

DIFFERENCES BETWEEN PERFECT POWERS: THE LEBESGUE-NAGELL EQUATION

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ABSTRACT. We develop a variety of new techniques to treat Diophantine equations of the shape $x^2 + D = y^n$, based upon bounds for linear forms in p -adic and complex logarithms, the modularity of Galois representations attached to Frey-Hellegouarch elliptic curves, and machinery from Diophantine approximation. We use these to explicitly determine the set of all coprime integers x and y , and $n \geq 3$, with the property that $y^n > x^2$ and $x^2 - y^n$ has no prime divisor exceeding 11.

1. INTRODUCTION

Understanding the gaps in the sequence of positive perfect powers

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, \dots$$

is a problem at once classical and fundamentally difficult. Mihăilescu’s Theorem [26] (née Catalan’s Conjecture) tells us that 8 and 9 are the only consecutive integers here, but it is not, for instance, a consequence of current technology that there are at most finitely many gaps of length k , for any fixed integer $k > 1$ (though this was conjectured to be the case by Pillai; see e.g. [29]). If we simplify matters by considering instead gaps between squares and other perfect powers, then we can show that such gaps, if nonzero, grow as we progress along the sequence. Indeed, the same is even true of the greatest prime factor of the gaps. Specifically, we have the following, a special case of Theorem 2 [10, Theorem 2]; here, by $P(m)$ we denote the greatest prime divisor of a nonzero integer m .

Theorem 1 (Bugeaud). *Let $n \geq 3$ be an integer. There exists an effectively computable positive constant $c = c(n)$ such that if x and y are coprime positive integers with $y \geq 2$, then*

$$P(x^2 - y^n) \geq c \log n$$

and, for suitably large x ,

$$P(x^2 - y^n) \geq \frac{\log \log y}{30n}.$$

This result is a consequence of bounds for linear forms in logarithms, complex and p -adic. As such, it can be made completely explicit and leads to an algorithm

Received by the editors September 14, 2021, and, in revised form, April 21, 2022.

2020 *Mathematics Subject Classification.* Primary 11D61; Secondary 11D41, 11F80, 11F03.

Key words and phrases. Exponential equation, Lucas sequence, shifted power, Galois representation, Frey curve, modularity, level lowering, Baker’s bounds, Hilbert modular forms, Thue equation.

The first author was supported by NSERC. The second author was supported by EPSRC Grant EP/S031537/1 “Moduli of elliptic curves and classical Diophantine problems”.

for solving the *Lebesgue-Nagell equation*

$$(1) \quad x^2 + D = y^n,$$

where we suppose that x and y are coprime nonzero integers, and that either

- (i) D is a fixed integer, or
- (ii) all the prime divisors of D belong to a fixed set of primes S .

The terminology here stems from the fact that equation (1) with $D = 1$ was first solved by V. A. Lebesgue [21], while T. Nagell [27], [28] was the first researcher to study such equations in a systematic fashion.

Regrettably, this algorithm is still, in most instances, not a practical one. Even in the very special case $D = -2$, we are not able to completely solve equation (1) (though there are a number of partial results available in the literature; see e.g. Chen [14]). Almost all the (very ample) literature on this problem concerns cases where $D > 0$ and y is odd in (1). Under these assumptions, we may solve the equation through appeal to a beautiful result of Bilu, Hanrot and Voutier [6] on primitive divisors in binary recurrence sequences, at least for all but a few small values of n . Proposition 5.1 of [12] (sharpening work of Cohn [15]) provides a very explicit summary of this approach – one bounds the exponent n in (1) in terms of the class numbers of a finite collection of imaginary quadratic fields, depending only upon the primes dividing D ; see Section 3 for details. Smaller values of n may be treated via techniques from elementary or algebraic number theory, or through machinery from Diophantine approximation. By way of example, in cases (i) and (ii), equation (1), for fixed n , reduces to finitely many *Thue* or *Thue-Mahler* equations, respectively. These can be solved through arguments of Tzanakis and de Weger [35], [36], [37] (see also [16] for recent refinements).

In case either $D > 0$ and y is even, or if $D < 0$, the literature on equation (1) is much sparser, primarily since the machinery of primitive divisors is no longer applicable. In these cases, other than bounds for linear forms in logarithms, the only general results that we know to apply to equation (1) are derived from the modularity of Galois representations arising from associated Frey-Hellegouarch curves. These are obtained by viewing (1) as a ternary equation of signature $(n, n, 2)$, i.e. as $y^n - D \cdot 1^n = x^2$. Such an approach can work to solve equation (1) in one of two ways, either by

- (a) producing an upper bound upon n that is sharper than that coming from linear forms in logarithms, leaving a feasible set of small n to treat, or
- (b) failing to produce such an upper bound, but, instead, providing additional arithmetic information that allows one to solve all the remaining Thue or Thue-Mahler equations below the bound coming from linear forms in logarithms.

An example of situation (a) is the case where D is divisible by only the primes in $S = \{5, 11\}$ and y is even. Then [5, Theorem 1.5] implies that equation (1) has no nontrivial solutions for all prime $n > 11$ and y even; work of Soydan and Tzanakis [34] treats smaller values of n and the case where y is odd (where the Primitive Divisor Theorem works readily). In general, we are potentially in situation (a) precisely when there fails to exist an elliptic curve E/\mathbb{Q} with nontrivial rational 2-torsion and conductor

$$N_{S^*} = 2 \prod_{p \in S^*} p,$$

for each subset $S^* \subseteq S$ with the property that the product of the primes in S^* is congruent to -1 modulo 8. Other examples of such sets S include

$$\{5, 19\}, \{11, 13, 41\}, \{11, 17, 29\}, \{11, 17, 37\}, \{17, 19, 37\} \text{ and } \{19, 29\}.$$

For situation (b), papers of Bugeaud, Mignotte and the second author [12], and of Barros [1] deal with a number of cases of equation (1) with D fixed and positive or negative, respectively.

In this paper, we will concentrate on the first of the two difficult cases, namely when $D > 0$ and y is even in (1) (so that necessarily $D \equiv -1 \pmod{8}$), under the additional hypothesis that D is divisible only by a few small primes. For completeness, we will also treat the easier situation where y is odd, under like hypotheses on D . In a companion paper [4], we will consider equation (1) in the other challenging situation where $D < 0$. Our main result in the paper at hand is the complete resolution of equation (1) in case $D > 0$, $P(D) < 13$, $\gcd(x, y) = 1$ and $n \geq 3$. We prove the following.

Theorem 2. *There are precisely 1240 triples of positive integers (x, y, n) with $n \geq 3$, $\gcd(x, y) = 1$, $y^n > x^2$ and*

$$P(x^2 - y^n) < 13.$$

They are distributed as follows.

n	$\#(x, y)$	n	$\#(x, y)$	n	$\#(x, y)$	n	$\#(x, y)$
3	755	7	5	12	4	26	1
4	385	8	17	13	1		
5	11	9	1	14	4		
6	51	10	4	15	1		

We provide the complete list of the 1240 solutions at

http://homepages.warwick.ac.uk/staff/S.Siksek/progs/lebnag/lebesgue_nagell_solutions.txt

Proving this result amounts to solving the equation

$$(2) \quad x^2 + 2^{\alpha_2} 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}} = y^n,$$

where x, y and n are positive integers, with $\gcd(x, y) = 1$, $n \geq 3$, and the α_i are nonnegative integers, i.e. equation (1), where $D > 0$ is supported only on primes in $S = \{2, 3, 5, 7, 11\}$. We note that earlier work along these lines typically either treat cases where there are no S -units congruent to $-1 \pmod{8}$, so that the analogous equations cannot have y even (see e.g. the paper of Luca [22] for $S = \{2, 3\}$), or simply exclude these cases (see Pink [30] for $S = \{2, 3, 5, 7\}$, where solutions with y even are termed *exceptional*). The only exceptions to this in the literature, of which we are aware, are the aforementioned paper of Soydan and Tzanakis [34] where $S = \{5, 11\}$ and work of Koutsianas [18] treating $S = \{7\}$ with prime exponent $n \equiv 13, 23 \pmod{24}$. A comprehensive survey of the extensive literature on this equation can be found in the paper of Le and Soydan [20].

To solve equation (2) completely, we are forced to introduce a variety of new techniques, many of which are applicable in rather more general settings. These include

- appeal to bounds for linear forms in two p -adic logarithms; what is interesting here is that the resulting inequalities are surprisingly strong, leading to

problems involving complex logarithms that are essentially the same level of difficulty as for the apparently easier case where $D > 0$ is fixed in equation (1) (as treated in [12])

- efficient sieving with Frey-Hellegouarch curves; on some level, this is likely the most important computational innovation in this paper
- refined use of lower bounds for linear forms in two and three complex logarithms
- a computationally efficient approach to treat the genus one curves encountered when solving equation (2) for $n \in \{3, 4\}$
- new practical techniques for solving Thue-Mahler equations of moderate ($n \leq 13$) degree.

The outline of this paper is as follows. In Section 2, we deal with the cases of exponents 3 and 4 in equation (2). In Section 3, we apply the Primitive Divisor Theorem to handle larger exponents in (2), under the assumption that the variable y is odd. Section 4 begins our treatment of the complementary (significantly harder) situation when y is even, showing how equation (2) with fixed exponent n reduces to solving a number of Thue-Mahler equations. From this, we are able to solve (2) completely for $n \leq 11$. In Section 5, we show how to associate to a putative solution of (2) a Frey-Hellegouarch elliptic curve. We then use this connection to develop a number of computational sieves that enable us to show that equation (2) has no solutions with prime exponents n between 17 and a reasonably large upper bound (which depends upon D , but is, in all cases, of order exceeding 10^8). This approach also deals with $n = 13$, except for one case that is solved through reduction to a Thue-Mahler equation. Finally, in Section 6, we apply inequalities for linear forms in p -adic and complex logarithms to show that (2) has no solutions for exponents n exceeding these upper bounds.

2. (VERY) SMALL VALUES OF n

We begin by treating equation (2) in case $n \in \{3, 4\}$. With these handled, we will thus be able to assume, without loss of generality, that $n \geq 5$ is prime. It is worth observing that our methods of proof in this section work equally well in the analogous situation where D is supported on $S = \{2, 3, 5, 7, 11\}$, but $D < 0$ (a conclusion that is far from true regarding our techniques for handling larger exponents).

2.1. Exponent $n = 3$. If we suppose that $n = 3$ in equation (2), then the problem reduces to one of determining S -integral points on

$$3^{\#S} = 3^5 = 243$$

Mordell elliptic curves of the shape $y^2 = x^3 - k$, where

$$k = 2^{\delta_2} 3^{\delta_3} 5^{\delta_5} 7^{\delta_7} 11^{\delta_{11}}, \quad \text{for } \delta_p \in \{0, 1, 2\}.$$

There are various ways to carry this out; if we try to do this directly using, say, the **Magma** computer algebra package [7], we very quickly run into problems arising from the difficulty of unconditionally certifying Mordell-Weil bases for some of the corresponding curves. We will instead argue somewhat differently.

Given a solution to equation (2) in coprime integers x and y , consider the Frey-Hellegouarch elliptic curve

$$E_{x,y} : Y^2 = X^3 - 3yX + 2x,$$

with corresponding discriminant

$$\Delta_{E_{x,y}} = 2^{\alpha_2+6} 3^{\alpha_3+3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}}.$$

This model has c -invariants

$$c_4 = 144y \quad \text{and} \quad c_6 = -1728x.$$

We may check via Tate's algorithm that this curve is minimal at all primes $p \geq 3$ and, while possibly not minimal at 2, the fact that x and y are coprime implies that a corresponding minimal model over \mathbb{Q} has either

$$c_4 = 144y, \quad c_6 = -1728x \quad \text{or} \quad c_4 = 9y, \quad c_6 = -27x,$$

with the latter case occurring only if xy is odd.

The isomorphism classes of elliptic curves over \mathbb{Q} with good reduction outside $\{2, 3, 5, 7, 11\}$ have recently been completely and rigorously determined using two independent approaches, by von Känel and Matschke [38] (via computation of S -integral points on elliptic curves, based upon bounds for elliptic logarithms), and by the first author, Gherga and Rechnitzer [3] (using classical invariant theory to efficiently reduce the problem to solutions of cubic Thue-Mahler equations). One finds that there are precisely 592192 isomorphism classes of elliptic curves over \mathbb{Q} with good reduction outside $\{2, 3, 5, 7, 11\}$; details are available at, e.g.

https://github.com/bmatschke/s-unit-equations/blob/master/elliptic-curve-tables/good-reduction-away-from-first-primes/K_deg_1/curves_K_1.1.1.1_S_2_3_5_7_11.txt

For each such class, we consider the corresponding c -invariants; if both $c_4 \equiv 0 \pmod{144}$ and $c_6 \equiv 0 \pmod{1728}$, we define

$$(3) \quad y = \frac{c_4}{144} \quad \text{and} \quad x = \frac{|c_6|}{1728},$$

while if at least one of $c_4 \equiv 0 \pmod{144}$ or $c_6 \equiv 0 \pmod{1728}$ fails to hold, but we have $c_4 \equiv 0 \pmod{9}$ and $c_6 \equiv 0 \pmod{27}$, we define

$$(4) \quad y = \frac{c_4}{9} \quad \text{and} \quad x = \frac{|c_6|}{27}.$$

For the resulting pairs (x, y) , we check that $y > 0$ and $\gcd(x, y) = 1$. We find 755 such pairs, corresponding to 812 triples (x, y, n) satisfying $3 \mid n$. There are 5 triples with $y > 10^9$, with the largest value of y corresponding to the identity

$$280213436582801^2 + 2^{16} \cdot 3^6 \cdot 5 \cdot 7^8 \cdot 11^2 = 4282124641^3.$$

2.2. Exponent $n = 4$. In this case, we may rewrite equation (2) as

$$(y^2 - x)(y^2 + x) = 2^{\alpha_2} 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}}$$

and so either $\alpha_2 = 0$, in which case

$$(5) \quad u_1 + u_2 = 2y^2,$$

where u_i are coprime $\{3, 5, 7, 11\}$ -units, or we have

$$(6) \quad u_1 + u_2 = y^2,$$

where u_i are coprime $\{2, 3, 5, 7, 11\}$ -units. In each case, since $xy \neq 0$, we may suppose that $u_1 > u_2$. To be precise, we have

$$u_1 u_2 = 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}}, \quad \sqrt{\frac{1}{2}(u_1 + u_2)} = y \quad \text{and} \quad \frac{1}{2}(u_1 - u_2) = x,$$

and

$$u_1 u_2 = 2^{\alpha_2 - 2} 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}}, \quad \sqrt{u_1 + u_2} = y \quad \text{and} \quad u_1 - u_2 = x,$$

in cases (5) and (6), respectively.

As for $n = 3$, we can write down corresponding Frey-Hellegouarch curves which have good reduction outside $\{2, 3, 5, 7, 11\}$ (and, additionally, in this situation, have nontrivial rational 2-torsion). It is easier to attack this problem more directly. Both equations (5) and (6) take the form $a + b = c^2$, where a and b are $\{2, 3, 5, 7, 11\}$ -units with $\gcd(a, b)$ square-free. Machinery for solving such problems has been developed by de Weger [39], [40]. Data from an implementation of this by von K nel and Matschke [38] is available at

https://github.com/bmatschke/solving-classical-diophantine-equations/blob/master/sums-of-units-equations/sumsOfUnitsBeingASquare_S_2_3_5_7_11.txt

We find that there are 1418 pairs (a, b) such that $a + b$ is a square, $\gcd(a, b)$ is square-free, $a \geq b$, and the only primes dividing a and b lie in $\{2, 3, 5, 7, 11\}$. We further restrict our attention to those with additionally $a > b \geq 1$ and either $\gcd(a, b) = 1$ (in which case we take $x = a - b$, $y = \sqrt{a + b}$), or $\gcd(a, b) = 2$ (whence we choose $x = \frac{1}{2}(a - b)$ and $y = \sqrt{\frac{1}{2}(a + b)}$). This gives 385 pairs of coprime, positive integers x, y with $y^4 > x^2$ and $P(y^4 - x^2) < 13$. These pairs actually lead to 406 triples (x, y, n) with $4 \mid n$, since 17 of the values of y are squares and four of them are cubes. However the four cubes have already appeared in our previous computation, so altogether we obtain 402 new triples (x, y, n) with $4 \mid n$. Together with the 812 triples satisfying $3 \mid n$ we have altogether 1214 triples (x, y, n) . The largest y with $n = 4$ corresponds to the identity

$$1070528159^2 + 2^{18} \cdot 3^3 \cdot 5 \cdot 7^4 \cdot 11^2 = 32719^4.$$

For the remainder of the paper, we may therefore assume that the exponent n in equation (2) is prime and ≥ 5 .

3. PRIMITIVE DIVISORS : EQUATION (2) WITH y ODD

In this section, we treat (2) under the assumption that y is odd, using the celebrated Primitive Divisor Theorem of Bilu, Hanrot and Voutier [6], and prove Proposition 3.1.

Proposition 3.1. *The only solutions to (2) with $n \geq 5$ prime, $\gcd(x, y) = 1$ and y odd correspond to the identities*

$$\begin{aligned} 1^2 + 2 \cdot 11^2 &= 3^5, & 241^2 + 2^3 \cdot 11^2 &= 9^5, & 401^2 + 2 \cdot 5^3 &= 11^5, \\ 4201^2 + 2 \cdot 3 \cdot 5^3 \cdot 11^4 &= 31^5 & \text{and} & & 4443^2 + 2^2 \cdot 7 \cdot 11^6 &= 37^5. \end{aligned}$$

If we consider solutions with $\gcd(x, y) = 1$, y odd, and n divisible by a prime ≥ 5 , then we must count one more solution corresponding to $241^2 + 2^3 \cdot 11^2 = 3^{10}$. Thus our total number of solutions to (2) for cases considered so far is $1214 + 6 = 1220$.

3.1. Lucas sequences and the Primitive Divisor Theorem. It is convenient to first introduce Lucas sequences as defined in [6]. A pair (γ, δ) of algebraic integers is called a **Lucas pair** if $\gamma + \delta$ and $\gamma\delta$ are non-zero coprime rational integers, and

γ/δ is not a root of unity. Given a Lucas pair (γ, δ) we define the corresponding **Lucas sequence** by

$$L_m = \frac{\gamma^m - \delta^m}{\gamma - \delta}, \quad m = 0, 1, 2, \dots$$

A prime ℓ is said to be a **primitive divisor** of the m -th term if ℓ divides L_m but ℓ does not divide $(\gamma - \delta)^2 \cdot u_1 u_2 \dots u_{m-1}$.

Theorem 3 (Bilu, Hanrot and Voutier [6]). *Let (γ, δ) be a Lucas pair and write $\{L_m\}$ for the corresponding Lucas sequence. If $m \geq 30$, then L_m has a primitive divisor. Moreover, if $m \geq 11$ is prime, then L_m has a primitive divisor.*

Let ℓ be a prime. We define the **rank of apparition** of ℓ in the Lucas sequence $\{L_m\}$ to be the smallest positive integer m such that $\ell \mid L_m$. We denote the rank of apparition of ℓ by m_ℓ . The following theorem of Carmichael [13] will be useful to us; for a concise proof see [2, Theorem 8].

Theorem 4 (Carmichael [13]). *Let (γ, δ) be a Lucas pair, and $\{L_m\}$ the corresponding Lucas sequence. Let ℓ be a prime.*

- (i) *If $\ell \mid \gamma\delta$ then $\ell \nmid L_m$ for all positive integers m .*
- (ii) *Suppose $\ell \nmid \gamma\delta$. Write $D = (\gamma - \delta)^2 \in \mathbb{Z}$.*
 - (a) *If $\ell \neq 2$ and $\ell \mid D$, then $m_\ell = \ell$.*
 - (b) *If $\ell \neq 2$ and $(\frac{D}{\ell}) = 1$, then $m_\ell \mid (\ell - 1)$.*
 - (c) *If $\ell \neq 2$ and $(\frac{D}{\ell}) = -1$, then $m_\ell \mid (\ell + 1)$.*
 - (d) *If $\ell = 2$, then $m_\ell = 2$ or 3 .*
- (iii) *If $\ell \nmid \gamma\delta$ then*

$$\ell \mid L_m \iff m_\ell \mid m.$$

3.2. Equation (2) with y odd. For the remainder of this section, $(x, y, n, \alpha_2, \dots, \alpha_{11})$ will denote a solution to the equation

$$(7) \quad x^2 + 2^{\alpha_2} 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}} = y^n \quad x > 0, y \text{ odd, } \gcd(x, y) = 1 \text{ and } n \geq 5 \text{ prime.}$$

We shall write

$$(8) \quad 2^{\alpha_2} 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}} = c^2 d, \quad \text{where } d \text{ is squarefree.}$$

Lemma 3.2. *There exist integers u and v such that*

$$x + c\sqrt{-d} = (u + v\sqrt{-d})^n, \quad \text{where } y = u^2 + dv^2, \quad u \mid x, \quad v \mid c \quad \text{and} \quad \gcd(u, dv) = 1.$$

If we define

$$\gamma = u + v\sqrt{-d} \quad \text{and} \quad \delta = u - v\sqrt{-d},$$

then (γ, δ) is a Lucas pair. Let $\{L_m\}$ be the corresponding Lucas sequence. Then

$$(9) \quad L_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} = \frac{c}{v}.$$

Proof. Write $M = \mathbb{Q}(\sqrt{-d})$. From (7), we have

$$(x + c\sqrt{-d})(x - c\sqrt{-d}) = y^n,$$

where the two factors on the left generate coprime ideals of \mathcal{O}_M . Thus $(x + c\sqrt{-d})\mathcal{O}_M = \mathfrak{A}^n$, for some ideal \mathfrak{A} of \mathcal{O}_M . There are 32 possible values of d and we checked, via **Magma**, that the corresponding quadratic fields $M = \mathbb{Q}(\sqrt{-d})$ have, in every case, class numbers h satisfying

$$h \in \{1, 2, 4, 8, 12, 32\}.$$

In particular n is coprime to h , and therefore \mathfrak{A} is principal. We deduce that $x + c\sqrt{-d} = \epsilon \cdot \gamma^n$ where $\epsilon \in \mathcal{O}_M^*$ and $\gamma \in \mathcal{O}_M$. The order of the unit group \mathcal{O}_M^* is either 4 (if $d = 1$), 6 (if $d = 3$) or 2 (in all other cases). Thus the unit group is n -divisible and we may absorb ϵ into the γ^n factor to obtain $x + c\sqrt{-d} = \gamma^n$ for some $\gamma \in \mathcal{O}_M$. We write δ for the conjugate of γ . Note that $\gamma + \delta$ is a divisor of $2x = \gamma^n + \delta^n$ and that $\gamma\delta = y$. It follows that $\gamma + \delta$ and $\gamma\delta$ are non-zero coprime rational integers. We claim that γ/δ is not a unit. If we suppose otherwise, then $(x + c\sqrt{-d})/(x - c\sqrt{-d}) = (\gamma/\delta)^n$ is a unit. By coprimality of the numerator and denominator, we obtain that $x + c\sqrt{-d}$ is a unit and therefore $y = 1$, a contradiction. Thus γ/δ is not a unit and (γ, δ) is a Lucas pair. Write $\{L_m\}$ for the corresponding Lucas sequence.

Since $\gamma \in \mathcal{O}_M$, we have $\gamma = u + v\sqrt{-d}$ with u and v are integers, or $\gamma = (u + v\sqrt{-d})/2$ where both u and v are odd integers. Suppose first that we are in the latter case (whence we note that necessarily $d \equiv 3 \pmod{4}$). Observe that $\gamma^n - \delta^n = 2c\sqrt{-d}$, so that $v\sqrt{-d} = \gamma - \delta$ divides $2c\sqrt{-d}$. As v is odd, we deduce that $v \mid c$ and that

$$L_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} = 2 \cdot \frac{c}{v}.$$

In particular, L_n is even. We note that $\gamma + \delta = u$ and $\gamma\delta = (u^2 + dv^2)/4 = y$. Thus the sequence $\{L_m\}$ satisfies the recurrence

$$L_0 = 0, \quad L_1 = 1, \quad L_{m+2} = uL_{m+1} - yL_m.$$

Using the fact that u and v are odd, one checks by induction that

$$L_m \equiv 0 \pmod{2} \iff 3 \mid m.$$

Thus, in particular, $3 \mid n$, contradicting the assumption that $n \geq 5$ is prime. It follows that $\gamma = u + v\sqrt{-d}$ where u and v are integers. Now observe that

$$2u = (\gamma + \delta) \mid (\gamma^n + \delta^n) = 2x \quad \text{and} \quad 2v\sqrt{-d} = (\gamma - \delta) \mid (\gamma^n - \delta^n) = 2c\sqrt{-d}.$$

Thus $u \mid x$ and $v \mid c$. Since $y = u^2 + dv^2$, we conclude that $\gcd(u, dv) = 1$. The lemma follows. \square

Lemma 3.3. *Let $(x, y, n, \alpha_2, \dots, \alpha_{11})$ be a solution to (7). Then $n = 5$ and*

$$(10) \quad 5u^4 - 10du^2v^2 + d^2v^4 = \pm 5^r \cdot 11^s,$$

for some $r \in \{0, 1\}$ and $s \geq 0$.

Proof. We continue with the notation of Lemma 3.2. By (9), we have $L_n = c/v$; this is coprime to $\gamma\delta = y$. If ℓ is any prime divisor of L_n then $m_\ell = n$, by part (iii) of Theorem 4 and the primality of n .

Suppose first that $n \geq 11$. By Theorem 3, L_n must have a primitive divisor, q say. By definition, this does not divide $D = (\gamma - \delta)^2 = -4v^2d$. Thus by part (ii) of Theorem 4, $n \mid (q - 1)$ if $(D/q) = 1$ and $n \mid (q + 1)$ if $(D/q) = -1$. As the possible values of $q \mid (c/v)$ are 2, 3, 5, 7 and 11, we obtain a contradiction. Thus $n = 5$ or $n = 7$.

Next we deal with the case $n = 7$. If we suppose that L_7 has a primitive divisor q then the above argument shows that $7 \mid (q - 1)$ or $7 \mid (q + 1)$ which is impossible as $q \in \{2, 3, 5, 7, 11\}$. Thus L_7 has no primitive divisor and our Lucas pair $(\gamma, \delta) = (u + v\sqrt{-d}, u - v\sqrt{-d})$ is 7-defective in the terminology of [6]. In particular, by the classification of defective Lucas pairs ([6, Theorem C]) we have

$u + v\sqrt{-d} = \pm(1 + \sqrt{-7})/2$ or $\pm(1 + \sqrt{-19})/2$. Both are impossible as u and v are integers. Hence there are no solutions to (7) with $n = 7$.

Finally we deal with $n = 5$. Since $5 \nmid \ell \cdot (\ell - 1)(\ell + 1)$ for any of $\ell \in \{2, 3, 7\}$, we see that $L_n = c/v = \pm 5^r \cdot 11^s$. Moreover, if $r \geq 1$ then $m_5 = 5$ and so $5 \mid dv$ by Theorem 4.

Substituting $\gamma = u + v\sqrt{-d}$ and $\delta = u - v\sqrt{-d}$ in (9) gives (10). We note that if $r \geq 2$ then $5 \mid u$, contradicting the coprimality of u and dv . Therefore $r \in \{0, 1\}$. \square

It remains to solve the quartic Thue–Mahler equations (10) for our 32 possible values of d . Appealing to the Thue–Mahler equation solver, implemented in **Magma** and associated to the paper [16], we obtain the following solutions:

$$(d, u, v) = (2, -1, 1), (2, 1, -2), (7, 3, -2), (10, 1, 1) \text{ and } (30, 1, 1).$$

These lead, respectively, to solutions of equation (2) with

$$(x, y, n) = (1, 3, 5), (241, 9, 5), (4443, 37, 5), (401, 11, 5) \text{ and } (4201, 31, 5),$$

completing the proof of Proposition 3.1.

4. REDUCTION TO THUE-MAHLER EQUATIONS: THE CASE OF EVEN y

From the results of the preceding sections, we are left to treat (2) with y even and $n \geq 5$ prime. It therefore remains to consider the equation

$$(11) \quad x^2 + 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}} = y^n \quad \text{with } y \text{ even, } \gcd(x, y) = 1 \text{ and } n \geq 5 \text{ prime.}$$

Let us define

$$(12) \quad N(d) = \begin{cases} 6 \times 10^8 & \text{if } d = 7, \\ 4 \times 10^8 & \text{if } d = 15, \\ 5 \times 10^8 & \text{if } d = 55, \\ 1.2 \times 10^9 & \text{if } d = 231. \end{cases}$$

The purpose of this section and the next is to prove Proposition 4.1.

Proposition 4.1. *The only solutions to (11) with $n < N(d)$ correspond to the identities*

$$\begin{aligned} 31^2 + 3^2 \cdot 7 &= 4^5, & 5^2 + 7 &= 2^5, & 181^2 + 7 &= 8^5, & 17^2 + 3 \cdot 5 \cdot 7^2 &= 4^5, \\ 23^2 + 3^2 \cdot 5 \cdot 11 &= 4^5, & 130679^2 + 3 \cdot 7^3 \cdot 11^7 &= 130^5, & 47^2 + 3^4 \cdot 5^2 \cdot 7 &= 4^7, \\ 11^2 + 7 &= 2^7, & 7^2 + 3^3 \cdot 5 \cdot 11^2 &= 4^7, & 117^2 + 5 \cdot 7^2 \cdot 11 &= 4^7, \\ 103^2 + 3 \cdot 5^2 \cdot 7 \cdot 11 &= 4^7, & \text{and } 8143^2 + 3^3 \cdot 5 \cdot 7^2 \cdot 11^2 &= 4^{13}. \end{aligned}$$

This gives 12 new solutions to (2) with $n \in \{5, 7, 13\}$ and, additionally, 8 further solutions with exponents 10, 14 and 26. Thus the total number of solutions we have found so far for (2) is $1220 + 12 + 8 = 1240$. We shall show in Section 6 that there are no further solutions, and that therefore (2) has precisely 1240 solutions as claimed in Theorem 2.

We assume without loss of generality that $x \equiv 1 \pmod{4}$. As before we shall write

$$(13) \quad 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}} = c^2 d, \text{ where } d \text{ is squarefree and } c = 3^{\beta_3} 5^{\beta_5} 7^{\beta_7} 11^{\beta_{11}}.$$

Since y is even, it follows from (11) that $d \equiv -1 \pmod{8}$, whence necessarily

$$(14) \quad d \in \{7, 15, 55, 231\}.$$

Let $M = M_d = \mathbb{Q}(\sqrt{-d})$. We note the structure of the class group of M :

$$\text{Cl}(M) \cong \begin{cases} 1 & d = 7 \\ C_2 & d = 15 \\ C_4 & d = 55 \\ C_2 \times C_6 & d = 231. \end{cases}$$

Lemma 4.2. *Let $c' = \pm c$ with the sign chosen so that $c' \equiv 1 \pmod{4}$. Let*

$$h = \begin{cases} 1 & d=7 \\ 2 & d=2 \\ 4 & d=55 \\ 6 & d=231 \end{cases} \quad \text{and} \quad \eta = r + s\sqrt{-d}, \quad \text{where} \quad (r, s) = \begin{cases} (1/4, -1/4) & d=7 \\ (1/8, -1/8) & d=15 \\ (3/32, 1/32) & d=55 \\ (5/128, -1/128) & d=231. \end{cases}$$

Let $0 \leq \kappa_n \leq h-1$ be the unique integer satisfying $\kappa_n \cdot n \equiv -2 \pmod{h}$. Then there is some non-zero $\mu \in \mathcal{O}_M$ such that

$$(15) \quad \frac{x + c'\sqrt{-d}}{2} = \eta^{(2+\kappa_n \cdot n)/h} \cdot \mu^n.$$

Moreover, η is supported only on prime ideals dividing 2 and μ is supported only on prime ideals dividing y .

Proof. As $d \equiv -1 \pmod{8}$, the prime 2 splits in \mathcal{O}_M as $2\mathcal{O}_M = \mathfrak{P} \cdot \overline{\mathfrak{P}}$, where

$$(16) \quad \mathfrak{P} = 2\mathcal{O}_M + \left(\frac{1 + \sqrt{-d}}{2} \right) \cdot \mathcal{O}_M.$$

We may rewrite (11) as

$$(17) \quad \left(\frac{x + c'\sqrt{-d}}{2} \right) \left(\frac{x - c'\sqrt{-d}}{2} \right) = \frac{y^n}{4}.$$

Note that the two factors on the left hand-side of this last equation are coprime elements of \mathcal{O}_M . Since $x \equiv c' \equiv 1 \pmod{4}$, we see that \mathfrak{P} divides the first factor on the left-hand-side. We thus deduce that

$$(18) \quad \left(\frac{x + c'\sqrt{-d}}{2} \right) \cdot \mathcal{O}_M = \mathfrak{P}^{-2} \cdot \mathfrak{A}^n,$$

where \mathfrak{A} is an integral ideal divisible by \mathfrak{P} , with $\mathfrak{A} \cdot \overline{\mathfrak{A}} = y\mathcal{O}_M$. The order of the class $[\mathfrak{P}]$ in $\text{Cl}(M)$ is h . Thus \mathfrak{P}^{-h} is principal, and η has been chosen so that $\mathfrak{P}^{-h} = \eta\mathcal{O}_M$. Let $\mathfrak{B} = \mathfrak{P}^{\kappa_n} \cdot \mathfrak{A}$. Then we may rewrite (18) as

$$\left(\frac{x + c'\sqrt{-d}}{2} \right) \cdot \mathcal{O}_M = \mathfrak{P}^{-(2+\kappa_n \cdot n)} \cdot \mathfrak{B}^n = \eta^{(2+\kappa_n \cdot n)/h} \cdot \mathfrak{B}^n.$$

Since n is a prime that does not divide the order of $\text{Cl}(M)$, the ideal \mathfrak{B} must be principal. Let μ be a generator for \mathfrak{B} . Then

$$\frac{x + c'\sqrt{-d}}{2} = \pm \eta^{(2+\kappa_n \cdot n)/h} \cdot \mu^n$$

and (15) follows on absorbing the \pm sign into μ . It is clear that η is supported on \mathfrak{P} only. Moreover \mathfrak{B} is an integral ideal with norm $2^{\kappa_n}y$. It follows, since y is even, that μ is supported only on prime ideals dividing y . \square

Lemma 4.3. *The only solutions to equation (11) with $n \in \{5, 7, 11\}$ are those corresponding to the identities in Proposition 4.1.*

Proof. We drop our requirement that $x > 0$ and replace it with the assumption $x \equiv 1 \pmod{4}$, so that we can apply Lemma 4.2. For each exponent n , there are four cases to consider depending on the value of $d \in \{7, 15, 55, 231\}$ in (13). For each pair (n, d) , Lemma 4.2 asserts that (x, c') satisfies (15) with $\mu \in \mathcal{O}_M$. We write

$$\mu = r + s(1 + \sqrt{-d})/2,$$

with r and s rational integers. We will show that $\gcd(r, s) = 1$. If $2 \mid r$ and $2 \mid s$ then $\overline{\mathfrak{P}} \mid \mu$ which contradicts the coprimality of the two factors in the left hand-side of (17). If ℓ is an odd prime with $\ell \mid r$ and $\ell \mid s$, then again we contradict the coprimality of those two factors. Hence $\gcd(r, s) = 1$.

From (15), we have

$$c' = \frac{1}{\sqrt{-d}} \left(\eta^m \cdot (r + s(1 + \sqrt{-d})/2)^n - \overline{\eta}^m \cdot (r + s(1 - \sqrt{-d})/2)^n \right),$$

where $m = (2 + \kappa_n \cdot n)/h$. The expression on the right has the form $2^{-hm}F(r, s)$ where $F \in \mathbb{Z}[X, Y]$ is homogeneous of degree n . We therefore, in each case, obtain a Thue-Mahler equation of the form

$$F(r, s) = 2^{hm} \cdot c' = \pm 2^{hm} \cdot 3^{\beta_3} 5^{\beta_5} 7^{\beta_7} 11^{\beta_{11}}.$$

We solved these Thue-Mahler equations using the Thue-Mahler solver associated with the paper [16]. This computation took around one day and resulted in the solutions in Proposition 4.1 for $n \in \{5, 7\}$; there were no solutions for $n = 11$. \square

5. FREY-HELLEGOUARCH CURVES AND RELATED OBJECTS

We continue to treat (2) with y even, i.e. equation (11), where we maintain the assumption that $x \equiv 1 \pmod{4}$. Although the results of the previous section allow us to assume more, for now we merely impose the following constraint on the exponent: $n \geq 7$ is prime. Following the first author and Skinner [5], we associate to a solution (x, y, n) the Frey-Hellegouarch elliptic curve $F = F(x, y, n)$ defined via

$$(19) \quad F : Y^2 + XY = X^3 + \left(\frac{x-1}{4} \right) X^2 + \frac{y^n}{64} X.$$

The model here is minimal, semistable, and we note the following invariants,

$$c_4 = x^2 - \frac{3}{4}y^n, \quad c_6 = -x^3 + \frac{9}{8}xy^n$$

and

$$\Delta_F = \frac{y^{2n}}{2^{12}}(x^2 - y^n) = -2^{-12} \cdot 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}} \cdot y^{2n}.$$

We invoke work of the first author and Skinner [5], building on the modularity of elliptic curves over \mathbb{Q} following Wiles and others [41], [8], Ribet's level lowering

theorem [31], and the isogeny theorem of Mazur [24]. Write N for the conductor of E and let

$$N' = \frac{N}{\prod_{\substack{\ell \mid N \\ n \mid \text{ord}_\ell(\Delta_F)}} \ell}.$$

The results of [5] assert the existence of a weight 2 newform f of level N' such that

$$(20) \quad \bar{\rho}_{F,n} \sim \bar{\rho}_{f,n},$$

with $\mathfrak{n} \mid n$ a prime ideal in the ring of integers \mathcal{O}_K of the Hecke eigenfield K of f .

Lemma 5.1. *We have $N' = 2R$ where $R \mid 3 \cdot 5 \cdot 7 \cdot 11$. Moreover, for $\ell \in \{3, 5, 7, 11\}$, we have*

$$(21) \quad \ell \nmid N' \iff \alpha_\ell \equiv 0 \pmod{n} \iff 2 \text{ord}_\ell(c) + \text{ord}_\ell(d) \equiv 0 \pmod{n},$$

where c and d are given in (13).

Proof. Since E is semistable, N is squarefree, and therefore N' is squarefree. Note that $\text{ord}_2(\Delta_F) = 2n \text{ord}_2(y) - 12$. Thus $2 \parallel N$ and $n \nmid \text{ord}_2(\Delta_F)$, whereby $2 \parallel N'$.

Next let $\ell \geq 13$. Then $\text{ord}_\ell(\Delta_F) = 2n \text{ord}_\ell(y)$ and hence $\ell \nmid N'$. It follows that $N' = 2R$ with $R \mid 3 \cdot 5 \cdot 7 \cdot 11$.

To prove the second part of the lemma, note that, for $\ell \in \{3, 5, 7, 11\}$,

$$\text{ord}_\ell(\Delta) = \alpha_\ell + 2n \text{ord}_\ell(y) = 2 \text{ord}_\ell(c) + \text{ord}_\ell(d) + 2n \text{ord}_\ell(y).$$

If $\alpha_\ell = 0$ and $\text{ord}_\ell(y) = 0$, then $\ell \nmid N$ and so $\ell \nmid N'$, and therefore (21) holds. Suppose $\alpha_\ell > 0$ or $\text{ord}_\ell(y) > 0$. Then $\ell \parallel N$. By the formula for N' , we have $\ell \nmid N'$ if and only if $n \mid \text{ord}_\ell(\Delta)$ which is equivalent to $n \mid \alpha_\ell$. This completes the proof. \square

Let f be the weight 2 newform of level N' satisfying (20). Write

$$(22) \quad f = \mathfrak{q} + \sum_{m=2}^{\infty} c_m \mathfrak{q}^m$$

for the usual q -expansion of f . Then $K = \mathbb{Q}(c_1, c_2, \dots)$, and the coefficients c_i belong to \mathcal{O}_K .

Lemma 5.2. *Let $\ell \nmid N'$ be a prime and write*

$$\mathcal{C}'_{f,\ell} = \begin{cases} (\ell+1)^2 - c_\ell^2 & \text{if } K = \mathbb{Q} \\ \ell \cdot ((\ell+1)^2 - c_\ell^2) & \text{if } K \neq \mathbb{Q}. \end{cases}$$

Let d be as in (13), and set

$$T_\ell(f) = \begin{cases} \{a \in \mathbb{Z} \cap [-2\sqrt{\ell}, 2\sqrt{\ell}] : \ell+1-a \equiv 0 \pmod{4}\} & \text{if } (-d/\ell) = 1 \\ \{a \in \mathbb{Z} \cap [-2\sqrt{\ell}, 2\sqrt{\ell}] : \ell+1-a \equiv 0 \pmod{2}\} & \text{if } (-d/\ell) = -1 \\ \emptyset & \text{if } \ell \mid d. \end{cases}$$

Let

$$\mathcal{C}_{f,\ell} = \mathcal{C}'_{f,\ell} \cdot \prod_{a \in T_\ell(f)} (a - c_\ell).$$

If $\bar{\rho}_{F,n} \sim \bar{\rho}_{f,n}$, then $\mathfrak{n} \mid \mathcal{C}_{f,\ell}$.

Proof. Suppose $\ell \nmid N'$ and write N for the conductor of F . Suppose $\bar{\rho}_{F,n} \sim \bar{\rho}_{f,n}$. A standard consequence [32, Propositions 5.1, 5.2] of this is that

$$\begin{cases} c_\ell \equiv a_\ell(F) \pmod{\mathfrak{n}} & \text{if } \ell \neq n \text{ and } \ell \nmid N \\ c_\ell \equiv \pm(\ell+1) \pmod{\mathfrak{n}} & \text{if } \ell \neq n \text{ and } \ell \mid N. \end{cases}$$

Here the restriction $\ell \neq n$ is unnecessary if $K = \mathbb{Q}$. It follows if $\ell \mid N$ that $\mathfrak{n} \mid \mathcal{C}'_{f,\ell}$. We observe that the discriminant of F can be written as

$$\Delta = (-d) \cdot (cy^n/2^6)^2.$$

If $\ell \mid d$ then $\ell \mid N$ and so we take $\mathcal{C}_{f,\ell} = \mathcal{C}'_{f,\ell}$.

Suppose $\ell \nmid N$ and so $\ell \nmid d$. Thus $c_\ell \equiv a_\ell(F) \pmod{\mathfrak{n}}$. To complete the proof it is sufficient to show that $a_\ell(F) \in T_\ell(f)$. The model for F given in (19) is isomorphic to

$$(23) \quad F : Y^2 = X^3 + xX^2 + \frac{y^n}{4}X,$$

and so has a point of order 2. Thus $\ell+1-a_\ell(F) = \#F(\mathbb{F}_\ell) \equiv 0 \pmod{2}$. Moreover, if $(-d/\ell) = 1$ then the discriminant is a square modulo ℓ , so F/\mathbb{F}_ℓ has full 2-torsion, whence $\#F(\mathbb{F}_\ell) \equiv 0 \pmod{4}$. It follows that $a_\ell(F) \in T_\ell(f)$. \square

There are a total of 76 conjugacy classes of newforms f at the levels $N' = 2R$ with $R \mid 3 \cdot 5 \cdot 7 \cdot 11$, of which 59 are rational (and so correspond to elliptic curves). Since there are four possible values of $d \in \{7, 15, 55, 231\}$, this gives $4 \times 76 = 304$ pairs (f, d) to consider. We apply Lemma 5.2 to each pair (f, d) , letting

$$\mathcal{C}_{f,d} = \sum \mathcal{C}_{f,\ell} \cdot \mathcal{O}_K,$$

where the sum is over all primes $3 \leq \ell < 500$ not dividing N' . It follows from Lemma 5.2 that $\mathfrak{n} \mid \mathcal{C}_{f,d}$. We let

$$C_{f,d} = \text{Norm}_{K/\mathbb{Q}}(\mathcal{C}_{f,d}).$$

Since $\mathfrak{n} \mid n$, we have that $n \mid C_{f,d}$. Of the 304 pairs (f, d) , the integer $C_{f,d}$ is identically zero for 114 pairs, and non-zero for the remaining 190 pairs. For the 190 pairs (f, d) where $C_{f,d} \neq 0$, we find that the largest possible prime divisor of any of these $C_{f,d}$ is 11. By the results of the previous section we know all the solutions to (11) with $n \in \{7, 11\}$ and hence can therefore eliminate these 190 pairs from further consideration. We focus on the 114 remaining pairs (f, d) . Here, each f satisfies $K = \mathbb{Q}$ and so corresponds to an elliptic curve E/\mathbb{Q} whose conductor is equal to the level N' of f . Moreover, each of these elliptic curve E has non-trivial rational 2-torsion. This is unsurprising in view of the remarks following [32, Proposition 9.1]. We observe that $\bar{\rho}_{f,n} \sim \bar{\rho}_{E,n}$. Thus we have 114 pairs (E, d) to consider, and if (x, y, n) is a solution to (11) with $n \geq 13$ prime then there is some pair (E, d) (among the 114) where d satisfies (13) and E/\mathbb{Q} is an elliptic curve such that $\bar{\rho}_{F,n} \sim \bar{\rho}_{E,n}$. In particular, for any prime $\ell \nmid N'$,

$$\begin{cases} a_\ell(E) \equiv a_\ell(F) \pmod{n} & \text{if } \ell \nmid N \\ a_\ell(E) \equiv \pm(\ell+1) \pmod{n} & \text{if } \ell \mid N. \end{cases}$$

5.1. The method of Kraus.

Lemma 5.3. *Let $c' = \pm c$ with the sign chosen so that $c' \equiv 1 \pmod{4}$. Let*

$$\gamma = u + v\sqrt{-d} \quad \text{where} \quad (u, v) = \begin{cases} (1/8, 3/8) & d = 7 \\ (7/8, 1/8) & d = 15 \\ (3/8, 1/8) & d = 55 \\ (5/16, -1/16) & d = 231. \end{cases}$$

Choose $\epsilon_n \in \{1, -1\}$ to satisfy $n \equiv \epsilon_n \pmod{3}$. Then there is some $\delta \in M^$ such that*

$$(24) \quad \frac{x + c'\sqrt{-d}}{x - c'\sqrt{-d}} = \begin{cases} \gamma \cdot \delta^n & \text{if } d = 7, 15, 55 \\ \gamma^{(2+\epsilon_n \cdot n)/3} \cdot \delta^n & \text{if } d = 231. \end{cases}$$

Moreover, δ is supported only on prime ideals dividing y .

Proof. From the proof of Lemma 4.2, and in particular (18), we have

$$(25) \quad \left(\frac{x' + c'\sqrt{-d}}{x' - c'\sqrt{-d}} \right) \cdot \mathcal{O}_M = (\overline{\mathfrak{P}}/\mathfrak{P})^2 \cdot \mathfrak{B}^n$$

with $\mathfrak{B} = \mathfrak{A}/\overline{\mathfrak{A}}$. Here \mathfrak{P} is given by (16), and \mathfrak{A} is an integral ideal dividing y . We observe that \mathfrak{B} is supported only on prime ideals dividing y . First let $d = 7, 15$ or 55 . In these cases the fractional ideal $(\overline{\mathfrak{P}}/\mathfrak{P})^2$ is principal, and we have chosen γ so that it is a generator. Since n is a prime not dividing the order of $\text{Cl}(M)$ we have that \mathfrak{B} is also principal. Let $\delta \in M^*$ be a generator of \mathfrak{B} . Then

$$\frac{x + c'\sqrt{-d}}{x - c'\sqrt{-d}} = \pm \gamma \cdot \delta^n,$$

and we complete the proof for $d = 7, 15$ and 55 by absorbing the \pm sign into δ .

Suppose now that $d = 231$. The class of the fractional ideal $\overline{\mathfrak{P}}/\mathfrak{P}$ has order 3, and we have chosen γ to be a generator of $(\overline{\mathfrak{P}}/\mathfrak{P})^3$. We may rewrite (25) as

$$\frac{x + c'\sqrt{-d}}{x - c'\sqrt{-d}} = (\overline{\mathfrak{P}}/\mathfrak{P})^{2+\epsilon_n \cdot n} \cdot \mathfrak{C}^n,$$

where $\mathfrak{C} = \mathfrak{B} \cdot (\mathfrak{P}/\overline{\mathfrak{P}})^{\epsilon_n}$. Note that $3 \mid (2 + \epsilon_n \cdot n)$ and hence

$$(\overline{\mathfrak{P}}/\mathfrak{P})^{2+\epsilon_n \cdot n} = \gamma^{(2+\epsilon_n \cdot n)/3} \cdot \mathcal{O}_M.$$

The ideal \mathfrak{C} must be principal and hence we complete the proof by letting δ be a suitably chosen generator for \mathfrak{C} . We note that, in all cases, δ is supported only on primes of \mathcal{O}_M dividing y . \square

Lemma 5.4. *Let $n \geq 13$ be a prime and (E, d) be one of the remaining 114 pairs. Let $q = kn + 1$ be a prime. Suppose that $(-d/q) = 1$, and choose a such that $a^2 \equiv -d \pmod{q}$. Let g_0 be a generator for \mathbb{F}_q^* and $g = g_0^n$. Let (u, v) be as in the statement of Lemma 5.3. If $d = 7, 15$ or 55 , then let*

$$\Theta'_q = \{(u + va) \cdot g^i : i = 0, 1, \dots, k-1\} \subset \mathbb{F}_q.$$

If $d = 231$, then set

$$\Theta'_q = \{(u + va)^{(2+\epsilon_n \cdot n)/3} \cdot g^i : i = 0, 1, \dots, k-1\} \subset \mathbb{F}_q$$

and, in all cases, let

$$\Theta_q = \Theta'_q \setminus \{0, 1\}.$$

Suppose the following two conditions hold:

- (i) $a_q(E)^2 \not\equiv 4 \pmod{n}$.
- (ii) $a_q(E)^2 \not\equiv a_q(H_\theta)^2 \pmod{n}$ for all $\theta \in \Theta_q$, where

$$H_\theta : Y^2 = X(X+1)(X+\theta).$$

Then $\bar{\rho}_{F,n} \approx \bar{\rho}_{E,n}$.

Proof. We suppose that $\bar{\rho}_{F,n} \sim \bar{\rho}_{E,n}$ and derive a contradiction. Since $n \geq 11$, we note that, in particular, $q \notin \{2, 3, 5, 7, 11\}$. Suppose first that $q \mid y$. Then $q+1 \equiv \pm a_q(E) \pmod{n}$. But $q+1 = kn+2 \equiv 2 \pmod{n}$ and hence $a_q(E)^2 \equiv 4 \pmod{n}$, contradicting hypothesis (i). We may therefore suppose that $q \nmid y$. In particular q is a prime of good reduction for the Frey curve F , and also for the curve E , whence $a_q(F) \equiv a_q(E) \pmod{n}$.

Since $a^2 \equiv -d \pmod{q}$, by the Dedekind-Kummer theorem, the prime q splits in \mathcal{O}_M as a product of two primes $q\mathcal{O}_M = \mathfrak{q} \cdot \bar{\mathfrak{q}}$ where we choose

$$(26) \quad \mathfrak{q} = q\mathcal{O}_M + (a - \sqrt{-d}) \cdot \mathcal{O}_M.$$

In particular $a \equiv \sqrt{-d} \pmod{\mathfrak{q}}$. Moreover, $\mathbb{F}_{\mathfrak{q}} = \mathbb{F}_q$. Since $\mathfrak{q} \mid q$ and $q \nmid 2y$, it follows from (17) that $\mathfrak{q} \nmid (x \pm c'\sqrt{-d})$. We let $\theta \in \mathbb{F}_q^*$ satisfy

$$(27) \quad \theta \equiv \frac{x + c'\sqrt{-d}}{x - c'\sqrt{-d}} \pmod{\mathfrak{q}}.$$

We will contradict hypothesis (ii), and complete the proof, by showing that $\theta \in \Theta_q$ and $a_q(F) = \pm a_q(H_\theta)$. If $\theta \equiv 1 \pmod{q}$ then $\mathfrak{q} \mid 2c'\sqrt{-d}$ giving that $q \mid 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, which is impossible. Therefore $\theta \not\equiv 1 \pmod{q}$. Let (u, v) , γ and δ be as in the statement of Lemma 5.3. Note that γ is supported only at the primes above 2 and that δ is supported at only the primes above y . Since $\mathfrak{q} \nmid y$, we may reduce γ and δ modulo \mathfrak{q} . In particular,

$$\gamma \equiv u + av \pmod{\mathfrak{q}}.$$

Moreover, $\delta^n \pmod{\mathfrak{q}}$ belongs to the subgroup of \mathbb{F}_q^* generated by $g = g_0^n$ of order k . The fact that θ belongs to Θ'_q (and therefore to Θ_q) follows from (27) and (24).

It remains to show that $a_q(F) = \pm a_q(H_\theta)$. The model for F in (19) is isomorphic to the model in (23). We note that the polynomial on the right hand-side of (23) can be factored as

$$(28) \quad X \left(X + \frac{x + c'\sqrt{-d}}{2} \right) \left(X + \frac{x - c'\sqrt{-d}}{2} \right).$$

Thus, $F \pmod{\mathfrak{q}}$ is a quadratic twist of H_θ , whence

$$a_q(F) = a_q(F) = \pm a_q(H_\theta) = \pm a_q(H_\theta),$$

completing the proof. \square

Remark. We know by Dirichlet's theorem that the natural density of primes q satisfying the conditions $q = kn+1$ and $(-d/q) = 1$ is $1/2n$. We now give a heuristic estimate for the probability of succeeding to show that $\bar{\rho}_{F,n} \approx \bar{\rho}_{E,n}$ using a single $q = kn+1$ that satisfies $(-d/q) = 1$. The set Θ'_q has size k , and so Θ_q has size close to k . For a given $\theta \in \Theta_q$, we expect the probability that $a_q(E)^2 \not\equiv a_q(H_\theta)^2 \pmod{n}$ to be roughly $(1 - 2/n)$. Thus the probability of the criterion succeeding is around

TABLE 1. The upper bounds $N(d)$ are as in (12). The table records the number of primes in the interval $13 \leq n < N(d)$ and the number of pairs (E, d) .

d	$N(d)$	Number of primes $13 \leq n < N(d)$	Number of pairs (E, d)
7	6×10^8	31324698	39
15	4×10^8	21336321	28
55	5×10^8	26355862	27
231	1.2×10^9	60454700	20

$(1 - 2/n)^k$. In particular, if k is large compared to $n/2$ then we expect failure, but if k is small compared to $n/2$ then we expect success. Moreover, if we fail with one particular value of q , we are likely to fail with larger values of q (which correspond to larger values of k).

However, this heuristic is likely to be inaccurate when \sqrt{q} is small compared to n , since $a_q(E)$ and $a_q(H_\theta)$ both belong to the Hasse interval $[-2\sqrt{q}, 2\sqrt{q}]$, and the probability of the criterion succeeding is around $(1 - 1/\sqrt{q})^k$.

We are working towards proving Proposition 4.1. Recall that we have 114 remaining pairs (E, d) with E/\mathbb{Q} an elliptic curve and $d \in \{7, 15, 55, 231\}$; these are distributed among the values of d according to Table 1. The table also records the upper bounds $N(d)$ of Proposition 4.1. We wrote a **Magma** script that applied the criterion of Lemma 5.4 to the

$$39 \cdot 31324698 + 28 \cdot 21336321 + 27 \cdot 26355862 + 20 \cdot 60454700 = 3739782484 \approx 3.7 \times 10^9$$

triples (E, d, n) . For each such triple, the script searches for a prime $q = kn + 1$ with $k < 10^3$ such that the hypotheses of Lemma 5.4 are satisfied. This computation took around 29000 hours, but was in fact distributed over 64 processors, and finished in around 20 days. For all but 1230 of the 3739782484 triples (E, d, n) the script found some q satisfying the hypotheses of Lemma 5.4. We are therefore reduced to considering the remaining 1230 triples (E, d, n) . While these are somewhat too numerous to record here, we note that the largest value of n appearing in any of these triples is $n = 1861$ and this corresponds to E being the elliptic curve with Cremona label 210A1 and $d = 15$.

5.2. A refined sieve. Our adaptation of the method of Kraus (Lemma 5.4) makes use of one auxiliary prime q satisfying $q = kn + 1$ and $(-k/q) = 1$. To treat the remaining 1230 triples (E, d, n) , we will use a refined sieve that combines information from several such primes q .

Lemma 5.5. *Let (E, d, n) be one of the remaining 1230 triples. Let $q = kn + 1$ be a prime. Suppose that $(-d/q) = 1$ and choose a such that $a^2 \equiv -d \pmod{q}$. Let $c', h, (r, s), \kappa_n$ be as in Lemma 4.2, $m = (2 + \kappa_n \cdot n)/h \in \mathbb{Z}$, and set*

$$\rho_1 = (r + sa)^m \quad \text{and} \quad \rho_2 = (r - sa)^m.$$

Let g_0 be a generator for \mathbb{F}_q^ and set $g = g_0^n$. Further, let us define*

$$\Upsilon_q'' = \{(\rho_1 \cdot g^i, \rho_2 \cdot g^j) : i = 0, 1, \dots, k-1, j = 0, 1\} \subset \mathbb{F}_q \times \mathbb{F}_q,$$

$$\Upsilon_q' = \{(\theta_1, \theta_2) \in \Upsilon_q'' : \theta_1 \theta_2 (\theta_1 - \theta_2) \neq 0\}$$

and

$$\Upsilon_q = \{(\theta_1, \theta_2) \in \Upsilon'_q : a_q(H_{\theta_1, \theta_2}) \equiv a_q(E) \pmod{n}\},$$

where $H_{\theta_1, \theta_2}/\mathbb{F}_q$ is the elliptic curve

$$H_{\theta_1, \theta_2} : Y^2 = X(X + \theta_1)(X + \theta_2).$$

Write

$$\Phi'_q = \{(\theta_1 - \theta_2)/a \cdot (\mathbb{F}_q^*)^{2n} : (\theta_1, \theta_2) \in \Upsilon_q\} \subset \mathbb{F}_q^*/(\mathbb{F}_q^*)^{2n}$$

and

$$\Phi_q = \begin{cases} \Phi'_q \cup \{(\omega/a) \cdot (\mathbb{F}_q^*)^{2n} : \omega \in \{\rho_1, \rho_1 g, -\rho_2, -\rho_2 g\}\} & \text{if } a_q(E)^2 \equiv (\text{mod } n) \\ \Phi'_q & \text{otherwise.} \end{cases}$$

If $\bar{\rho}_{F,n} \sim \bar{\rho}_{E,n}$, then necessarily

$$(29) \quad c' \cdot (\mathbb{F}_q^*)^{2n} \in \Phi_q.$$

Proof. Let $M = \mathbb{Q}(\sqrt{-d})$ and $\mathfrak{q} \mid q$ be the prime ideal of \mathcal{O}_M given by (26), so that $\mathcal{O}_M/\mathfrak{q} = \mathbb{F}_q$ and $\sqrt{-d} \equiv a \pmod{\mathfrak{q}}$. Let μ be as in Lemma 4.2. From (15) and its conjugate, we have

$$(30) \quad \frac{x + c'\sqrt{-d}}{2} \equiv \rho_1 \cdot \mu^n \pmod{\mathfrak{q}} \quad \text{and} \quad \frac{x - c'\sqrt{-d}}{2} \equiv \rho_2 \cdot \bar{\mu}^n \pmod{\mathfrak{q}}.$$

Suppose first that $q \nmid y$. Thus both F and E have good reduction at q , and so $a_q(F) \equiv a_q(E) \pmod{n}$. It follows from (17) that $\mathfrak{q} \nmid ((x \pm c'\sqrt{-d})/2)$ and that $\mathfrak{q} \nmid \mu, \bar{\mu}$. Recall that $g = g_0^n$ where g_0 is a generator for \mathbb{F}_q^* ; in particular, g is a non-square, it generates $(\mathbb{F}_q^*)^n$, and has order k . We note that the class of $\bar{\mu}^n$ modulo \mathfrak{q} is either in $(\mathbb{F}_q^*)^{2n}$ or in $g \cdot (\mathbb{F}_q^*)^{2n}$. Hence there is some $\phi \in (\mathbb{F}_q^*)^{2n}$ and some $0 \leq j \leq 1$ such that

$$\frac{x - c'\sqrt{-d}}{2} \equiv \rho_2 \cdot g^j \cdot \phi \pmod{\mathfrak{q}}.$$

Now the class of μ^n/ϕ modulo \mathfrak{q} belongs to $(\mathbb{F}_q^*)^n$ and so is equal to g^i for some $0 \leq i \leq k-1$. We note that

$$\frac{x + c'\sqrt{-d}}{2} \equiv \rho_1 \cdot g^i \cdot \phi \pmod{\mathfrak{q}}.$$

Hence

$$\left(\frac{x + c'\sqrt{-d}}{2}, \frac{x - c'\sqrt{-d}}{2} \right) \equiv (\theta_1 \cdot \phi, \theta_2 \cdot \phi) \pmod{\mathfrak{q}},$$

where $(\theta_1, \theta_2) \in \Upsilon''_q$. Since $\mathfrak{q} \nmid ((x \pm c'\sqrt{-d})/2)$, we see that $\theta_1 \theta_2 \neq 0$. Moreover, $\theta_1 - \theta_2 = c'\sqrt{-d}/\phi \in \mathbb{F}_q^*$. Thus $(\theta_1, \theta_2) \in \Upsilon'_q$. Now recall that the model for the Frey curve F in (19) is isomorphic to the model given in (23). The polynomial on the right hand-side of the latter model factors as in (28). Thus F/\mathbb{F}_q is isomorphic to the elliptic curve

$$Y^2 = X(X + \theta_1 \phi)(X + \theta_2 \phi).$$

As ϕ is a square in \mathbb{F}_q , we see that this elliptic curve is in turn isomorphic to the elliptic curve H_{θ_1, θ_2} . Then $a_q(F) = a_q(E) = a_q(H_{\theta_1, \theta_2})$. Since

$a_q(E) \equiv a_q(F) \pmod{n}$, it follows that $(\theta_1, \theta_2) \in \Upsilon_q$. Moreover,

$$c' = \frac{1}{\sqrt{-d}} \cdot \left(\frac{x + c'\sqrt{-d}}{2} - \frac{x - c'\sqrt{-d}}{2} \right) \equiv \frac{\theta_1 - \theta_2}{a} \cdot \phi \pmod{\mathfrak{q}}.$$

Since $\phi \in (\mathbb{F}_q^*)^{2n}$, this proves (29).

So far we have considered only the case $q \nmid y$. We know that if $q \mid y$, then

$$a_q(E) \equiv \pm(q+1) \equiv \pm 2 \pmod{n}.$$

Thus if $a_q(E)^n \not\equiv 4 \pmod{n}$, then $q \nmid y$ and the proof is complete. Suppose $a_q(E)^2 \equiv 4 \pmod{n}$ and that $q \mid y$. In particular, either $\mathfrak{q} \mid \mu$ or $\mathfrak{q} \mid \bar{\mu}$, but not both (by the coprimality of the factors on the right hand-side of (17)). Suppose $\mathfrak{q} \mid \bar{\mu}$. Then $x \equiv c'\sqrt{d} \pmod{\mathfrak{q}}$ and so from (30) we have

$$c' \equiv \frac{\rho_1}{\sqrt{-d}} \cdot \mu^n \equiv \frac{\rho_1}{a} \cdot \mu^n \pmod{\mathfrak{q}}.$$

However, the class of μ^n modulo \mathfrak{q} belongs to either $(\mathbb{F}_q^*)^{2n}$ or $g \cdot (\mathbb{F}_q^*)^{2n}$, establishing (29). The case $\mathfrak{q} \mid \mu$ is similar. This completes the proof. \square

Lemma 5.6. *Let (E, d, n) be one of the remaining 1230 triples. Let $q = kn + 1$ be a prime. Suppose that $(-d/q) = -1$. Let $M = \mathbb{Q}(\sqrt{-d})$ and let $\mathfrak{q} = q\mathcal{O}_M$. Write $\mathbb{F}_{\mathfrak{q}} = \mathcal{O}_M/\mathfrak{q} \cong \mathbb{F}_{q^2}$. Let $c', h, (r, s), \kappa_n$ be as in Lemma 4.2, and set $m = (2 + \kappa_n \cdot n)/h \in \mathbb{Z}$. Define $\rho_1 = (r + sa)^m$, choose g_0 to be a generator for $\mathbb{F}_{\mathfrak{q}}^*$, and set $g = g_0^n$. Define*

$$\Upsilon_q'' = \{\rho_1 \cdot g^i : i = 0, 1, \dots, 2q+1\} \subset \mathbb{F}_{\mathfrak{q}}^*,$$

$$\Upsilon_q' = \{\theta \in \Upsilon_q'' : \theta \neq \theta^q\}$$

and

$$\Upsilon_q = \{\theta \in \Upsilon_q' : a_q(H_{\theta}) \equiv a_q(E) \pmod{n}\},$$

where H_{θ}/\mathbb{F}_q is the elliptic curve

$$H_{\theta} : Y^2 = X(X + \theta)(X + \theta^q).$$

Let

$$\Phi_q = \left\{ (\theta - \theta^q)/\sqrt{-d} \cdot (\mathbb{F}_q^*)^{2n} : \theta \in \Upsilon_q \right\} \subset \mathbb{F}_q^*/(\mathbb{F}_q^*)^{2n}.$$

If $\bar{\rho}_{F,n} \sim \bar{\rho}_{E,n}$ then necessarily (29) holds.

Proof. We note that in $\mathbb{F}_{\mathfrak{q}}$ Galois conjugation agrees with the action of Frobenius. Thus if $\alpha \in \mathcal{O}_M$ and $\bar{\alpha}$ denotes its conjugate, then $\bar{\alpha} \equiv \alpha^q \pmod{\mathfrak{q}}$.

Since $(-d/q) = -1$ and $x^2 + c^2d = y^n$ we observe that $q \nmid y$. Thus F and E both have good reduction at q , and so $a_q(F) \equiv a_q(E) \pmod{n}$. Let μ be as in Lemma 4.2. Thus $\mathfrak{q} \nmid \mu, \bar{\mu}$. Recall that $g = g_0^n$ where g_0 is a generator for $\mathbb{F}_{\mathfrak{q}}^*$, whence $\mu^n \equiv g^j$ for some integer j . From (15),

$$\frac{x + c'\sqrt{-d}}{2} \equiv \rho_1 \cdot g^j \pmod{\mathfrak{q}} \quad \text{and} \quad \frac{x - c'\sqrt{-d}}{2} \equiv (\rho_1 \cdot g^j)^q \pmod{\mathfrak{q}}.$$

Write $j = i + (2q+2)t$, where $i \in \{0, 1, \dots, 2q+1\}$ and t is an integer. We note that

$$g^{2q+2} = (g_0^{q+1})^{2n}.$$

Moreover, $g_0^{q+1} = g_0 g_0^q \in \mathbb{F}_q^*$. Thus there is some $\theta \in \Upsilon_q''$ and some $\phi \in (\mathbb{F}_q^*)^{2n}$ such that

$$\frac{x + c'\sqrt{-d}}{2} \equiv \theta \cdot \phi \pmod{\mathfrak{q}} \quad \text{and} \quad \frac{x - c'\sqrt{-d}}{2} \equiv \theta^q \cdot \phi \pmod{\mathfrak{q}}.$$

Since $\mathfrak{q} \nmid c'\sqrt{-d}$, we see that $\theta \neq \theta^q$ and so $\theta \in \Upsilon_q'$. We note that the model for F in (23) can, over \mathbb{F}_q , be written as

$$Y^2 = X(X^2 + \phi \cdot (\theta + \theta^q)X + \phi \cdot (\theta\theta^q)),$$

where the coefficients are fixed by Frobenius and so do indeed belong to \mathbb{F}_q . This model is a twist by ϕ of H_θ . As ϕ is a square in \mathbb{F}_q^* , we have $a_q(H_\theta) = a_q(F) \equiv a_q(E) \pmod{n}$. Thus $\theta \in \Upsilon_q$. Finally,

$$c' = \frac{1}{\sqrt{-d}} \cdot \left(\frac{x + c'\sqrt{-d}}{2} - \frac{x - c'\sqrt{-d}}{2} \right) \equiv \frac{\theta - \theta^q}{\sqrt{-d}} \cdot \phi \pmod{\mathfrak{q}}.$$

Since $\phi \in (\mathbb{F}_q^*)^{2n}$, this proves (29). \square

Lemma 5.7. *Let (E, d, n) be one of the remaining 1230 triples. Let q_1, q_2, \dots, q_r be primes satisfying $q_i \equiv 1 \pmod{n}$. Let*

$$(31) \quad \psi_q : (\mathbb{Z}/2n\mathbb{Z})^4 \rightarrow \mathbb{F}_q^*/(\mathbb{F}_q^*)^{2n},$$

$$(32) \quad \text{psi}_q(x_1, x_2, x_3, x_4) = (-3)^{x_1} 5^{x_2} (-7)^{x_3} (-11)^{x_4} \cdot (\mathbb{F}_q^*)^{2n}.$$

If $(-d/q) = 1$, let Φ_{q_i} be as in Lemma 5.5 and if $(-d/q) = -1$, let Φ_{q_i} be as in Lemma 5.6. Suppose

$$\bigcap_{i=1}^r \psi_{q_i}^{-1}(\Phi_{q_i}) = \emptyset.$$

Then $\bar{\rho}_{F,n} \approx \bar{\rho}_{E,n}$.

Proof. Recall, from (11) and (13), that

$$c = 3^{\beta_3} 5^{\beta_5} 7^{\beta_7} 11^{\beta_{11}}.$$

Thus $c \equiv (-1)^{\beta_3 + \beta_7 + \beta_{11}} \pmod{4}$ and hence, since we choose $c' = \pm c$ so that $c' \equiv 1 \pmod{4}$,

$$c' = (-1)^{\beta_3 + \beta_7 + \beta_{11}} \cdot 3^{\beta_3} 5^{\beta_5} 7^{\beta_7} 11^{\beta_{11}} = (-3)^{\beta_3} 5^{\beta_5} (-7)^{\beta_7} (-11)^{\beta_{11}}.$$

Suppose $\bar{\rho}_{F,n} \sim \bar{\rho}_{E,n}$. Thus

$$\psi_q(\beta_3, \beta_5, \beta_7, \beta_{11}) = c' \cdot (\mathbb{F}_{q_i}^*)^{2n} \in \Phi_{q_i}$$

by (29). Therefore

$$((\beta_3, \beta_5, \beta_7, \beta_{11}) \pmod{2n}) \in \bigcap_{i=1}^r \psi_{q_i}^{-1}(\Phi_{q_i})$$

giving a contradiction. \square

We wrote a **Magma** script which for each of the 1230 remaining triples (E, d, n) recursively computes the intersections

$$\psi_{q_1}^{-1}(\Phi_{q_1}), \quad \bigcap_{i=1}^2 \psi_{q_i}^{-1}(\Phi_{q_i}), \quad \bigcap_{i=1}^3 \psi_{q_i}^{-1}(\Phi_{q_i}), \dots$$

where the q_i are primes $\equiv 1 \pmod{n}$. It stops when the intersection is empty, or when we have used 200 primes q_i , whichever comes first. If the intersection is

TABLE 2. This table gives the six triples (E, d, n) such that the intersection $\bigcap_{i=1}^{200} \psi_{q_i}^{-1}(\Phi_{q_i})$ is non-empty. Here the elliptic curve E is given in the first column in Cremona notation. We note that $n = 13$ for all six triples. Therefore the intersection given in the last column is a subset of $(\mathbb{Z}/26\mathbb{Z})^4$.

Elliptic Curve	d	n	$\bigcap_{i=1}^{200} \psi_{q_i}^{-1}(\Phi_{q_i})$
462b1	231	13	$\{ (7, 2, 19, 3), (9, 1, 24, 9) \}$
462f1	231	13	$\{ (0, 15, 25, 13), (15, 18, 5, 0) \}$
2310j1	231	13	$\{ (11, 6, 6, 18), (24, 19, 19, 5) \}$
2310l1	231	13	$\{ (10, 5, 22, 8) \}$
2310m1	231	13	$\{ (5, 14, 11, 21), (7, 21, 19, 19) \}$
2310o1	15	13	$\{ (1, 0, 1, 1) \}$

empty, then we know from Lemma 5.7 that $\bar{\rho}_{F,n} \approx \bar{\rho}_{E,n}$ and we may eliminate the particular triple (E, d, n) from further consideration. We reached an empty intersection in 1224 cases. Table 2 gives the details for the six triples (E, d, n) where the intersection is non-empty.

5.3. Proof of Proposition 4.1. We now complete the proof of Proposition 4.1. To summarise, Lemma 4.3 showed that the only solutions to (11) with exponent $n \in \{5, 7, 11\}$ are the ones given in the statement of Proposition 4.1. In view of the results of this section, it only remains to consider the six triples (E, d, n) given in Table 2. To eliminate further cases, we make use of the following result of Halberstadt and Kraus [17, Lemme 1.6].

Theorem 5 (Halberstadt and Kraus). *Let E_1 and E_2 be elliptic curves over \mathbb{Q} and write Δ_j for the minimal discriminant of E_j . Let $n \geq 5$ be a prime such that $\bar{\rho}_{E_1,n} \sim \bar{\rho}_{E_2,n}$. Let $q_1, q_2 \neq n$ be distinct primes of multiplicative reduction for both elliptic curves such that $\text{ord}_{q_i}(\Delta_j) \not\equiv 0 \pmod{n}$ for $i, j \in \{1, 2\}$. Then*

$$\frac{\text{ord}_{q_1}(\Delta_1) \cdot \text{ord}_{q_2}(\Delta_1)}{\text{ord}_{q_1}(\Delta_2) \cdot \text{ord}_{q_2}(\Delta_2)}$$

is congruent to a square modulo n .

We shall use Theorem 5 and Lemma 5.1 to eliminate the first five of the six outstanding triples (E, d, n) given in Table 2. In all these cases $n = 13$. We know from the proof of Lemma 5.7 that $(\beta_3, \beta_5, \beta_7, \beta_{11}) \pmod{26}$ belongs to the intersection in the last column of Table 1.

Consider the first triple, corresponding to the first row of the table. The $\beta_5 \equiv 1$ or $2 \pmod{26}$. But $\beta_5 = \text{ord}_5(c)$. Thus $2 \text{ord}_5(c) + \text{ord}_5(d) \equiv 2\beta_5 + \text{ord}_5(231) \equiv 2$ or $4 \pmod{13}$ and so by Lemma 5.1, 5 must divide the conductor of E which is 462 giving a contradiction. The same argument eliminates the second triple.

Next we consider the third triple. Here $\beta_7 \equiv 6$ or $19 \pmod{26}$, and so $\text{ord}_7(c) \equiv \beta_7 \equiv 6 \pmod{13}$. Then $2 \text{ord}_7(c) + \text{ord}_7(d) \equiv 2\beta_7 + \text{ord}_7(231) \equiv 0 \pmod{13}$. By Lemma 5.1, 7 does not divide the conductor of E which is 2310, again a contradiction.

We next consider the fourth triple. Here the elliptic curve E with Cremona reference 231011 has minimal discriminant

$$\Delta_E = 2^4 \times 3^{12} \times 5^3 \times 7 \times 11.$$

We apply Theorem 5 with $E_1 = F$, $E_2 = E$, $q_1 = 2$ and $q_2 = 3$. From the proof of Lemma 5.1 we have

$$\begin{aligned} \text{ord}_2(\Delta_F) &\equiv -12 \equiv 1 \pmod{13}, \\ \text{ord}_3(\Delta_F) &= 2\beta_3 + \text{ord}_3(231) \equiv 2 \times 10 + 1 \equiv 8 \pmod{13}. \end{aligned}$$

Hence

$$\frac{\text{ord}_2(\Delta_F) \cdot \text{ord}_3(\Delta_F)}{\text{ord}_2(\Delta_E) \cdot \text{ord}_3(\Delta_E)} \equiv \frac{1 \times 8}{4 \times 12} \equiv 11 \pmod{13}$$

which is a non-square modulo 13, contradicting Theorem 5.

Next we consider the fifth triple. Here there are two possibilities for $(\beta_3, \beta_5, \beta_7, \beta_{11})$. In the second possibility we have $\beta_7 \equiv 19 \pmod{26}$ which leads to a contradiction via Lemma 5.1. We focus on the first possibility. The minimal discriminant of the curve E is

$$\Delta_E = 2^4 \times 3^8 \times 5 \times 7^3 \times 11.$$

We obtain a contradiction by applying Theorem 5 with $q_1 = 2$ and $q_2 = 3$.

We are left with the last triple, which we have been unable to eliminate by appealing to Theorem 5 or Lemma 5.1, or by further sieving. In fact, (11) has the solution

$$(33) \quad 8143^2 + 3^3 \cdot 5 \cdot 7^2 \cdot 11^2 = 4^{13}.$$

Here $n = 13$, $d = 15$ and $c = 3 \cdot 7 \cdot 11$. We note that the vector of exponents for this value of c is $(\beta_3, \beta_5, \beta_7, \beta_{11}) = (1, 0, 1, 1)$ which agrees with the prediction in the last column of the table. Moreover, letting $x = -8143 \equiv 1 \pmod{4}$, and $y^n = 4^{13}$ in the Frey curve F gives the elliptic curve 2310o1. To complete the proof, we need to solve (11) with $d = 15$ and $n = 13$. We do this by reducing this case to a Thue-Mahler equation using the approach in the proof of Lemma 4.3. After possibly changing the sign of x so that $x \equiv 1 \pmod{4}$, we have that

$$\frac{x + c'\sqrt{-15}}{2} = \left(\frac{1 - \sqrt{-15}}{8} \right) \left(r + s \cdot \frac{(1 + \sqrt{-15})}{2} \right)^{13},$$

where $y = r^2 + rs + 4s^2$ for some integers r and s . Equating imaginary parts leads to the conclusion that

$$F_{13}(r, s) = \sum_{i=0}^{13} a_i r^{13-i} s^i = \pm 4 \cdot 3^{\beta_3} \cdot 5^{\beta_5} \cdot 7^{\beta_7} \cdot 11^{\beta_{11}},$$

where

i	a_i	i	a_i	i	a_i
0	1	5	36036	10	195624
1	0	6	-34320	11	-95160
2	-312	7	-226512	12	-51428
3	-1144	8	-66924	13	924.
4	8580	9	340340		

We solved this Thue-Mahler equation using the **Magma** package associated to the paper [16]. The only solution is with

$$r = 0, \quad s = \pm 1, \quad \beta_3 = 1, \quad \beta_5 = 0, \quad \beta_7 = 1 \quad \text{and} \quad \beta_{11} = 1.$$

This corresponds to the identity (33) and completes the proof of Proposition 4.1.

Remark. It is natural to ask if the case $n = 13$ could have been dealt with entirely using the Thue-Mahler approach, just as we did for $n \in \{5, 7, 11\}$ in Lemma 4.3. The Thue-Mahler solver that we are using can quickly deal with the Thue-Mahler equations associated to the pairs $(d, n) = (7, 13)$ and $(55, 13)$. However, the Thue-Mahler equation for the pair $(d, n) = (231, 13)$ appears to be somewhat beyond its capabilities. The approach in [16] reduces solving a Thue-Mahler equation to a certain number of S -unit equations. By way of example, the Thue-Mahler equation for the pair $(d, n) = (15, 13)$ reduces to solving four S -unit equations. The Thue-Mahler equation for the pair $(d, n) = (231, 13)$, on the other hand, corresponds to 2240 S -unit equations. This explains the effort we invested into eliminating $(d, n) = (231, 13)$ via sieving and appeal to Theorem 5 and Lemma 5.1.

6. EQUATION (2) WITH y EVEN : LARGE EXPONENTS

From the results of the preceding sections, it remains to solve equation (2) with y even and exponent n prime and

$$(34) \quad n > N(d),$$

where $N(d)$ is as defined in (12). We will accomplish this through (quite careful) application of bounds for linear forms in logarithms.

6.1. Upper bounds for n : linear forms in logarithms, complex and q -adic.

Our first order of business will be to produce an upper bound for the exponent n ; initially it will be somewhat larger than $N(d)$. To this end, as it transpires, it will prove useful to have at our disposal a lower bound upon y . From the discussion following Lemma 5.2, we have that $\bar{\rho}_{E,n} \sim \bar{\rho}_{E,n}$ for E/\mathbb{Q} with nontrivial rational 2-torsion.

To begin, we will need to treat the case where y in equation (2) has no odd prime divisors. Suppose that we have a solution to equation (11) with $y = 2^\kappa$ for κ a positive integer. For the time being, we will relax our assumptions upon n and suppose only that $n \geq 7$ is prime. Then the Frey-Hellegouarch curve F has nontrivial rational 2-torsion and conductor

$$N = 2 \cdot 3^{\delta_3} 5^{\delta_5} 7^{\delta_7} 11^{\delta_{11}} \quad \text{where} \quad \delta_i \in \{0, 1\},$$

so that

$$N \in \{14, 30, 42, 66, 70, 154, 210, 330, 462, 770, 2310\},$$

and minimal discriminant

$$-2^{2\kappa n - 12} 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} 11^{\alpha_{11}}.$$

A quick check of Cremona's tables reveals that we find such curves with minimal discriminant negative and divisible by precisely $2^{2\kappa n - 12}$, with $n \geq 7$ prime, only for 18 isomorphism classes of curves, given, in Cremona's notation, by

$$14a4, 210b5, 210e1, 210e6, 330c1, 330c6, 330e4, 462a1, 462d1, 462e1, \\ 462g3, 770a1, 770e1, 770g3, 2310d4, 2310n1, 2310n6, 2310o1.$$

Most of these have $2\kappa n - 12 = 2$ and so $\kappa = 1$ and $n = 7$. Since $P(2^7 - x^2) > 11$ for $1 \leq x < 11$ odd, only the curve 14a4 with $\Delta = -2^2 \cdot 7$ corresponds to a solution, arising from the identity $11^2 + 7 = 2^7$. Four more curves have $2\kappa n - 12 = 16$ and so $\kappa = 2$ and $n = 7$. Corresponding identities are

$$7^2 + 3^3 \cdot 5 \cdot 11^2 = 2^{14}, \quad 47^2 + 3^4 \cdot 5^2 \cdot 11 = 2^{14}, \quad 103^2 + 3 \cdot 5^2 \cdot 7 \cdot 11 = 2^{14}, \quad 117^2 + 5 \cdot 7^2 \cdot 11 = 2^{14},$$

arising from the curves 330c1, 210e1, 2310n1 and 770e1, with discriminants

$$-2^{16} \cdot 3^3 \cdot 5 \cdot 11^2, \quad -2^{16} \cdot 3^4 \cdot 5^2 \cdot 7, \quad -2^{16} \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \quad \text{and} \quad -2^{16} \cdot 5 \cdot 7^2 \cdot 11,$$

respectively. Neither 462d1 nor 462e1 lead to any solutions while 2310o1, with discriminant $-2^{40} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11^2$, corresponds to the identity

$$8143^2 + 3^3 \cdot 5 \cdot 7^2 \cdot 11^2 = 2^{26}.$$

We may thus suppose that y is divisible by an odd prime factor, provided $n \geq 17$.

Lemma 6.1. *If $n \geq 17$ and y is even, we have*

$$y > 4n - 4\sqrt{2n} + 2.$$

Proof. By our preceding remarks, there necessarily exists an odd prime $p \mid y$. Since $\bar{\rho}_{F,n} \sim \bar{\rho}_{E,n}$ where E/\mathbb{Q} has nontrivial rational 2-torsion, the fact that $\gcd(x, y) = 1$ thus allows us to conclude that

$$a_p(E) \equiv \pm(p+1) \pmod{n}.$$

From the Hasse-Weil bounds, we have that $a_p(E)$ is bounded in modulus by $2\sqrt{p}$, so that, using the fact that $a_p(E)$ is even,

$$n < \frac{1}{2}(\sqrt{p} + 1)^2 \leq \frac{1}{2}(\sqrt{y/2} + 1)^2.$$

The desired inequality follows. \square

As before, define c and d via (13), where, since y is even, $d \in \{7, 15, 55, 231\}$, and let $c' = \pm c$ with the sign chosen so that $c' \equiv 1 \pmod{4}$. To derive an upper bound upon n , we will begin by using (24) to find a “small” linear form in logarithms. Specifically, let us define

$$(35) \quad \Lambda = \log \left(\frac{x + c' \sqrt{-d}}{x - c' \sqrt{-d}} \right).$$

We prove

Lemma 6.2. *If we suppose that*

$$(36) \quad y^n > 100 c^2 d,$$

then

$$\log |\Lambda| < 0.75 + \log c + \frac{1}{2} \log d - \frac{n}{2} \log y.$$

Proof. Assumption (36), together with, say, Lemma B.2 of Smart [33], implies that

$$|\Lambda| \leq -10 \log(9/10) \left| \frac{x + c' \sqrt{-d}}{x - c' \sqrt{-d}} - 1 \right| = 20 \log(10/9) \frac{c\sqrt{d}}{y^{n/2}},$$

whence the lemma follows. \square

To show that $\log |\Lambda|$ here is indeed small, we first require an upper bound upon the exponents α_q in equation (11). From (24), we have that

$$(37) \quad \frac{2 \cdot c' \sqrt{-d}}{x - c' \sqrt{-d}} = \begin{cases} \gamma \cdot \delta^n - 1 & \text{if } d \in \{7, 15, 55\} \\ \gamma^{(2+\epsilon_n \cdot n)/3} \cdot \delta^n - 1 & \text{if } d = 231. \end{cases}$$

For prime q , let $\overline{\mathbb{Q}_q}$ denote an algebraic closure of the q -adic field \mathbb{Q}_q , and define ν_q to be the unique extension to $\overline{\mathbb{Q}_q}$ of the standard q -adic valuation over \mathbb{Q}_q , normalized so that $\nu_q(q) = 1$. For any algebraic number α of degree d over \mathbb{Q} , we define the *absolute logarithmic height* of α via the formula

$$(38) \quad h(\alpha) = \frac{1}{d} \left(\log |a_0| + \sum_{i=1}^d \log \max \left(1, |\alpha^{(i)}| \right) \right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over \mathbb{Z} and the $\alpha^{(i)}$ are the conjugates of α in \mathbb{C} . Since $\gcd(x, q) = 1$, it follows from (37) that, if we set

$$\Lambda_1 = \begin{cases} \delta^n - (1/\gamma) & \text{if } d \in \{7, 15, 55\} \\ \delta^n - (1/\gamma)^{(2+\epsilon_n \cdot n)/3} & \text{if } d = 231, \end{cases}$$

then $\nu_q(\Lambda_1) \geq \alpha_q/2$, for $q \in \{3, 5, 7, 11\}$.

To complement this with an upper bound for linear forms in q -adic logarithms, we will appeal to Théorème 4 of Bugeaud and Laurent [11], with, in the notation of that result, the choices $(\mu, \nu) = (10, 5)$.

Theorem 6 (Bugeaud-Laurent). *Let q be a prime number and let α_1, α_2 denote algebraic numbers which are q -adic units. Let f be the residual degree of the extension $\mathbb{Q}_q(\alpha_1, \alpha_2)/\mathbb{Q}_q$ and put $D = [\mathbb{Q}_q(\alpha_1, \alpha_2) : \mathbb{Q}_q]/f$. Let b_1 and b_2 be positive integers and put*

$$\Lambda_1 = \alpha_1^{b_1} - \alpha_2^{b_2}.$$

Denote by $A_1 > 1$ and $A_2 > 1$ real numbers such that

$$\log A_i \geq \max \left\{ h(\alpha_i), \frac{\log q}{D} \right\}, \quad i \in \{1, 2\},$$

and put

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

If α_1 and α_2 are multiplicatively independent, then we have the bound

$$\nu_q(\Lambda_1) \leq \frac{24q(q^f - 1)}{(q - 1) \log^4(q)} D^4 \left(\max \left\{ \log b' + \log \log q + 0.4, \frac{10 \log q}{D}, 5 \right\} \right)^2 \cdot \log A_1 \cdot \log A_2.$$

We will choose $q \in \{3, 5, 7, 11\}$ and apply this result with the following choices of parameters :

$$f = 1, \quad D = 2, \quad \alpha_1 = \delta, \quad \alpha_2 = 1/\gamma, \quad b_1 = n$$

and

$$(39) \quad b_2 = \begin{cases} 1 & \text{if } d \in \{7, 15, 55\} \\ (2 + \epsilon_n \cdot n)/3 & \text{if } d = 231. \end{cases}$$

We have

$$h(1/\gamma) = \begin{cases} \log 2 & \text{if } d \in \{7, 15, 55\} \\ \frac{3}{2} \log 2 & \text{if } d = 231 \end{cases}$$

and

$$h(\delta) \leq \frac{1}{2} \log(y/2),$$

and hence, from (34) and Lemma 6.1, may choose

$$\log A_1 = \frac{1}{2} \log(y/2) < \frac{1}{2} \log y$$

and

$$\log A_2 = \begin{cases} \log 2 & \text{if } d \in \{7, 15, 55\} \text{ and } q = 3, \\ \frac{3}{2} \log 2 & \text{if } d = 231 \text{ and } q \in \{3, 5, 7\}, \\ \frac{1}{2} \log q & \text{otherwise.} \end{cases}$$

Once again appealing to (34) and Lemma 6.1, we have, in all cases, that $b' > 5 \log q$ and

$$b' \leq \frac{n}{2 \log 2} + \frac{1}{\log y} < 0.722n.$$

We thus have

$$\log b' + \log \log q + 0.4 < 1.05 \log n,$$

whence, from Theorem 6,

$$\nu_q(\Lambda_1) < c(d, q) \cdot 1.05^2 \log^2 n \log y,$$

where

$$c(d, q) = \begin{cases} \frac{576 \log 2}{\log^4 3} & \text{if } d \in \{7, 15, 55\} \text{ and } q = 3, \\ \frac{288q \log 2}{\log^4 q} & \text{if } d = 231 \text{ and } q \in \{3, 5, 7\}, \\ \frac{96q}{\log^3 q} & \text{otherwise.} \end{cases}$$

It follows that

$$(40) \quad \sum_{q \in \{3, 5, 7, 11\}} \alpha_q \log q < C(d) \cdot 1.05^2 \log^2 n \log y,$$

where

$$C(d) = 2 \sum_{q \in \{3, 5, 7, 11\}} c(d, q) \log q.$$

We have

$$C(7) = C(15) = C(55) = 2 \left(\frac{576 \log 2}{\log^3 3} + \frac{480}{\log^2 5} + \frac{672}{\log^2 7} + \frac{1056}{\log^2 11} \right) < 1696$$

and

$$C(231) = 2 \left(\frac{864 \log 2}{\log^3 3} + \frac{1440 \log 2}{\log^3 5} + \frac{2016 \log 2}{\log^3 7} + \frac{1056}{\log^2 11} \right) < 2129.$$

From (24) and (35), we can write

$$(41) \quad \Lambda = n \log(\tau\delta) + b_2 \log(\gamma) + j\pi i,$$

with b_2 as in (39), while, if $d = 231$, we also have

$$(42) \quad \Lambda' = 3\Lambda = n \log(\tau'\delta^3\gamma^{\epsilon_n}) + 2 \log(\gamma) + j'\pi i.$$

In each case, we take the principal branches of the logarithms, choose $\tau, \tau' \in \{-1, 1\}$ so that $\text{Im}(\log(\tau\delta))$ and $\text{Im}(\log(\tau'\delta^3\gamma^{\epsilon_n}))$ have opposite signs to $\text{Im}(\log(\gamma))$, and take integers j and j' so that $|\Lambda|$ and $|\Lambda'|$ are minimal. Notice that, with these choices,

$$(43) \quad n|\log(\tau\delta)| = |\log(\gamma)| + |j|\pi \pm |\Lambda|, \quad \text{if } d \in \{7, 15, 55\}$$

and

$$(44) \quad n |\log(\tau' \delta^3 \gamma^{\epsilon_n})| = 2 |\log(\gamma)| + |j'| \pi \pm |\Lambda'|.$$

Note further that we have

d	$ \log(\gamma) $
7	$\arccos(1/8)$
15	$\arccos(7/8)$
55	$\arccos(3/8)$
231	$\arccos(5/16)$

Assume first that inequality (36) fails to hold. Then, from (40), we have

$$n < \frac{2 \log 10}{\log y} + C(d) \cdot 1.05^2 \log^2 n,$$

contradicting Lemma 6.1, (34) and $C(d) < 2129$. It follows that inequality (36) holds and hence we may conclude, from Lemma 6.2, that

$$\log |\Lambda| < 0.75 + \frac{1}{2} C(d) \cdot 1.05^2 \log^2 n \log y - \frac{n}{2} \log y.$$

From Lemma 6.1, (34) and $C(d) < 2129$, we find, in all cases, that

$$(45) \quad \log |\Lambda| < -0.499 n \log y.$$

6.1.1. Linear forms in three logarithms. To deduce an initial lower bound upon the linear form in logarithms $|\Lambda|$, we will use the following, the main result (Theorem 2.1) of Matveev [23].

Theorem 7 (Matveev). *Let \mathbb{K} be an algebraic number field of degree D over \mathbb{Q} and put $\chi = 1$ if \mathbb{K} is real, $\chi = 2$ otherwise. Suppose that $\alpha_1, \alpha_2, \dots, \alpha_{n_0} \in \mathbb{K}^*$ with absolute logarithmic heights $h(\alpha_i)$ for $1 \leq i \leq n_0$, and suppose that*

$$A_i \geq \max\{D h(\alpha_i), |\log \alpha_i|\}, \quad 1 \leq i \leq n_0,$$

for some fixed choice of the logarithm. Define

$$\Lambda = b_1 \log \alpha_1 + \dots + b_{n_0} \log \alpha_{n_0},$$

where the b_i are integers and set

$$B = \max\{1, \max\{|b_i| A_i / A_{n_0} : 1 \leq i \leq n_0\}\}.$$

Define, with $e := \exp(1)$, further,

$$\Omega = A_1 \cdots A_{n_0},$$

$$C(n_0) = C(n_0, \chi) = \frac{16}{n_0! \chi} e^{n_0} (2n_0 + 1 + 2\chi)(n_0 + 2)(4n_0 + 4)^{n_0+1} (en_0/2)^\chi,$$

$$C_0 = \log(e^{4.4n_0+7} n_0^{5.5} D^2 \log(eD)) \quad \text{and} \quad W_0 = \log(1.5eBD \log(eD)).$$

Then, if $\log \alpha_1, \dots, \log \alpha_{n_0}$ are linearly independent over \mathbb{Z} and $b_{n_0} \neq 0$, we have

$$\log |\Lambda| > -C(n_0) C_0 W_0 D^2 \Omega.$$

We apply Theorem 7 to Λ as given in (41), with

$$D = 2, \chi = 2, n_0 = 3, b_3 = n, \alpha_3 = \tau\delta, \alpha_2 = \gamma, b_1 = j, \alpha_1 = -1,$$

and b_2 as in (39).

We may thus take

$$A_3 = \log y, A_2 = 3 \log 2, A_1 = \pi \text{ and } B = n.$$

Since

$$4C(3)C_0 = 2^{18} \cdot 3 \cdot 5 \cdot 11 \cdot e^5 \cdot \log(e^{20.2} \cdot 3^{5.5} \cdot 4 \log(2e)) < 1.80741 \times 10^{11},$$

and

$$W_0 = \log(3en \log(2e)) < 2.63 + \log n,$$

we may therefore conclude that

$$\log |\Lambda| > -1.181 \times 10^{12} (2.63 + \log n) \log y.$$

It thus follows from (45) that

$$n < 2.37 \times 10^{12} (\log n + 2.63),$$

whence

$$(46) \quad n < 8.22 \times 10^{13}.$$

To improve this inequality, we appeal to a sharper but less convenient lower bound for linear forms in three complex logarithms, due to Mignotte ([25, Theorem 2]).

Theorem 8 (Mignotte). *Consider three non-zero algebraic numbers α_1, α_2 and α_3 , which are either all real and > 1 , or all complex of modulus one and all $\neq 1$. Further, assume that the three numbers α_1, α_2 and α_3 are either all multiplicatively independent, or that two of the numbers are multiplicatively independent and the third is a root of unity. We also consider three positive rational integers b_1, b_2, b_3 with $\gcd(b_1, b_2, b_3) = 1$, and the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3,$$

where the logarithms of the α_i are arbitrary determinations of the logarithm, but which are all real or all purely imaginary. We assume that

$$0 < |\Lambda| < 2\pi/w,$$

where w is the maximal order of a root of unity in $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Suppose further that

$$(47) \quad b_2 |\log \alpha_2| = b_1 |\log \alpha_1| + b_3 |\log \alpha_3| \pm |\Lambda|$$

and put

$$d_1 = \gcd(b_1, b_2), \quad d_3 = \gcd(b_3, b_2) \text{ and } b_2 = d_1 b'_2 = d_3 b''_2.$$

Let $K, L, R, R_1, R_2, R_3, S, S_1, S_2, S_3, T, T_1, T_2, T_3$ be positive rational integers with

$$K \geq 3, L \geq 5, R > R_1 + R_2 + R_3, S > S_1 + S_2 + S_3 \text{ and } T > T_1 + T_2 + T_3.$$

Let $\rho \geq 2$ be a real number. Let a_1, a_2 and a_3 be real numbers such that

$$a_i \geq \rho |\log \alpha_i| - \log |\alpha_i| + 2D h(\alpha_i), \quad i \in \{1, 2, 3\},$$

where $D = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}] / [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}]$, and set

$$U = \left(\frac{KL}{2} + \frac{L}{4} - 1 - \frac{2K}{3L} \right) \log \rho.$$

Assume further that

$$(48) \quad U \geq (D+1) \log(K^2 L) + gL(a_1 R + a_2 S + a_3 T) + D(K-1) \log b - 2 \log(e/2),$$

where

$$g = \frac{1}{4} - \frac{K^2 L}{12 R S T} \quad \text{and} \quad b = (b'_2 \eta_0)(b''_2 \zeta_0) \left(\prod_{k=1}^{K-1} k! \right)^{-\frac{4}{K(K-1)}},$$

with

$$\eta_0 = \frac{R-1}{2} + \frac{(S-1)b_1}{2b_2} \quad \text{and} \quad \zeta_0 = \frac{T-1}{2} + \frac{(S-1)b_3}{2b_2}.$$

Put

$$\mathcal{V} = \sqrt{(R_1+1)(S_1+1)(T_1+1)}.$$

If, for some positive real number χ , we have

- (i) $(R_1+1)(S_1+1)(T_1+1) > K\mathcal{M}$,
- (ii) $\text{Card}\{\alpha_1^r \alpha_2^s \alpha_3^t : 0 \leq r \leq R_1, 0 \leq s \leq S_1, 0 \leq t \leq T_1\} > L$,
- (iii) $(R_2+1)(S_2+1)(T_2+1) > 2K^2$,
- (iv) $\text{Card}\{\alpha_1^r \alpha_2^s \alpha_3^t : 0 \leq r \leq R_2, 0 \leq s \leq S_2, 0 \leq t \leq T_2\} > 2KL$, and
- (v) $(R_3+1)(S_3+1)(T_3+1) > 6K^2 L$,

where

$$\mathcal{M} = \max\left\{R_1 + S_1 + 1, S_1 + T_1 + 1, R_1 + T_1 + 1, \chi \mathcal{V}\right\},$$

then either

$$(49) \quad |\Lambda| \cdot \frac{L S e^{L S |\Lambda| / (2b_2)}}{2|b_2|} > \rho^{-KL},$$

or at least one of the following conditions **(C1)**, **(C2)**, **(C3)** holds :

$$\textbf{(C1).} \quad |b_1| \leq R_1 \text{ and } |b_2| \leq S_1 \text{ and } |b_3| \leq T_1,$$

$$\textbf{(C2).} \quad |b_1| \leq R_2 \text{ and } |b_2| \leq S_2 \text{ and } |b_3| \leq T_2,$$

(C3). *either* there exist non-zero rational integers r_0 and s_0 such that

$$(50) \quad r_0 b_2 = s_0 b_1$$

with

$$(51) \quad |r_0| \leq \frac{(R_1+1)(T_1+1)}{\mathcal{M} - T_1} \quad \text{and} \quad |s_0| \leq \frac{(S_1+1)(T_1+1)}{\mathcal{M} - T_1},$$

or there exist rational integers r_1, s_1, t_1 and t_2 , with $r_1 s_1 \neq 0$, such that

$$(52) \quad (t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1,$$

which also satisfy

$$|r_1 s_1| \leq \gcd(r_1, s_1) \cdot \frac{(R_1+1)(S_1+1)}{\mathcal{M} - \max\{R_1, S_1\}},$$

$$|s_1 t_1| \leq \gcd(r_1, s_1) \cdot \frac{(S_1+1)(T_1+1)}{\mathcal{M} - \max\{S_1, T_1\}}$$

and

$$|r_1 t_2| \leq \gcd(r_1, s_1) \cdot \frac{(R_1 + 1)(T_1 + 1)}{\mathcal{M} - \max\{R_1, T_1\}}.$$

Moreover, when $t_1 = 0$ we can take $r_1 = 1$, and when $t_2 = 0$ we can take $s_1 = 1$.

We will apply this result to our Λ (if $d \in \{7, 15, 55\}$) or Λ' (if $d = 231$). To do this, we must distinguish between a number of cases, depending on d and the signs of the coefficients in (41) or (42). By way of example, suppose first that $d = 231$. If we have $j' = 0$ or $j' = \pm n$, then Λ' reduces to a linear form in two logarithms and we may appeal to Corollary 1 of Laurent [19]; actually, what we state here is specialized for our purposes and follows from the arguments of [19] (see pages 346 and 347) after a short computation (in each case using, in the notation of [19], values of μ in $[0.555, 0.562]$ and ρ in $[6.12, 6.31]$).

Theorem 9 (Laurent). *Consider the linear form*

$$\Lambda = c_2 \log \beta_2 - c_1 \log \beta_1,$$

where c_1 and c_2 are positive integers, and β_1 and β_2 are multiplicatively independent algebraic numbers. Define $D = [\mathbb{Q}(\beta_1, \beta_2) : \mathbb{Q}] / [\mathbb{R}(\beta_1, \beta_2) : \mathbb{R}]$ and set

$$b' = \frac{c_1}{D \log B_2} + \frac{c_2}{D \log B_1},$$

where $B_1, B_2 > 1$ are real numbers such that

$$\log \beta_i \geq \max\{h(\beta_i), |\log \beta_i|/D, 1/D\}, \quad i \in \{1, 2\}.$$

Then

$$\log |\Lambda| \geq -CD^4 (\max\{\log b' + 0.21, m_1/D, 1\})^2 \log B_1 \log B_2,$$

for each pair (m_1, C) in the following set

$$\begin{aligned} &\{(14, 28.161), (14.5, 27.812), (15, 27.486), (15.5, 27.182), (16, 26.896), \\ &(16.5, 26.627), (17, 26.374), (17.5, 26.136), (18, 25.911), (18.5, 25.697), (19, 25.495), \\ &(19.5, 25.303), (20, 25.120)\}. \end{aligned}$$

If $j' = 0$, we apply this with

$$c_2 = n, \beta_2 = \tau' \delta^3 \gamma^{\epsilon_n}, c_1 = 2, \beta_1 = 1/\gamma, D = 1,$$

whence

$$h(\beta_2) \leq \frac{3}{2} \log(y), \quad h(\beta_1) = \frac{3 \log 2}{2},$$

and we can take

$$\log B_2 = \frac{3}{2} \log(y) \quad \text{and} \quad \log B_1 = \arccos(5/16).$$

Choosing $(m_1, C) = (18, 25.911)$, it follows that

$$\log |\Lambda'| \geq -39 \arccos(5/16) (\log(n) + 0.18)^2 \log y,$$

whereby, from (45) and $\Lambda' = 3\Lambda$, $n < 8300$, contradicting (34). Similarly, if $j' = \pm n$, we may apply Theorem 9 with

$$c_2 = n, c_1 = 2, \beta_1 = 1/\gamma, D = 1,$$

where

$$\log \beta_2 = \log(\tau' \delta^3 \gamma^{\epsilon_n}) \pm \pi i.$$

We can once again choose

$$\log B_2 = \frac{3}{2} \log(y) \quad \text{and} \quad \log B_1 = \arccos(5/16),$$

and derive a contradiction from taking $(m_1, C) = (18, 25.911)$. Note that from (42),

$$|j'|\pi < \pi n + 2 \arccos(5/16) + 3 \cdot y^{-0.499n} < \pi n + 2.51,$$

whereby $|j'| \leq n$. We may thus suppose that $|j'| < n$ and $j' \neq 0$ (so that, in particular, we have $\gcd(j', n) = 1$).

We will now apply Theorem 8. From (44), we can take, in the notation of Theorem 8 and writing $v = -j'/|j'|$,

$$(53) \quad b_1 = 2, \alpha_1 = \gamma^{-v}, b_2 = n, \alpha_2 = (\tau' \delta^3 \gamma^{\epsilon_n})^v, b_3 = |j'| \quad \text{and} \quad \alpha_3 = -1.$$

It follows that

$$h(\alpha_1) = \frac{3 \log 2}{2}, \quad h(\alpha_2) \leq \frac{3}{2} \log(y) \quad \text{and} \quad h(\alpha_3) = 0.$$

We can thus choose

$$a_1 = \rho \arccos(5/16) + 3 \log(2), \quad a_2 = \rho\pi + 3 \log(y) \quad \text{and} \quad a_3 = \rho\pi.$$

As noted in [12], if we suppose that $m \geq 1$ and define

$$(54) \quad K = [mLa_1a_2a_3], \quad R_1 = [c_1a_2a_3], \quad S_1 = [c_1a_1a_3], \quad T_1 = [c_1a_1a_2], \quad R_2 = [c_2a_2a_3],$$

$$S_2 = [c_2a_1a_3], \quad T_2 = [c_2a_1a_2], \quad R_3 = [c_3a_2a_3], \quad S_3 = [c_3a_1a_3] \quad \text{and} \quad T_3 = [c_3a_1a_2],$$

where

$$(55) \quad c_1 = \max\{(\chi mL)^{2/3}, (2mL/a_1)^{1/2}\}, \quad c_2 = \max\{2^{1/3}(mL)^{2/3}, (m/a_1)^{1/2}L\}$$

$$\text{and } c_3 = (6m^2)^{1/3}L,$$

then conditions (i)–(v) are automatically satisfied. It remains to verify inequality (48).

Define

$$R = R_1 + R_2 + R_3 + 1, \quad S = S_1 + S_2 + S_3 + 1 \quad \text{and} \quad T = T_1 + T_2 + T_3 + 1.$$

We choose

$$\rho = 5.9, \quad L = 206, \quad m = 25 \quad \text{and} \quad \chi = 2.89,$$

so that

$$c_1 = (\chi mL)^{2/3}, \quad c_2 = 2^{1/3}(mL)^{2/3},$$

and we have

$$K = [K_1 + K_2 \log(y)],$$

where

$$K_1 = 16759141.618\dots \quad \text{and} \quad K_2 = 2712508.708\dots$$

We thus have

$$S_1 = 106229, \quad S_2 = 65966 \quad \text{and} \quad S_3 = 561893.$$

Since Lemma 6.1, (34) and $N(231) = 1.2 \times 10^9$ together imply that

$$(56) \quad \log y > 22.2,$$

we find, after a little work, that $\mathcal{M} = \chi\mathcal{V}$ and that $g < 0.2438$.

Since $\gcd(j', n) = 1$, we have

$$d_1 = d_3 = 1, \quad b'_2 = b''_2 = n,$$

and it follows that

$$\eta_0 = \frac{1}{2}(R_1 + R_2 + R_3) + \frac{1}{n}(S_1 + S_2 + S_3) < 718258 + 116252 \log y,$$

and, from $|j'| < n$,

$$\zeta_0 = \frac{1}{2}(T_1 + T_2 + T_3) + \frac{-j'}{2n}(S_1 + S_2 + S_3) < \frac{1}{2}(T_1 + T_2 + T_3 + S_1 + S_2 + S_3),$$

whereby

$$\zeta_0 < 734089 + 59408 \log y.$$

From [25, Lemma 3.4], we have the inequality

$$(57) \quad \log \left(\prod_{k=1}^{K-1} k! \right)^{\frac{4}{K(K-1)}} \geq 2 \log K - 3 + \frac{2 \log (2\pi K / e^{3/2})}{K-1} - \frac{2 + 6\pi^{-2} + \log K}{3K(K-1)},$$

whence, from $K > 10^6$,

$$\log \left(\prod_{k=1}^{K-1} k! \right)^{\frac{4}{K(K-1)}} > 2 \log K - 3.$$

It follows, from (56), that

$$b < e^3 n^2 \frac{(718258 + 116252 \log y)(734089 + 59408 \log y)}{(16759141.6 + 2712508.7 \log y)^2} < 0.023062 n^2 < 1.559 \times 10^{26},$$

where the last inequality is a consequence of (46). The right-hand-side of inequality (48) is thus bounded above by

$$4 \log(K) + 2.051 \times 10^9 + 3.319 \times 10^8 \log(y) + 60.312K$$

while the left-hand-side satisfies

$$U > 182.814K + 89.635.$$

If inequality (48) fails to hold, it follows that

$$122.502K < 4 \log(K) + 2.051 \times 10^9 + 3.318 \times 10^8 \log(y),$$

contradicting

$$K > 16759141 + 2712508 \log y$$

and (56).

Note that we have

$$\frac{LSe^{LS|\Lambda|/(2b_2)}}{2|b_2|} = \frac{75611167 e^{75611167|\Lambda|/n}}{n}$$

and hence, from (45),

$$\frac{LSe^{LS|\Lambda|/(2b_2)}}{2|b_2|} < \frac{75611167 \exp\left(\frac{75611167}{ny^{0.499n}}\right)}{n} < 0.127,$$

where the last inequality is a consequence of Lemma 6.1 and (34). If we have inequality (49), it thus follows that

$$\log |\Lambda| > 2 - 365.65K.$$

Once again appealing to (45), we find that

$$0.499n \log y < 365.65K - 2 < 365.65(16759141.62 + 2712508.71 \log y)$$

and so

$$n < 1.988 \times 10^9 + \frac{1.229 \times 10^{10}}{\log y},$$

whence, from (56),

$$(58) \quad n < 2.6 \times 10^9.$$

If, on the other hand, inequality (49) fails to be satisfied, from inequality (34) and our choices of S_1 and S_2 , necessarily **(C3)** holds. If (50) holds then $n \mid s_0$, where

$$\begin{aligned} |s_0| &\leq \frac{(S_1+1)(T_1+1)}{\mathcal{M}-T_1} \leq \frac{(S_1+1)(T_1+1)}{R_1+1} \\ &< \frac{106230(106230.73 + 17193.55 \log y)}{207878 + 33645 \log y} < 54287, \end{aligned}$$

from calculus. It follows that necessarily $s_0 = 0$, a contradiction. We thus have (52). In particular,

$$(59) \quad (2t_1 + r_1|j'|)s_1 = r_1t_2n,$$

for integers r_1, s_1, t_1, t_2 with $\gcd(r_1, t_1) = \gcd(s_1, t_2) = 1$,

$$(60) \quad |s_1t_1| \leq \gcd(r_1, s_1) \cdot \frac{(S_1+1)(T_1+1)}{\chi\mathcal{V}-T_1} < \gcd(r_1, s_1) \cdot 81$$

and

$$(61) \quad |r_1s_1| \leq \gcd(r_1, s_1) \cdot \frac{(R_1+1)(S_1+1)}{\chi\mathcal{V}-R_1} < \gcd(r_1, s_1) \cdot 158,$$

again via calculus. It follows that

$$(62) \quad |t_1| \leq 80 \quad \text{and} \quad |r_1| \leq 157.$$

Since r_1 is coprime to t_1 , necessarily $r_1 \mid 2s_1$, while the fact that $\gcd(s_1, t_2) = 1$ while $n > 1.2 \times 10^9$ is prime, together imply that $s_1 \mid r_1$. We thus have that $r_1 = \pm s_1$ or $r_1 = \pm 2s_1$, whence, from (59),

$$(63) \quad r_1j' = t_3n \pm 2t_1,$$

where $t_3 = \pm t_2$ or $t_3 = \pm t_2/2$. We can thus rewrite $r_1\Lambda'$ as a linear form in two logarithms,

$$r_1\Lambda' = n \log \alpha - 2 \log \beta,$$

where

$$\log \alpha = r_1 \log(\tau' \delta^3 \gamma^{\epsilon_n}) + t_3 \pi i \quad \text{and} \quad \log \beta = -r_1 \log(\gamma) \pm t_1 \pi i.$$

We apply Theorem 9 with

$$D = 1, \quad c_2 = n, \quad \beta_2 = \alpha, \quad c_1 = 2 \quad \text{and} \quad \log \beta_1 = \log \beta.$$

We may take

$$\log B_2 = \frac{3}{2}|r_1| \log(y) \quad \text{and} \quad \log B_1 = |r_1| \arccos(5/16) + |t_1| \pi,$$

whence

$$b' = \frac{n}{|r_1| \arccos(5/16) + |t_1| \pi} + \frac{4}{3|r_1| \log(y)}.$$

From (62), we thus have

$$\log B_1 < 448.05.$$

Again choosing $(m_1, C) = (18, 25.911)$, we conclude that

$$\log |\Lambda'| > -39 \times 157 \times 448.05 \log^2(n) \log(y) - \log 157.$$

From (45) and $\Lambda' = 3\Lambda$, we thus have

$$0.499n < \frac{\log 471}{\log y} + 2.744 \times 10^6 \log^2(n),$$

whence, from (56), we once again obtain inequality (58).

We now iterate this argument, using slightly more care and, in each case, assuming that $n > N(231) = 1.2 \times 10^9$. We begin by taking

$$\rho = 5.8, \quad L = 144, \quad m = 27 \quad \text{and} \quad \chi = 3.14,$$

and, arguing as previously, find that either

$$(64) \quad n < 1.3 \times 10^9,$$

or that we have (63) with

$$(65) \quad |t_1| \leq 68 \quad \text{and} \quad |r_1| \leq 133.$$

From (58), we have that

$$b' = \frac{n}{|r_1| \arccos(5/16) + |t_1| \pi} + \frac{4}{3|r_1| \log(y)} < e^{18-0.21},$$

provided, crudely, $|r_1| \geq 39$. Applying Theorem 9 as previously, once again with $(m_1, C) = (18, 25.911)$, if $39 \leq |r_1| \leq 133$, we find that

$$\log |\Lambda'| > -39 \times 133 \times 380.28 \times 18^2 \log(y) - \log 133,$$

while, in case $1 \leq |r_1| \leq 38$,

$$(66) \quad \log |\Lambda'| > -39 \times 38 \times 261.25 \log^2(n) \log(y) - \log 38.$$

Once more, we have, in either case, inequality 64.

To finish the case $d = 231$, we repeat this argument, only now using

$$\rho = 6.3, \quad L = 125, \quad m = 27 \quad \text{and} \quad \chi = 3.28,$$

and appealing to Theorem 9 with $(m_1, C) = (17, 26.38)$. We conclude then, in all cases with $d = 231$, that

$$n < 1.2 \times 10^9 = N(231).$$

We argue similarly for $d \in \{7, 15, 55\}$, applying Theorem 8 to Λ as in (41). As in the case $d = 231$, we either have inequality (49), or we find ourselves in the degenerate case where, analogous to (63), we deduce the existence of “small” integers t_1, t_3 and r_1 , the last nonzero, such that $r_1 j = t_3 n \pm 2t_1$. In the latter case, we rewrite $r_1 \Lambda$ as a linear form in two logarithms,

$$r_1 \Lambda = n \log \alpha - \log \beta,$$

where

$$\log \alpha = r_1 \log(\tau \delta) + t_3 \pi i \quad \text{and} \quad \log \beta = -r_1 \log(\gamma) \pm 2t_1 \pi i,$$

and apply Theorem 9 with

$$D = 1, \ c_2 = n, \ \beta_2 = \alpha, \ c_1 = 1, \ \log \beta_1 = \log \beta,$$

$$\log B_2 = \frac{1}{2}|r_1|\log(y), \ \log B_1 = |r_1||\log \gamma| + 2|t_1|\pi$$

and

$$b' = \frac{n}{|r_1||\log \gamma| + 2|t_1|\pi} + \frac{2}{|r_1|\log(y)}.$$

We make our parameter choices for Theorems 8 and 9 as in the following table. In practice, once we have a reasonable upper bound upon n , $|r_1|$ and $|t_1|$, we loop over $|r_1|$ and $|t_1|$, compute upper and lower bounds upon b' and, in case b' potentially exceeds $\exp(m_1 - 0.21)$, deduce sharpened versions of inequality (66).

d	ρ	L	m	χ	(m_1, C)	upper bound upon n
7	4.8	205	45	2.96	(20, 25.120)	1.27×10^9
7	5.1	142	40	3.91	(20, 25.120)	6.44×10^8
7	6.2	100	40	3.00	(14, 28.161)	$6 \times 10^8 = N(7)$
15	5.8	180	36	2.77	(20, 25.120)	6.00×10^8
15	5.1	140	40	3.00	(20, 25.120)	$4 \times 10^8 = N(15)$
55	5.7	189	33	3.00	(20, 25.120)	1.11×10^9
55	5.2	144	35	3.41	(18, 25.911)	5.22×10^8
55	6.2	100	40	3.00	(14, 28.161)	$5 \times 10^8 = N(55)$

This, with Proposition 4.1, completes the proof of Theorem 2.

As mentioned previously, of note here is that the bounds we obtain upon the exponent n for the equation $x^2 + c^2d = y^n$, with $d \in \{7, 15, 55, 231\}$ and c an S -unit, $S = \{3, 5, 7, 11\}$, are essentially identical to those deduced for the simpler equation $x^2 + d = y^n$. This is admittedly not immediately apparent from perusal of [12, Section 15], where the treatment of the degenerate cases (which reduce to linear forms in two logarithms) requires some modification.

7. CONCLUDING REMARKS

Extending the results of this paper to the more general equation

$$x^2 + D = y^n, \ \gcd(x, y) = 1, \ D > 0, \ P(D) \leq 13$$

is probably computationally feasible with current technology, if one is suitably enthusiastic, while the equation

$$x^2 + D = y^n, \ \gcd(x, y) = 1, \ D > 0, \ P(D) \leq 17$$

is certainly out of reach without the introduction of fundamentally new ideas. The two main obstructions arise from both large exponents n (where the corresponding spaces of modular forms have extremely large dimensions) and moderately small ones (where one will encounter Thue-Mahler equations with very many associated S -unit equations). Additionally, it is possible to relax the restriction that $\gcd(x, y) = 1$ in (2) (at least provided this gcd is odd), though the computational difficulties increase substantially since, once again, the spaces of modular forms one encounters have significantly higher dimensions.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referee, Vandita Patel and Pedro-Jose Carzola Garcia for numerous helpful suggestions that improved both the exposition and correctness of this paper.

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