+\eta-c}{2} - (\pi - \theta_{-\eta})\left(c - \frac{1}{2}\right) + \frac{r\,(\sin\theta_{-\eta} + \sin\theta_{1+\eta})}{2}.$$

For a positive integer J_1 (we will actually take $J_1 = 64$), set

$$\kappa_2(J_1) := \frac{\pi}{4J_1}\left(\log\zeta(c+r) + 2\sum_{j=1}^{J_1-1}\log\zeta(c + r\cos\tfrac{\pi j}{2J_1})\right).$$

For a positive integer J_2 (we will actually take $J_2 = 24$), set

$$\kappa_3(J_2) := \frac{\pi - \theta_{1-c}}{2J_2}\left(\log\zeta(1-c+r) + 2\sum_{j=1}^{J_2-1}\log\zeta\big(1-c-r\cos(\tfrac{\pi j}{J_2} + (1-\tfrac{j}{J_2})\theta_{1-c})\big)\right).$$

Finally, define

$$\begin{aligned}
\kappa_4 &:= \frac{1}{4}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} (1+\eta-\sigma)L_1^\star(\theta)\,d\theta,\\
\kappa_5 &:= \frac{1}{4}\int_{\theta_{-\eta}}^{\theta_{-1/2}} (1-2\sigma)L_1^\star(\theta)\,d\theta, \quad \text{and}\\
\kappa_{6,j} &:= \frac{1}{4}\int_{\theta_{-j+1/2}}^{\theta_{-j-1/2}} \left((1-2\sigma-2j)L_{j+1}^\star + 2\sum_{k=1}^{j} L_{k-1}^\star(\theta)\right) d\theta.
\end{aligned}$$

Note that the integrands involved here are polynomials in $\sin\theta$ and $\cos\theta$, and so we can evaluate the integrals exactly. These evaluations are not enlightening to examine, but they are computationally important. For details, the reader may consult the files *ZerosOfLFunctions-Largeell.nb* and *ZerosOfLFunctions-Middleell.nb* at

`http://www.nt.math.ubc.ca/BeMaObRe2/`

From the fact that $F_{c,r}$ is an even function, the Jensen integral is then bounded as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{m}\log|f_m(c+re^{i\theta})|\,d\theta \le \frac{1}{\pi}\int_0^\pi F_{c,r}(\theta)\,d\theta.$$

We evaluate the main term of this integral (as q or T go to ∞) with the fundamental theorem of calculus, while the minor terms require labourious bounding.

Proposition 5.13. *Let c, r and η be positive real numbers satisfying*

$$1+\eta \le c < r-\eta, \tag{5.9}$$

and suppose that $q \ge 3$ and $T \ge 5/7$. Then $\int_0^\pi F_{c,r}(\theta)\,d\theta$ is at most

$$\begin{aligned}
&\kappa_1\ell + (\theta_{-\eta}-\theta_{1+\eta})\log\zeta(1+\eta) + \int_0^{\theta_{1+\eta}}\log\zeta(c+r\cos\theta)\,d\theta\\
&\qquad + \int_{\theta_{-\eta}}^{\pi}\log\zeta(1-c-r\cos\theta)\,d\theta + \frac{\kappa_4+\kappa_5}{T+2} + \frac{1}{T+2}\sum_{j=1}^{\infty}\kappa_{6,j}.
\end{aligned}$$

The infinite sum is cosmetic: if $j \ge r-c+\frac{1}{2}$, then $\theta_{-j-1/2} = \theta_{-j+1/2} = \pi$ and so $\kappa_{6,j} = 0$.

Proof. By the hypothesized inequalities, we have $\theta_{-\eta} < \theta_{-1/2}$. Rearranging terms,

$$\int_0^{\pi} F_{c,r}(\theta)\,d\theta = \int_0^{\theta_{1+\eta}} \log\zeta(\sigma)\,d\theta + \int_{\theta_{-\eta}}^{\pi} \log\zeta(1-\sigma)\,d\theta$$

$$+ \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \log\zeta(1+\eta)\,d\theta \tag{5.10}$$

$$+ \ell\left(\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2}\,d\theta + \int_{\theta_{-\eta}}^{\pi} \frac{1-2\sigma}{2}\,d\theta\right) \tag{5.11}$$

$$+ \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{4} L_1(\theta)\,d\theta + \int_{\theta_{-\eta}}^{\theta_{-1/2}} \frac{1-2\sigma}{4} L_1(\theta)\,d\theta \tag{5.12}$$

$$+ \sum_{j=1}^{\infty} \int_{\theta_{-j+1/2}}^{\theta_{-j-1/2}} \left(\frac{1-2\sigma-2j}{4} L_{j+1}(\theta) + \frac{1}{2}\sum_{k=1}^{j} L_{k-1}(\theta)\right) d\theta. \tag{5.13}$$

The integrand on line (5.10) is constant; the integral is $(\theta_{-\eta} - \theta_{1+\eta})\log\zeta(1+\eta)$. The sum of the two integrals on line (5.11) is exactly κ_1.

For $\theta_{1+\eta} < \theta < \theta_{-\eta}$, we have $-\eta < \sigma < 1+\eta$ and so $1+\eta-\sigma > 0$. For $\theta_{-\eta} < \theta < \theta_{-1/2}$, we have $-\frac{1}{2} < \sigma < -\eta < 0$, whence $1-2\sigma > 0$. Thus, line (5.12) is bounded (using $T \geq 5/7$) by

$$\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{4} L_1^{\star}(\theta)\,d\theta + \int_{\theta_{-\eta}}^{\theta_{-1/2}} \frac{1-2\sigma}{4} L_1^{\star}(\theta)\,d\theta = \frac{\kappa_4+\kappa_5}{T+2}.$$

Likewise, if $\theta_{-j+1/2} < \theta < \theta_{-j-1/2}$, then $-j-\frac{1}{2} < \sigma < -j+\frac{1}{2}$, and so

$$1-2\sigma-2j > 0$$

and we can appeal to the inequality $L_j(\theta) \leq L_j^{\star}(\theta)/(T+2)$. This bounds line (5.13) by

$$\frac{1}{T+2}\sum_{j=1}^{\infty} \kappa_{6,j},$$

as claimed. □

Our goal in the remainder of this section is to provide upper bounds for the two integrals appearing in the statement of Proposition 5.13. In both cases, these bounds will take the form of a small finite sum of reasonably manageable (that is, easily optimized) functions.

Lemma 5.14. *Let c, r and η be positive real numbers satisfying* (5.9)*, $\sigma = c + r\cos\theta$ and J_1 be a positive integer. If $\theta_{1+\eta} \leq 2.1$, then*

$$\int_0^{\theta_{1+\eta}} \log\zeta(\sigma)\,d\theta \leq \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1+\eta}-\frac{\pi}{2}) + \frac{\pi}{4J_1}\log\zeta(c) + \kappa_2(J_1).$$

Proof. As the map $\theta \mapsto \log\zeta(c+r\cos\theta)$ is increasing for $0 \leq \theta \leq \theta_{1+\eta}$, we could use right endpoints to overestimate the integral $\int_0^{\theta_{1+\eta}} \log\zeta(\sigma)\,d\theta$. We can get the needed accuracy using many fewer terms, however, by showing that the map is convex, whereby the trapezoid rule provides an overestimate.

To see that the map is convex, observe that

$$\begin{aligned}\frac{d^2}{d\theta^2}\log\zeta(c+r\cos\theta) &= \frac{d^2}{d\theta^2}\sum_p -\log(1-p^{-c-r\cos\theta})\\ &= \sum_p \frac{r\log p}{(1-p^{c+r\cos\theta})^2}\left(p^{c+r\cos\theta}(\cos\theta+r\log(p)\sin^2\theta)-\cos\theta\right)\\ &\geq \sum_p \frac{r\log p}{(1-p^{c+r\cos\theta})^2}\left(p^{c+r\cos\theta}(\cos\theta+\log(2)\sin^2\theta)-\cos\theta\right). \qquad (5.14)\end{aligned}$$

Here, the sums are over primes p. Since $0\leq\theta\leq\theta_{1+\eta}\leq 2.1$, Definition 5.11 and (5.9) together imply that

$$1<1+\eta=c+r\cos\theta_{1+\eta}\leq c+r\cos\theta$$

(which in particular justifies the use of the Dirichlet series for $\log\zeta$), while $0\leq\theta\leq 2.1$ yields the inequality $\cos\theta+\log(2)\sin^2\theta>0$. It thus follows that

$$p^{c+r\cos\theta}(\cos\theta+\log(2)\sin^2\theta)>(\cos\theta+\log(2)\sin^2\theta),$$

whence

$$p^{c+r\cos\theta}(\cos\theta+\log(2)\sin^2\theta)-\cos\theta>\log(2)\sin^2\theta,$$

and so each term in (5.14) is positive.

Singling out the part between $\theta_c=\pi/2$ and $\theta_{1+\eta}\geq\theta_c$, we obtain the claimed bound. □

In a nearly identical fashion, we prove the next lemma. The hypotheses on c,r and η guarantee that $c-r<1-c\leq-\eta$, whence $\pi>\theta_{1-c}\geq\theta_{-\eta}$.

Lemma 5.15. *Let J_2 be a positive integer. If $r>2c-1$ and $1+\eta\leq c$, then*

$$\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)\,d\theta\leq\frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta})+\frac{\pi-\theta_{1-c}}{2J_2}\log\zeta(c)+\kappa_3(J_2).$$

6. Assembling the bound

We begin this section by describing how to assemble the results in the proceeding sections to produce an explicit bound for $\ell\geq 27.02$. In Section 6.2, we adjust this argument to treat values for ℓ with $5.98\leq\ell\leq 28$. Finally, in Section 6.3 we outline the rigorous explicit computations of zeros that allows us to handle small ℓ with $\ell\leq 6$.

6.1. Large values of ℓ.

Let us assume that $\ell\geq 27.02$ and set

$$\begin{aligned}\ell &\coloneqq \log\frac{q(T+2)}{2\pi}\subseteq[27.02,\infty), & \eta &\coloneqq \frac{18}{10+9\ell}\subseteq(0,0.08),\\ c &\coloneqq 1+\frac{391}{74\ell+683}\subseteq(1,1.15), & r &\coloneqq \frac{149}{140}+\frac{769}{30\ell+512}\subseteq(1.06,1.65),\\ \sigma_1 &\coloneqq c+\frac{(c-1/2)^2}{r}\subseteq(1.23,1.40), & \delta &\coloneqq 2c-\sigma_1-\frac{1}{2}\subseteq(0.26,0.40).\end{aligned}$$

These definitions guarantee the chain of inequalities

$$-\frac{1}{2}<c-r<1-c<-\eta<0<1<1+\eta<c<\sigma_1<c+r,$$

which is simply (5.9) along with the extra condition that $c - r < 1 - c < -\eta$. We set $E_\delta := E(a_\chi, \delta, T)$. The values for c and r were chosen after extensive numerical work, with $\frac{149}{140} = 1.06429\cdots$ being a good approximation to our numerically determined "ideal" value of r. Numerical work suggests that we should choose $c = 1 + O(\frac{\log \ell}{\ell})$, but the improvement in the final values is slight, while the added complexity in producing a bound would be considerable.

The value of η can be motivated, however, and some words on why we define η in this way are appropriate. To apply Lemma 5.8, we require $\eta \le 1/2$. In an ideal world, we could choose η optimally for each σ, so as to make the right side of (5.7) as small as possible. Experiments indicate that the numerical advantage in doing so is slight, albeit noticeable, and not justifying the added complexity. The derivative with respect to η of (5.7) at $\sigma = 1/2$ is

$$\frac{1}{2}\log\frac{q}{2\pi} + \frac{1}{4}\log\left((\sigma+1)^2 + (t+T)^2\right) + \frac{\zeta'(1+\eta)}{\zeta(1+\eta)}.$$

For η between 0 and $1/2$, we know that $\frac{\zeta'}{\zeta}(1+\eta) + \frac{1}{\eta}$ is nearly linear, decreasing from $\gamma \doteq 0.577216$ to just below $1/2$; we choose $5/9$ as a convenient rational in the desired range. We handle small ℓ (for which η is near $1/2$) by direct computation of zeros, so we find it reasonable to replace $\frac{\zeta'}{\zeta}(1+\eta)$ with $\frac{5}{9} - \frac{1}{\eta}$. The value of t will cover a range, but $t = 2$ seems roughly typical. The critical value of η is then estimated as the solution to

$$\frac{1}{2}\log\frac{q}{2\pi} + \frac{1}{4}\log\left((2+T)^2\right) + \frac{5}{9} - \frac{1}{\eta} = 0,$$

which is $\eta = \frac{18}{10+9\ell}$. Setting η in this way allows us to give the single bound in Theorem 1.1 instead of a table of bounds for various settings of η (as in [13]) or a bound that depends continuously on η (as in [6]).

With these choices of parameters, we now return to inequality (2.6). Appealing to Propositions 5.2 and 5.4, and letting $m \to \infty$, we find that

$$\left|N(T,\chi) - \left(\frac{T}{\pi}\log\frac{qT}{2\pi e} - \frac{\chi(-1)}{4}\right)\right| \le |g(a,T)| + \frac{2}{\pi}\log\zeta(\sigma_1) + \frac{E_\delta}{\pi} + \frac{\log\zeta(c) - \log\zeta(2c)}{\log r/(c-1/2)} + \frac{1/\pi}{\log r/(c-1/2)}\int_0^\pi F_{c,r}(\theta)\,d\theta. \tag{6.1}$$

Lemma 3.2 bounds $g(a,T)$ and Lemma 3.4 bounds E_δ. Combining those bounds, whose sum is monotone in d and rational in T, we can prove that

$$|g(a,T)| + \frac{E_\delta}{\pi} \le \frac{1}{14(T-1/5)} + \frac{1}{2^{10}}.$$

Using interval analysis,

$$\frac{2}{\pi}\log\zeta(\sigma_1) - \frac{\log\zeta(2c)}{\log r/(c-1/2)} \le \frac{178\ell^2 + 17909\ell + 80807}{4\,(128\ell^2 + 9637\ell + 164296)}.$$

We now consider the integral term in equation (6.1). We apply Proposition 5.13 to break the integral $\int F_{c,r}$ into pieces. With our settings for c, r, η and bound on ℓ, the hypotheses are satisfied and $\theta_{j-1/2} = \pi$ for all $j \ge 0$, whence the "infinite" sum is 0. We use Lemmata 5.14 and 5.15 to bound the pieces. The main term is

bounded as

$$\frac{\kappa_1\ell/\pi}{\log r/(c-1/2)} \le \frac{238413}{2^{20}}\ell + \frac{798\ell^2+135589\ell+80396}{16\,(32\ell^2+3105\ell+38735)}.$$

Collecting the various $\log\zeta(c)$ terms, we have a total of

$$\left(\frac{\theta_{1-c}-\theta_{-\eta}+\theta_{1+\eta}}{\pi} - \frac{\theta_{1-c}}{\pi J_2} + \frac{1}{2J_1} + \frac{1}{J_2} + \frac{3}{2}\right)\frac{\log\zeta(c)}{2\log\left(\frac{r}{c-1/2}\right)},$$

where θ_σ is defined in Definition 5.11. With $J_1 = 64, J_2 = 24$, these terms contribute at most

$$\frac{-1135\ell^2-214796\ell+149201}{512\ell^2+75117\ell+496726} + \frac{\ell}{2^{20}} + \frac{1365\log(\ell+1)}{2^{10}}.$$

Collecting the various $\log\zeta(1+\eta)$ terms, we have a total of

$$\left(\frac{\theta_{1-c}+\theta_{-\eta}-\theta_{1+\eta}}{\pi} - \frac{1}{2}\right)\frac{\log\zeta(1+\eta)}{2\log\frac{r}{c-1/2}}$$
$$\le \frac{-182\ell^2-118430\ell+79045}{512\ell^2+91562\ell+599789} + \frac{\ell}{2^{22}} + \frac{529\log(\ell+1)}{2^{10}}.$$

For the absolutely bounded terms, we obtain the inequalities

$$\frac{\kappa_2/\pi}{\log r/(c-1/2)} \le \frac{635}{1024} - \frac{9(113745\ell+25384532)}{64\,(512\ell^2+150141\ell+7149852)},$$
$$\frac{\kappa_3/\pi}{\log r/(c-1/2)} \le \frac{491}{1024} - \frac{3346893\ell+33179656}{512\,(512\ell^2+21113\ell+208616)}.$$

Since the $O(1/T)$ terms contribute

$$\frac{1/\pi}{\log\left(\frac{r}{c-1/2}\right)}\frac{\kappa_4}{T+2} \le \frac{1}{T+2}\left(\frac{-50\ell^2-1411\ell+18281}{512\ell^2+63962\ell+800695}\right)$$
$$\frac{1/\pi}{\log\left(\frac{r}{c-1/2}\right)}\frac{\kappa_5}{T+2} \le \frac{1}{T+2}\left(\frac{-42\ell^2-15293\ell-961048}{512\ell^2+113665\ell+3255348}\right),$$

we are led to conclude that

$$|g(a,T)| + \frac{E_\delta}{\pi} + \frac{1/\pi}{\log\frac{r}{c-1/2}}\frac{\kappa_4}{T+2} + \frac{1/\pi}{\log\frac{r}{c-1/2}}\frac{\kappa_5}{T+2} \le \frac{75}{2^{10}}.$$

The remaining terms involve only ℓ, and we find (rigorously, as with all the inequalities in this article) that they are at most

$$0.22737\ell + 2\log(1+\ell) - 0.5.$$

This establishes Theorem 1.1 for $\ell \ge 27.02$.

6.2. **Middle values of ℓ.** For $5.98 \le \ell \le 28$, we set

$$c := 1 + \frac{505}{111\ell+430} \quad\text{and}\quad r := \frac{149}{140} + \frac{747}{36\ell+283},$$

and find that

$$-\frac{3}{2} \le c-r \le -\frac{1}{2},\ c \ge 1+\eta,\ \theta_{1+\eta} \le 1.62 \quad\text{and}\quad r \ge 2c-1.$$

A similar fully rigorous analysis yields

$$\left|N(T,\chi)-\left(\frac{T}{\pi}\log\frac{qT}{2\pi e}-\frac{\chi(-1)}{4}\right)\right|\le 0.22737\ell+2\log(1+\ell)-0.5.$$

6.3. **Small values of T and ℓ.** We first attempted to use Rubinstein's `LCALC`, and then gp/PARI, to compute all zeros of all L-functions up to conductor 10000 and $\ell:=\log\frac{q(T+2)}{2\pi}\le 8$. However, both programs were found to miss pairs of zeros occasionally. Using `Arb` for interval arithmetic, for each primitive character (we actually only concern ourselves with one from each conjugate pair) with conductor $1<q<935$ and $\ell\le 6$, we rigorously bounded the expression in equation (2.2). To do so we used the identity

$$\arg L\left(\frac{1}{2}+iT,\chi\right)=\arg L(3+iT,\chi)+\Im\int_{3+iT}^{1/2+iT}\frac{L'(s,\chi)}{L(s,\chi)}\,ds$$

to rigorously bound $\arg L(\frac{1}{2}+iT,\chi)$; the term $\arg L(3+iT,\chi)$ is the principal value of the argument (it is easy to show from the Euler product that $|\arg L(3+iT,\chi)|\le\sum_p \arcsin p^{-3}<0.176$). We thereby computed $N(T,\chi)$ for some T greater than or equal to $2\pi e^6/q-2$ (for some characters, the integrand is highly oscillatory and it can be advantageous to let T be slightly larger). Then, again using `Arb` for the rigorous computation, we found the L-function zeros by locating sign changes in the appropriate Hardy Z-function (see e.g. Edwards [4] for details on the latter object). From this approach, we rigorously located (and stored) every zero of every nontrivial primitive Dirichlet L-function with conductor at most 934 and $\ell\le 6$, accurate to within 10^{-12}. By only considering one from each pair of complex characters and only the positive imaginary parts for real characters, we examined 80818 characters and found a total of 403272 zeros.

With this dataset, we have proved the following lemma. The code to generate the dataset (in C), to process the dataset into Mathematica format, and Mathematica code to pull the following lemma out of the data, are all available on the website. Additional commentary on each item is provided below.

Lemma 6.1. *Let $1<q<935$, suppose that χ is a primitive character with conductor q , and set $\ell:=\log\frac{q(T+2)}{2\pi}$. Then*

(a) *All of the zeros of $L(s,\chi)$ with real part between 0 and 1 and imaginary part between $-2(e^6\pi-q)/q$ and $2(e^6\pi-q)/q$ have real part equal to $1/2$.*
(b) *If $T\ge 0$ and $\ell\le 1.567$, then $N(T,\chi)=0$.*
(c) *If $T\ge 0$ and $1.567\le\ell\le 6$, then*

$$\left|N(T,\chi)-\left(\frac{T}{\pi}\log\frac{qT}{2\pi e}-\frac{\chi(-1)}{4}\right)\right|\le\frac{\ell}{\log(2+\ell)}.$$

(d) *Let $a:=(1-\chi(-1))/2$ and $T\in\{\frac{5}{7},1,2\}$. Let $0\le k\le 4$, or $(T,k)=(2,5)$, or $(a,T,k)=(1,2,6)$. Let $q_a(T,k)$ be the integer corresponding to a,T and k in Table 1. Then:*

$$\text{If } q\le q_a(T,k) \text{ and } a_\chi=a, \text{ then } N(T,\chi)\le k.$$

Moreover, if $q_a(T,k)$ is one of the boldface entries of Table 1, then there is a character τ with conductor $q_a(T,k)+1$, $a_\chi=a_\tau$, and $N(T,\tau)>k$.

Proof. Lemma 6.1(a) is a partial verification of the generalized Riemann hypothesis. Although Platt [8] has made similar computations to much greater height with many more conductors, we independently confirm GRH to this level and make our rigorous zeros openly available at

`http://www.nt.math.ubc.ca/BeMaObRe2/`

The conditions in Lemma 6.1(b) imply that $q \le 15$. The 40 primitive characters with $q \le 15$ are covered in our dataset, and for each, the lowest-height zero is excluded by $\log \frac{q(T+2)}{2\pi} \le 1.567$.

Lemma 6.1(c) requires many cases. For each of the 80818 relevant characters, the zeros are known to within 10^{-12}. Between each pair of consecutive zeros u, v (or before the first zero), we know the value of $N(T,\chi)$. This gives a range of T over which the inequalities can be proved by our Moore–Skelboe-style interval arithmetic algorithm.

Lemma 6.1(d), concerning the boldface and asterisked entries in Table 1, is also straightforward to pull from our dataset. □

The other entries in Table 1 can be verified as follows. For a given a, T, k, we find the values of c and r from Table 2. We then use equation (2.2), evaluating the first two terms to many digits. The last term of equation (2.2) is bounded using Proposition 2.2 and Proposition 5.5. To use Proposition 5.5, we need to confirm that the restrictive inequalities hypothesized there are satisfied. Both E_δ and E_{σ_1} can be explicitly computed, leaving only S. In Proposition 5.2, the quantity S is bounded in terms of the Jensen integral. The integrand in the Jensen integral is bounded in Lemma 5.8, and the bound is restated in Definition 5.10. The resulting integral is then rigorously bounded above using interval arithmetic, subdividing the region of integration until trivial bounds give the needed precision. Finally, as $N(T,\chi)$ must be an integer, we take a floor.

Example 6.2. For example, to verify the statement

$$\text{if the conductor of } \chi \text{ is at most } 25252 \text{ and } a_\chi = 0, \text{ then } N(1,\chi) \le 7,$$

we take $T = 1, q = 25252, a = 0$ and $c = \frac{2694}{2048} \approx 1.315$, $r = \frac{4651}{2048} \approx 2.271$, with values of c and r being pulled from Table 2. Looking ahead to Proposition 5.5, we set

$$\sigma_1 = \frac{1}{2} + \frac{835}{512\sqrt{2}} \approx 1.653$$

and

$$\delta = \frac{835}{512} - \frac{835}{512\sqrt{2}} \approx 0.478.$$

We find that

$$\frac{T}{\pi}\log\frac{q}{\pi} + \frac{2}{\pi}\Im \ln \Gamma(\tfrac{1}{4} + \tfrac{a}{2} + i\,\tfrac{T}{2}) \le 2.1013434, \quad \frac{1}{\pi}\cdot 2\log\zeta(\sigma_1) \le 0.4883702$$

and

$$E_\delta \le 0.1616976, \quad E_{\sigma_1} - E_\delta \le 0.5119502, \quad \log\left(\frac{r}{c-1/2}\right) = \log\left(\frac{4651}{1670}\right).$$

TABLE 1. If χ has sign a and conductor $q \leq q_a(T,k)$, then $N(T,\chi) \leq k$. For example, if χ has sign $a = 1$ and conductor less than 9007, then $N(5/7,\chi) \leq 6$. The numbers in boldface are best possible.

	$T = 5/7$		$T = 1$		$T = 2$	
k	$a = 0$	$a = 1$	$a = 0$	$a = 1$	$a = 0$	$a = 1$
0	**42**	**16**	**36**	**12**	**16**	**10**
1	**172**	**66**	**148**	**42**	**28**	**18**
2	934*	934*	844*	**408**	**120**	**64**
3	934*	934*	844*	844*	**330**	**210**
4	934*	934*	844*	844*	634*	**630**
5	3289	1909	1616	905	634*	634*
6	15991	9007	6256	3425	660	634*
7	82233	45137	25252	13554	1669	1050
8	443412	238003	105597	55727	4289	2677
9	2489523	1310445	455195	236710	11185	6932

At this point, as $r > (1+\sqrt{2})(c-\frac{1}{2})$, we can appeal to Proposition 5.5 to find that

$$N(T,\chi) \leq 2.1013434 + 0.4883702 + \frac{2}{\pi}\left(\frac{\pi S}{2\log\frac{4651}{1670}} + \frac{0.1616976}{2} + \frac{0.5119502}{2}\left(1 - \frac{\log(1+\sqrt{2})}{\log\frac{4651}{1670}}\right)\right),$$

whereby we may conclude that

$$N(T,\chi) \leq 2.6639165 + 0.9763160 \cdot S.$$

From Proposition 5.2,

$$S \leq \log\frac{\zeta(c)}{\zeta(2c)} + \frac{1}{\pi}\int_0^\pi F_{c,r}(\theta)\,d\theta,$$

with $F_{c,r}(\theta)$ made explicit in Definition 5.10. Easily computing $\log\frac{\zeta(c)}{\zeta(2c)} \leq 1.0682664$, and using interval arithmetic branch-and-bound, we find that

$$\int_0^\pi F_{c,r}(\theta)\,d\theta \leq 13.8132592.$$

Thus,

$$N(T,\chi) \leq 2.663915 + 0.9763160\left(1.0682664 + \frac{13.8132592}{\pi}\right) \leq 7.9997.$$

As $N(T,\chi)$ must be an integer, necessarily $N(T,\chi) \leq 7$.

For the entries in Table 1 with asterisks, the method just described yields bounds that are inferior to the results of our brute-force computations recorded in Lemma 6.1, and so the entries that appear are taken from those computations instead of the theoretical bound.

TABLE 2. Values of pairs $(c^\star, r^\star)$ where $c = c^\star/2^{11}$ and $r = r^\star/2^{11}$ that can be used to justify the entries in Table 1.

	$T = 5/7$		$T = 1$		$T = 2$	
k	$a = 0$	$a = 1$	$a = 0$	$a = 1$	$a = 0$	$a = 1$
5	(2822, 5006)	(2896, 5176)	(2886, 5212)	(2961, 5388)		
6	(2719, 4694)	(2770, 4836)	(2778, 4902)	(2831, 5046)	(2956, 5481)	
7	(2640, 4447)	(2677, 4566)	(2694, 4651)	(2734, 4771)	(2861, 5221)	(2906, 5346)
8	(2577, 4246)	(2606, 4348)	(2628, 4444)	(2660, 4546)	(2785, 5001)	(2822, 5107)
9	(2527, 4081)	(2550, 4168)	(2575, 4272)	(2600, 4358)	(2723, 4812)	(2753, 4904)

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