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INTERSECTIONS OF RECURRENCE SEQUENCES

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Abstract. We derive sharp upper bounds for the size of the intersection of certain linear recurrence sequences. As a consequence of these, we partially resolve a conjecture of Yuan on simultaneous Pellian equations, under the condition that one of the parameters involved is suitably large.

1. Introduction

Let $\{u_m\}_{m=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be integral linear recurrence sequences. That is, let us suppose that there exist positive integers h and k, and rational integers $a_1, a_2, \ldots, a_h$, $u_0, u_1, \ldots u_{h-1}$, $b_1, b_2, \ldots, b_k$ and $v_0, v_1, \ldots v_{k-1}$, such that

$$u_{m+h} = a_{h-1}u_{m+h-1} + a_{h-2}u_{m+h-2} + \cdots + a_0 u_m, \quad \text{for } m = 0, 1, 2, \ldots$$

and

$$v_{n+k} = b_{k-1}v_{n+k-1} + b_{k-2}v_{n+k-2} + \cdots + b_0 v_n, \quad \text{for } n = 0, 1, 2, \ldots$$

Then, as is well-known, there further exist algebraic integers

$$\alpha_1, \ldots, \alpha_h,\ \beta_1, \ldots, \beta_k$$

and polynomials $P_1, \ldots, P_h$, $Q_1, \ldots, Q_k$, with algebraic coefficients, such that we may write

$$u_m = P_1(m)\alpha_1^m + \cdots + P_h(m)\alpha_h^m, \quad P_1 \neq 0,$$

and

$$v_n = Q_1(n)\beta_1^n + \cdots + Q_k(n)\beta_k^n,\ Q_1 \neq 0,$$

for each pair of nonnegative integers m and n. It is certainly possible that such sequences may share values, even infinitely many, but, typically, our expectation is that their intersection is finite. Our goal in this paper is to derive a very sharp estimate for the size of such an intersection, for a specific class of recurrences. This problem is studied in detail in much greater generality by, for example, Laurent [8] and Schlickewei and Schmidt [13], [14].

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As a qualitative example of the type of theorem available in the literature, let us mention the main result of [11]:

Theorem 1.1. *(Mignotte) For* $\{u_m\}_{m=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$ *as above with the additional assumptions that*

$$|\alpha_1| > \max\{1, |\alpha_2|, \dots, |\alpha_h|\} \quad \text{and} \quad |\beta_1| > \max\{1, |\beta_2|, \dots, |\beta_k|\},$$

there exists an effectively computable constant m_0 *such that if* $u_m = v_n$ *with* $m \geq m_0$*, then necessarily*

$$P_1(m)\alpha_1^m = Q_1(n)\beta_1^n.$$

If this last relation occurs infinitely often, then there exist positive integers x *and* y *such that* $\alpha_1^x = \beta_1^y$*. If, further, the polynomials* P_1 *and* Q_1 *are actually constant, then the set of pairs of integers* (m, n) *for which* $u_m = v_n$ *lie in the union of a finite set with a finite number of arithmetic progressions.*

If we can rule out the presence of such progressions, then the corresponding intersection is necessarily finite. In this context, our main result quantifies the size of such an intersection, at least under favourable circumstances. While our arguments lead to a more general statement, we will restrict our attention somewhat in the interests of simplicity. Here and henceforth, by $h(\alpha)$ we mean the *absolute logarithmic Weil height* of an algebraic number of degree d, given by the formula

$$h(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\max\left(1, |\alpha^{(i)}|\right)\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$ and the $\alpha^{(i)}$s are the conjugates of α in the field of complex numbers.

Theorem 1.2. *Suppose that* $\{u_m\}_{m=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$ *are integral linear recurrence sequences, that*

$$\alpha_1, \dots, \alpha_h, \beta_1, \dots, \beta_k$$

are algebraic integers, and that

$$P_1, \dots, P_h, Q_1, \dots, Q_k$$

are algebraic numbers, for which

$$u_m = P_1\alpha_1^m + \cdots + P_h\alpha_h^m, \quad P_1 \neq 0, \tag{1.1}$$

and

$$v_n = Q_1\beta_1^n + \cdots + Q_k\beta_k^n, \; Q_1 \neq 0 \tag{1.2}$$

hold, and we have

$$|\alpha_1| > \max\{1, |\alpha_2|, \dots, |\alpha_h|\} \quad \text{and} \quad |\beta_1| > \max\{1, |\beta_2|, \dots, |\beta_k|\}. \tag{1.3}$$

Let us assume further that α_1, β_1, P_1 *and* Q_1 *are real, that* α_1 *and* β_1 *are multiplicatively independent and that* $P_1 \neq Q_1$*. Defining*

$$M = \max\{h(P_i), h(Q_j) : 1 \leq i \leq h, 1 \leq j \leq k\}$$

and

$$N = \max\{h, k, M, \log|\beta_1|, 3\},$$

there exists an effectively computable absolute constant C *such that if*

$$\log|\alpha_1| \geq C\, M \, \log|\beta_1| \log^3 N \tag{1.4}$$

then there is at most one pair of positive integers (m, n) *with*

$$u_m = v_n \quad \text{and} \quad P_1\alpha_1^m \neq Q_1\beta_1^n.$$

It is worth observing that the *dominant root* condition (1.3) is one that occurs somewhat naturally in a variety of contexts in the theory of recurrence sequences.

In the case where the two recurrences under consideration are both binary, there are many results in the literature establishing absolute bounds upon the size of their intersections, under various restrictions. One of the simplest cases is that of simultaneous Pellian equations, where, given distinct nonsquare positive integers a and b, we find that the number of positive integral triples (x, y, z) satisfying

$$x^2 - az^2 = 1, \ \ y^2 - bz^2 = 1 \tag{1.5}$$

is at most two (a bound that is achieved for infinitely many pairs (a, b); see [2] and [17]). In the case of the similar simultaneous equations

$$x^2 - ay^2 = 1, \ \ y^2 - bz^2 = 1, \tag{1.6}$$

the number of positive solutions has also been shown (see [6] and [7]) to be at most two. In this situation, however, we know of no pair (a, b) for which two such solutions actually exist and Yuan (Conjecture 1.1 of [18]) suggests that (1.6) has, in fact, at most a single positive solution (x, y, z) for a fixed pair (a, b). We can verify this conjecture (in a rather stronger form), provided b is sufficiently large as a function of a. Indeed, a somewhat straightforward corollary of Theorem 1.2 in this case is the following

Corollary 1.3. *Let* a *and* b *be nonsquare positive integers and let* ε_a *and* ε_b *denote the fundamental units in* $\mathbb{Q}(\sqrt{a})$ *and* $\mathbb{Q}(\sqrt{b})$*, respectively. Then there exists an effectively computable absolute constant* κ *such that if*

$$\log \varepsilon_b > \kappa \log a \log \varepsilon_a \left(\log \max\{\log \varepsilon_a, 3\}\right)^3, \tag{1.7}$$

the system of simultaneous equations

$$\left|x^2 - ay^2\right| = \left|y^2 - bz^2\right| = 1. \tag{1.8}$$

has at most one solution in positive integers x, y *and* z.

It is easy to observe that this result is sharp. Defining

$$T_k = \frac{\left(3 + 2\sqrt{2}\right)^k - \left(3 - 2\sqrt{2}\right)^k}{2\sqrt{2}},$$

if we choose $(a, b) = (2, T_k^2 - 1)$ for k suitably large, then inequality (1.7) is satisfied and equations (1.8) have the positive integer solution $(x, y, z) = (U_k, T_k, 1)$, where

$$U_k = \frac{\left(3 + 2\sqrt{2}\right)^k + \left(3 - 2\sqrt{2}\right)^k}{2}.$$

In what follows, our principal tool will be lower bounds for linear forms in complex logarithms of algebraic numbers. Our hope is that this paper will serve as a small advertisement for the theory of simultaneous linear forms in logarithms (indeed, Theorem 1.2 depends fundamentally upon such estimates). These results have been around for many years, dating to the early days of development of the general theory, but are neither widely known, nor widely used. One has the sense that they could find application rather more broadly than is currently the case. Interested readers are directed to [3], [4], [5], [9], [12], [15] and [16].

2. Proof of Theorem 1.2

We begin by proving Theorem 1.2. Suppose that we have

$$u_{m_1} = v_{n_1} \quad \text{and} \quad u_{m_2} = v_{n_2},$$

where (m_1, n_1) and (m_2, n_2) are distinct pairs of positive integers. Suppose further that

$$P_1 \alpha_1^{m_1} \neq Q_1 \beta_1^{n_1} \text{ and } P_1 \alpha_1^{m_2} \neq Q_1 \beta_1^{n_2}. \tag{2.1}$$

Define

$$\delta = \frac{1}{2} \min \left\{ 1 - \max_{2 \le i \le h} \left\{ \frac{\log |\alpha_i|}{\log |\alpha_1|} \right\}, 1 - \max_{2 \le j \le k} \left\{ \frac{\log |\beta_j|}{\log |\beta_1|} \right\} \right\}.$$

Then, assuming (1.4) for suitably large C, it follows from (1.3) that

$$|P_1 \alpha_1^{m_i} - Q_1 \beta_1^{n_i}| < \frac{1}{2} \min\{|P_1 \alpha_1^{m_i}|, |Q_1 \beta_1^{n_i}|\}^{1-\delta}.$$

Since P_1, Q_1, α_1 and β_1 are real, if we consider the linear forms

$$\Lambda_i = m_i \log |\alpha_1| - n_i \log |\beta_1| + \log |P_1/Q_1|, \quad i = 1, 2,$$

we therefore have that

$$\log |\Lambda_i| < -\delta \min\{\log |P_1 \alpha_1^{m_i}|, \log |Q_1 \beta_1^{n_i}|\}. \tag{2.2}$$

Assumption (2.1) ensures further that $\Lambda_i \neq 0$. Since the α_i and the β_j are roots of the companion polynomials of the recurrences defining $\{u_m\}_{m=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, respectively, monic polynomials with integer coefficients, it follows from (1.3) that

$$h(\alpha_1) \le \log |\alpha_1| \quad \text{and} \quad h(\beta_1) \le \log |\beta_1|.$$

We appeal to standard bounds for linear forms in logarithms to derive a lower bound upon $|\Lambda_i|$. Specifically, we use the main result of [1].

Theorem 2.1. *(Baker-Wüstholz) Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers different from* 0 *and* 1*, in a fixed number field $\mathbb{K}$ of degree d. Define the modified height h' by*

$$h'(\alpha) = \max \left\{ h(\alpha), \frac{|\log \alpha|}{d}, \frac{1}{d} \right\},$$

for every nonzero α in $\mathbb{K}$, where $h(\alpha)$ is the usual logarithmic Weil height. Let $b_1, \ldots, b_n$ be rational integers, not all 0*, and with absolute values less than $B \ge 3$. Setting*

$$\Lambda = b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n \neq 0,$$

we have

$$\log |\Lambda| > -C(n, d) \cdot h'(\alpha_1) \cdots h'(\alpha_n) \log B,$$

with

$$C(n, d) = 18(n + 1)!\, n^{n+1} (32d)^{n+2} \log(2nd).$$

We will also have need of a simultaneous analogue of this result, due to Loxton [9], which provides a sharper lower bound for linear combinations of logarithms of algebraic numbers.

Theorem 2.2. *(Loxton) Set*

$$\Lambda_i = b_{i1}\log\alpha_1 + \ldots + b_{in}\log\alpha_n, (1 \le i \le t),$$

where $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent elements of a fixed number field $\mathbb{K}$ of degree d, the matrix of rational integers (b_{ij}) has rank t and the $\log\alpha_j$ are the principal values. Let $A_j \ge 4$ be an upper bound for $\exp(h(\alpha_j))$, $B \ge 4$ be an upper bound for $\max\{|b_{ij}|\}$ and put $\Omega = \log A_1 \cdots \log A_n$. Then

$$\max_{1\le i\le t} |\Lambda_i| > \exp\{-\mathfrak{C}(\Omega\log\Omega)^{1/t}\log(B\Omega)\} \quad \textit{with} \quad \mathfrak{C} = (16nd)^{200n}.$$

Applying Theorem 2.1 thus yields a lower bound of the shape

$$\log|\Lambda_i| \gg -M\log n_i\,\log|\beta_1|\,\log|\alpha_1|,$$

whereby we reach the conclusion that

$$\frac{n_i}{\log n_i} \ll M\,\log|\alpha_1|$$

and hence, once again appealing to (1.4),

$$n_i \ll M\,\log|\alpha_1|\log\log|\alpha_1|. \tag{2.3}$$

Next, applying Theorem 2.2 (which we may do since we assume that $P_1 \ne Q_1$ and that α_1 and β_1 are multiplicatively independent) and writing

$$\Omega = M\,\log|\alpha_1|\log|\beta_1|,$$

we obtain the inequality

$$\max\{\log|\Lambda_1|, \log|\Lambda_2|\} \gg -(\Omega\log\Omega)^{1/2}\log\max\{n_1\Omega, n_2\Omega\}.$$

From (2.3), it follows that

$$\max\{\log|\Lambda_1|, \log|\Lambda_2|\} \gg -\Omega^{1/2}\log^{3/2}\Omega.$$

Combining this with inequality (2.2), we therefore have

$$\log|\alpha_1| \ll \Omega^{1/2}\log^{3/2}\Omega,$$

whence

$$\log|\alpha_1| \ll M\,\log|\beta_1|\log^3\Omega,$$

contradicting (1.4) if the constant C is chosen suitably large. This completes the proof of Theorem 1.2.

3. Proof of Corollary 1.3

We next turn our attention to Corollary 1.3. If we have two solutions in positive integers to equation (1.8), say (x_1, y_1, z_1) and (x_2, y_2, z_2), with $x_1 < x_2$, then

$$y_i = \frac{\varepsilon_a^{n_i} - \overline{\varepsilon_a}^{n_i}}{2\sqrt{a}} = \frac{\varepsilon_b^{m_i} + \overline{\varepsilon_b}^{m_i}}{2}, \quad \text{for } i \in \{1, 2\}, \tag{3.1}$$

where the n_i and m_i are positive integers and $\overline{\varepsilon_a}$ and $\overline{\varepsilon_b}$ are the conjugates of ε_a and ε_b in $\mathbb{Q}(\sqrt{a})$ and $\mathbb{Q}(\sqrt{b})$, respectively. That is, we have $u_{m_i} = v_{n_i}$ for $i = 1, 2$, where the recurrences satisfy $h = k = 2$,

$$\alpha_1 = \varepsilon_b,\ \alpha_2 = \overline{\varepsilon_b},\ \beta_1 = \varepsilon_a,\ \beta_2 = \overline{\varepsilon_a},$$

$$P_1 = P_2 = 1/2,\ Q_1 = \frac{1}{2\sqrt{a}} \text{ and } Q_2 = -\frac{1}{2\sqrt{a}}.$$

If we have
$$P_1\alpha_1^{m_i} = Q_1\beta_1^{n_i}$$
for either $i = 1$ or 2, then (3.1) implies that
$$\varepsilon_a^{n_i} = \sqrt{a}\,\varepsilon_b^{m_i} \quad \text{and} \quad -\overline{\varepsilon_a}^{n_i} = \sqrt{a}\,\overline{\varepsilon_b}^{m_i},$$
whence $a = \pm 1$, an immediate contradiction. Since $|\overline{\varepsilon_a}| = |\varepsilon_a|^{-1}$ and $|\overline{\varepsilon_b}| = |\varepsilon_b|^{-1}$, these recurrences satisfy (1.3) and hence, applying Theorem 2.2, we conclude as stated, at least provided the fundamental units ε_a and ε_b are multiplicatively independent.

Let us now suppose that ε_a and ε_b are multiplicatively dependent, satisfying, say,
$$\varepsilon_a^r = \varepsilon_b^s,$$
for r and s coprime, positive integers. If we have even a single solution to equation (1.8) in positive integers (x, y, z), then there exist positive integers m and n such that
$$y = \frac{\varepsilon_a^n - \overline{\varepsilon_a}^n}{2\sqrt{a}} = \frac{\varepsilon_b^m + \overline{\varepsilon_b}^m}{2}. \tag{3.2}$$
The corresponding linear form
$$\Lambda = m \log \varepsilon_b - n \log \varepsilon_a + \log(\sqrt{a})$$
can be rewritten as
$$\Lambda = \left(\frac{mr}{s} - n\right) \log \varepsilon_a + \log(\sqrt{a}).$$
Since we have
$$|\Lambda| = \log \left|\overline{\varepsilon_a}^n + \sqrt{a}\,\overline{\varepsilon_b}^m\right| \ll \sqrt{a}\,\varepsilon_b^{-m} \ll a\,\varepsilon_a^{-n},$$
it follows that
$$\log|\Lambda| \ll \log a - n \log \varepsilon_a. \tag{3.3}$$

In the other direction, let us begin by noting that $mr \geq ns$ implies the inequality $|\Lambda| \geq \log(\sqrt{a})$, contradicting (1.7) and (3.3). We may thus assume that $\max\{mr, ns\} = ns$. Applying part C of Theorem 3 of Loxton and van der Poorten [10], we have that
$$s \leq \frac{2}{\log\left(\frac{1}{2}(1+\sqrt{5})\right)} \log(\varepsilon_a), \tag{3.4}$$
whence, from Theorem 2.1,
$$\log|\Lambda| \gg -\log \varepsilon_a \log a \, \log(ns)$$
and so, appealing to (3.3) and (3.4),
$$n \ll \log a \, \log(ns) \ll \log a \log(n \log(\varepsilon_a)). \tag{3.5}$$
Combining this with (3.2), which implies that
$$\log(\varepsilon_b) \ll n \log(\varepsilon_a),$$
we contradict (1.7), provided κ is chosen to be suitably large. This completes the proof of Corollary 1.3.

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