SIMULTANEOUS RATIONAL APPROXIMATION TO BINOMIAL FUNCTIONS

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Abstract. We apply Padé approximation techniques to deduce lower bounds for simultaneous rational approximation to one or more algebraic numbers. In particular, we strengthen work of Osgood, Fel'dman and Rickert, proving, for example, that

$$\max\left\{\left|\sqrt{2}-p_1/q\right|,\left|\sqrt{3}-p_2/q\right|\right\} > q^{-1.79155}$$

for $q > q_0$ (where the latter is an effective constant). Some of the Diophantine consequences of such bounds will be discussed, specifically in the direction of solving simultaneous Pell's equations and norm form equations.

0. Introduction

In 1964, Baker [1,2] utilized the method of Padé approximation to hypergeometric functions to obtain explicit improvements upon Liouville's theorem on rational approximation to algebraic numbers. By way of example, he showed that

$$\left|\sqrt[3]{2}-\frac{p}{q}\right| > 10^{-6}q^{-2.955} \tag{0.1}$$

for all positive integers p and q and used such bounds to solve related Diophantine equations. Chudnovsky [6] subsequently refined Baker's results,

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primarily through a detailed analysis of the arithmetical properties of certain Padé approximants. Analogous to (0.1), he proved that

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| > q^{-2.42971} \tag{0.2}$$

for all integers p and q with q greater than some effectively computable constant q_0. By working out the implicit constants in (0.2), Easton [8] deduced

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| > 6.6 \times 10^{-6} q^{-2.795}$$

for positive integers p and q (as well as related bounds for other cubic irrationalities).

Similar results exist for simultaneous approximation to several algebraic numbers. In particular, Baker [3] derived bounds of the form

$$\max_{1 \le u \le m} \left\{ \left| \theta_u - \frac{p_u}{q} \right| \right\} > q^{-\lambda} \tag{0.3}$$

for certain algebraic numbers $\theta_1, \theta_2, \dots \theta_m$ with $1, \theta_1, \theta_2, \dots \theta_m$ linearly independent over the rationals, $\lambda = \lambda(\theta_1, \dots \theta_m)$ an explicit real number and $p_1, \dots p_m, q$ positive integers with q greater than an effective $q_0(\lambda, \theta_1, \dots \theta_m)$. To be precise, he considered

$$(\theta_1, \theta_2, \dots \theta_m) = (r^{\nu_1}, r^{\nu_2}, \dots r^{\nu_m}) \tag{0.4}$$

with $r, \nu_1, \nu_2, \dots \nu_m$ rational, via approximation to the system of binomial functions

$$1, (1+x)^{\nu_1}, \dots (1+x)^{\nu_m}.$$

Chudnovsky [6], generalizing his approach from the case of a single approximation, sharpened these inequalities. Along somewhat different lines, Osgood [13], Fel'dman [9] and Rickert [15] obtained results like (0.3) with

$$(\theta_1, \theta_2, \dots \theta_m) = (r_1^{\nu}, r_2^{\nu}, \dots r_m^{\nu}) \tag{0.5}$$

for $r_1, r_2, \dots r_m$ and ν rational. These also utilized Padé approximation, this time to the functions

$$1, (1+a_1 x)^{\nu}, \dots (1+a_m x)^{\nu}$$

where $a_1, \dots a_m$ are distinct integers. Through use of an elegant contour integral representation for the desired Padé approximants, Rickert proved the inequality

$$\max\left\{\left|\sqrt{2}-\frac{p_1}{q}\right|,\left|\sqrt{3}-\frac{p_2}{q}\right|\right\} > 10^{-7}q^{-1.913} \tag{0.6}$$

for p_1, p_2 and q integral.

In this paper, we will strengthen the work of Osgood, Fel'dman and Rickert on simultaneous approximation to algebraic numbers satisfying (0.5), in analogy to Chudnovsky's results for those with (0.4). This is primarily accomplished through careful estimation of both "analytic" and "arithmetic" asymptotics (in the same sense as Chudnovsky [6]) of Padé approximants to binomial functions. A particularly striking result along these lines (with rather different approximating forms) is due to Hata [11] who proved that

$$\left|\pi-\frac{p}{q}\right| \geq q^{-8.0161}$$

for sufficiently large positive integers p and q.

In the special case $m = 1$, we obtain Chudnovsky's Theorem 5.3 of [6] on approximation to a single algebraic number (see section §7). For larger values of m, we can prove, for example, that

$$\max\left\{\left|\sqrt{2}-\frac{p_1}{q}\right|,\ \left|\sqrt{3}-\frac{p_2}{q}\right|\right\} > q^{-1.79155}$$

for p_1 and p_2 integral and $q \geq q_0$ effectively computable (compare to (0.6)). Similarly, we have

$$\max\left\{\left|\sqrt{3}-\frac{p_1}{q}\right|,\left|\sqrt{5}-\frac{p_2}{q}\right|\right\} > q^{-1.82227}$$

for $q \geq q_1$ effective. Optimally, one would like to derive (0.3) for any $\lambda > 1+1/m$. Theorems of Roth [16] ($m = 1$) and Schmidt [17] ($m > 1$) assert that such bounds exist for any independent algebraic $\theta_1, \dots \theta_m$, but are ineffective in that they do not permit the explicit calculation of q_0. For specific classes of algebraic numbers, however, we will be able to obtain effective bounds with λ arbitrarily close to $1 + 1/m$. These correspond to the situations described by previous authors where the rationals r or $r_1, r_2, \dots r_m$ in (0.4) or (0.5), respectively, are suitably close to 1. We will also prove a theorem on linear forms, of the type

$$|x_0 + x_1\theta_1 + \cdots + x_m\theta_m| > X^{-\lambda_1} \tag{0.7}$$

for $x_0, \dots x_m$ integers, $\theta_1, \dots \theta_m$ as in (0.5) and $X = \max_{0 \le i \le m} |x_i|$ satisfying $X \ge X_0(\lambda_1, \theta_1, \dots \theta_m)$. Standard transference arguments (see e.g. Cassels [5]) ensure that (0.3) implies (0.7) with exponent

$$\lambda_1 = \frac{m(\lambda - 1)}{m(-\lambda + 1) + \lambda}$$

provided $\lambda < 1 + 1/(m-1)$. Our result, however, is somewhat stronger. These results have direct applications to Diophantine equations which we will address in §8 and §9. For example, they permit solution of the norm form equation

$$N_{K/\mathbb{Q}}\left(x + y\sqrt[4]{M^4 - 1} + z\sqrt[4]{M^4 + 1}\right) = u$$

(where $K = \mathbb{Q}(\sqrt[4]{M^4 - 1}, \sqrt[4]{M^4 + 1}), x, y$ and z are integers and u is constant) for $M \ge 6$.

1. A Pair of Theorems on Rational Approximation

Henceforth, we will suppose that $a_0, a_1, \cdots a_m$ are distinct integers $(m \ge 1)$ with one of them equal to zero, satisfying

$$a_0 < a_1 < \dots < a_m.$$

Let us also assume that N is a positive integer with

$$N > \max_{0 \le u \le m} |a_u|$$

and that s and n are integers with $1 \le s < n$ and $(s, n) = 1$. Define

$$c_1 = \operatorname{lcm}\left\{ \prod_{\substack{l=0 \\ l \ne v}}^{m} |a_l - a_v| : 0 \le v \le m \right\} \tag{1.1}$$

$$(1.2) \qquad c_2 = \text{ lcm } \{|a_l - a_v| : 0 \le v < l \le m\}$$

$$(1.3) \qquad c_3 = \prod_{p|n} p^{\max\{ord_p(n/c_2)+\frac{1}{p-1},0\}}$$

and

$$c_4 = c_1 \cdot c_2 \cdot c_3.$$

If, following Rickert [15], we set $A(z) = \prod_{u=0}^{m} (z - a_u)$, then the polynomial

$$(1.4) \qquad A(z) - (z + N)A'(z)$$

(where we write $A'(z)$ for $dA(z)/dz$) is readily seen to have $m + 1$ real zeros, one of them, say z_0, satisfying $z_0 < -N$ and the remaining m, say $z_1, z_2, \ldots z_m$, lying between successive values of the a_i's. Without loss of generality, we suppose that $a_{u-1} < z_u < a_u$ $(1 \le u \le m)$ and define

$$c_5 = |A'(z_0)|$$

$$c_6(v) = \begin{cases} |A'(z_1)| & \text{if } v = 0 \\ \min\{|A'(z_v)|, |A'(z_{v+1})|\} & \text{if } 1 \le v < m \\ |A'(z_v)| & \text{if } v = m \end{cases}$$

$$c_7 = \min_{1 \le u \le m} |A'(z_u)|$$

and

$$c_8 = \exp\left(-\gamma - \frac{1}{\phi(n)} \sum_{\substack{1 \le r < n \\ (r,n)=1}} \psi\left(\max\left\{ \frac{nm - r}{nm}, \frac{r}{n} \right\} \right) \right)$$

where γ is Euler's constant, $\phi(n)$ is Euler's totient function and $\psi(z) = d \log \Gamma(z)/dz$ is the digamma function. We may conclude

Theorem 1.1 *Let $a_0, a_1, \ldots a_m$ be distinct integers with $a_r = 0$ for some r and $N > \max_{0 \le u \le m} |a_u|$ an integer. If, further, s and n are relatively prime with $1 \le s < n, \epsilon > 0$ and $c_7 \cdot c_8 < c_4 < c_5 \cdot c_8$, then*

$$\max_{\substack{0 \le u \le m \\ a_u \ne 0}} \left\{ \left| \left(1 + \frac{a_u}{N}\right)^{s/n} - \frac{p_u}{q} \right| \right\} > q^{-\lambda-\epsilon}$$

for all integers $p_0, \ldots p_m$ *and* q *with* $q \geq q_0(\epsilon, s, n, a_0, \ldots, a_m, N)$, *where* $\lambda = 1 + \dfrac{\log(c_4/c_7 \cdot c_8)}{\log(c_5 \cdot c_8/c_4)}$ *and* q_0 *is effectively computable.*

As mentioned previously, in section §7 we will show that, in the case $m = 1$, the above theorem implies Chudnovsky's result [6, Theorem 5.3] (see also Heimonen, et. al. [12]).

For linear forms, we will prove

Theorem 1.2 *If* $a_0 \ldots a_m, N, s$ *and* n *are integers satisfying the hypotheses of the previous theorem,* $x_0, \ldots x_m$ *integers,* $X = \max_{0 \leq u \leq m} |x_u|, \epsilon > 0$ *and*

$$\prod_{v=1}^{m} c_6(v) < (c_4/c_8)^m < c_5 \cdot \min_{1 \leq l \leq m} \prod_{\substack{v=1 \\ v \neq l}}^{m} c_6(v)$$

then

$$\left| \sum_{u=0}^{m} x_u \cdot \left(1 + \frac{a_u}{N}\right)^{s/n} \right| > X^{-\lambda_1 - \epsilon}$$

for all $X \geq X_0(\epsilon, s, n, a_0, \ldots a_m, N)$, *where*

$$\lambda_1 = \frac{m \log(c_4/c_8) - \sum_{1 \leq v \leq m} \log(c_6(v))}{m \log(c_8/c_4) + \log(c_5) + \min_{1 \leq l \leq m} \sum_{\substack{1 \leq v \leq m \\ v \neq l}} \log(c_6(v))}$$

and X_0 *is effectively computable.*

Examples and applications of this result will be briefly described in §8 and §9.

2. The Nature of the Approximating Forms

To construct our approximants to the system of binomial functions (0.3), we consider the contour integral

$$I_u(x) = \frac{1}{2\pi i} \int_\gamma \frac{(1+zx)^k (1+zx)^\nu}{(z-a_u)(A(z))^k} \, dz \ (0 \leq u \leq m). \tag{2.1}$$

Here, k is some fixed positive integer, ν a nonintegral positive rational, γ a closed, counter-clockwise contour enclosing the poles of the integrand and x a real satisfying

$$|x|^{-1} > \max_{0 \le u \le m} |a_u|. \tag{2.2}$$

By application of Cauchy's residue theorem, we can write

$$I_u(x) = \sum_{v=0}^{m} P_{uv}(x)(1 + a_v x)^\nu \ (0 \le u \le m) \tag{2.3}$$

where the $P_{uv}(x)$ are polynomials with rational coefficients and degree at most k in x. Explicitly, from Rickert [15, Lemma 3.3], we have
(2.4)

$$P_{uv}(x) = \sum \binom{k+\nu}{h_v} (1 + a_v x)^{k-h_v} x^{h_v} \prod_{\substack{0 \le l \le m \\ l \ne v}} \binom{-k_{ul}}{h_l} (a_v - a_l)^{-k_{ul}-h_l}$$

where $\sum$ denotes summation over all nonnegative integers $h_0, \dots h_m$ with sum $k_{uv} - 1$, for $k_{ab} = k + \delta_{ab}$ and δ_{ab} the Kronecker delta. To guarantee the "independence" of the approximants, we require that $\det_{0 \le u,v \le m} (P_{uv}(x))$ does not vanish for nonzero x, a consequence of Rickert's Lemma 3.4. To be precise, one may write

$$\det_{0 \le u,v \le m} (P_{uv}(x)) = \left(\prod_{v=-1}^{m-1} \binom{\nu - vk}{k} \right) \cdot \left(\prod_{l=0}^{m} \prod_{\substack{s=0 \\ s \ne l}}^{m} (a_s - a_l)^{-k} \right) x^{(m+1)k}.$$

In the sections that follow, we will find asymptotics for $|P_{uv}(1/N)|$ and $|I_u(1/N)|$ and then study the arithmetic properties of the coefficients of $P_{uv}(x)$.

3. Contour Integral Estimates

To begin, we note that the value $P_{uv}(x)(1+a_v x)^\nu$ $(0 \le u \le m)$ is obtained from the integral (2.1), only with the contour γ changed so as to enclose the integer a_v and no other a_l's (for $l \ne v$). Setting $x = 1/N$, one sees that

(2.2) is satisfied and it follows that the integrand of (2.1) is analytic in a suitable deleted neighbourhood of a_v. Following Hata [11], we may apply the saddle-point method as described in Dieudonné [7, chapter IX] to estimate the principal part of $P_{uv}\left(\frac{1}{N}\right)(1+a_v/N)^\nu$ for large values of k. Explicitly, we set

$$F(z) = \log\left(1+\frac{z}{N}\right) - \log\left(|A(z)|\right)$$

and

$$G(z) = \left(1+\frac{z}{N}\right)^\nu (z-a_u)^{-1}$$

so that

$$P_{uv}\left(\frac{1}{N}\right)(1+a_v/N)^\nu = \int_\gamma G(z)e^{kF(z)}dz. \tag{3.1}$$

The saddles of the surface $|F(z)|$ are given by the zeros of the derivative of $F(z)$ which, since $x = 1/N$, are the zeros of the polynomial (1.4) (say $z_0, z_1, \ldots z_m$ as in §1). Since $G(z)e^{kF(z)}$ vanishes as z tends to $-N$ or ∞ (avoiding the real branch cut from $-N$ to $-\infty$), the saddle-point method yields

Lemma 3.1 *As $k \to \infty$, the principal part of* (3.1) *is given by*

$$P_{uv}\left(\frac{1}{N}\right)\left(1+\frac{a_v}{N}\right)^\nu \sim \sum_{v \le t \le v+1} e^{kF(z_t)}\, G(z_t)\sqrt{\frac{-2\pi}{kF''(z_t)}}.$$

where the summation is restricted to $t \in [1, m]$. In particular, since the roots of (1.4) *satisfy*

$$e^{F(z_l)} = |NA'(z_l)|^{-1}\ (0 \le l \le m)$$

we may conclude that

$$\lim_{k\to\infty} \frac{1}{k}\log\left|P_{uv}\left(\frac{1}{N}\right)\right| = -\log(c_6(v)\cdot N) \le -\log(c_7 \cdot N)$$

for all $0 \le u, v \le m$.

To find asymptotics for $|I_u(1/N)|$ requires a more delicate analysis. Since the integrand of (2.1) has a branch point at $z = -N$, we cannot simply apply the saddle-point method for the saddle z_0 without justification (recall that $z_0 < -N$ is real). If, however, we make the change of variables $1+\frac{z}{N} \to -w$, then we may write

$$I_u\left(\frac{1}{N}\right) = \frac{(-1)^{mk}e^{\pi i\nu}}{2\pi i N^{(m+1)k}} \int_{\gamma'} \frac{w^k w^\nu dw}{(w+1+\frac{a_u}{N})(B(w))^k} \tag{3.2}$$

where $B(w) = \prod_{l=0}^{m}\left(w+1+\frac{a_l}{N}\right)$ and $\gamma' = \gamma_1+\gamma_2+\gamma_3+\gamma_4$ is a contour containing the poles of the integrand of (3.2) while avoiding a branch cut along the nonnegative real axis (see Figure 1).

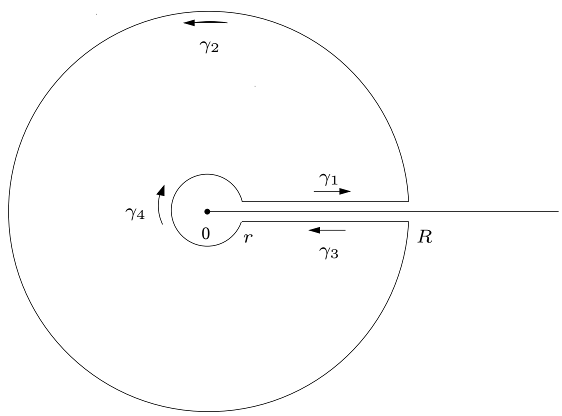

Figure 1:

Since

$$\left|\int_{\gamma_l} \frac{w^k w^\nu dw}{(w+1+\frac{a_u}{N})(B(w))^k}\right| \le \int_0^{2\pi} \left|\frac{w^k w^\nu}{(w+1+\frac{a_u}{N})(B(w))^k}\right| d\theta\ ,\ l = 2 \text{ or } 4$$

(where $w = Re^{i\theta}$ or $re^{i\theta}$ respectively) we have that the contribution to (3.2) associated with the arcs γ_2 and γ_4 becomes negligible as $r \to 0$ and $R \to \infty$.

Therefore, from

$$\frac{w^k w^\nu}{(w+1+\frac{a_u}{N})(B(w))^k} = \begin{cases} \dfrac{x^k x^\nu}{(x+1+\frac{a_u}{N})(B(x))^k} & \text{on } \gamma_1 \\ \\ \dfrac{e^{2\pi i\nu} x^k x^\nu}{(x+1+\frac{a_u}{N})(B(x))^k} & \text{on } \gamma_3 \end{cases}$$

we may conclude, letting $r \to 0$ and $R \to \infty$, that

$$I_u\left(\frac{1}{N}\right) = \frac{(-1)^{mk} e^{\pi i\nu}(1-e^{2\pi i\nu})}{2\pi i N^{(m+1)k}} \int_0^\infty \frac{x^k x^\nu dx}{\left(x+1+\frac{a_u}{N}\right)(B(x))^k}.$$

This is readily evaluated for large k via Laplace's method (see e.g. Dieudonné [7, Chapter IV §2]). Since the function $x/B(x)$ has only one critical point on the positive real axis, say x_0, and vanishes as $x \to 0^+$ or $x \to \infty$, we conclude that

$$\lim_{k\to\infty} \frac{1}{k} \log\left|I_u\left(\frac{1}{N}\right)\right| = \log\left|\frac{x_0}{N^{m+1}B(x_0)}\right|. \tag{3.3}$$

Changing variables back to our original z, we find that

$$\left|\frac{x_0}{N^{m+1}B(x_0)}\right| = \left|\frac{1+z_0/N}{A(z_0)}\right| = \left|\frac{1}{NA'(z_0)}\right|$$

for z_0 as in §1. Combining this with (3.3) yields

Lemma 3.2 $\lim_{k\to\infty} \frac{1}{k} \log\left|I_u\left(\frac{1}{N}\right)\right| = -\log(c_5 \cdot N)$ $(0 \le u \le m)$.

4. Coefficients of the Approximating Polynomials

We wish to determine certain arithmetic properties of the coefficients of the polynomials $P_{uv}(x)$ defined in §2. Let us write

$$P_v = \prod_{\substack{0\le l\le m \\ l \ne v}} \binom{k+h_l-1}{h_l} (a_l - a_v)^{-k-h_l}.$$

¿From (2.4), if $u = v$, then

$$P_{uv}(x) = (-1)^{mk} \sum \binom{k+\nu}{h_v} (1+a_v x)^{k-h_v} x^{h_v} P_v \tag{4.1}$$

where $\sum$ implies the sum over nonnegative $h_0, \ldots h_m$ satisfying $\sum_{l=0}^{m} h_l = k$. If, however, $u \neq v$, we have

$$P_{uv}(x) = (-1)^{mk} \sum \binom{k+\nu}{h_v} (1+a_v x)^{k-h_v} x^{h_v} \left(\frac{k+h_u}{k(a_u - a_v)} \right) P_v \tag{4.2}$$

where in this latter case, the summation is over nonnegative $h_0, \ldots h_m$ with $\sum_{l=0}^{m} h_l = k - 1$. ¿From here on we will fix $\nu = s/n$. The following elementary lemma concerning primes dividing binomial coefficients will be the chief tool in determining our "arithmetic" asymptotics. It enables us to identify certain classes of prime numbers which are guaranteed to divide the numerators of the coefficients of the polynomials $P_{uv}(x)$ defined in (2.4). Similar results have been utilized by Chudnovsky [6], Hata [11] and Heimonen, et al. [12], amongst others, in the pursuit of irrationality and linear independence measures. Suppressing dependence on m, n, s and k, let

$$\begin{aligned} S(r) = \ & \Big\{ p \text{ prime} : p > \sqrt{nk+s}, (p, nk) = 1 \text{ (and, if } m = 1, \\ & (p, nk - s - n) = 1) \text{ and } \left\{ \frac{k-1}{p} \right\} > \max \left(\frac{nm-r}{nm}, \frac{r}{n} \right) \\ & \text{for } r \text{ with } 1 \leq r < n \text{ and } pr \equiv s \ (\text{mod} n) \Big\} \end{aligned} \tag{4.3}$$

where we adopt the notation $\{x\} = x - [x]$ for the fractional part of a real number x.

Lemma 4.1 *If $p \in S(r)$ then*

$$\text{ord}_p \left(\binom{k+s/n}{h_0} \binom{k+h_1-1}{h_1} \cdots \binom{k+h_m-1}{h_m} \right) \geq 1$$

for all nonnegative integers $h_0, \ldots h_m$ with $\sum_{l=0}^{m} h_l = k$ or $k-1$.

Proof. Suppose that $p \in S(r)$ does not divide the product

$$\binom{k+h_1-1}{h_1} \cdots \binom{k+h_m-1}{h_m}.$$

Then it follows that

$$\left\{\frac{h_l}{p}\right\} < 1 - \left\{\frac{k-1}{p}\right\} \ (1 \le l \le m)$$

and thus, since $p \not| \, k$, that

$$\left\{\frac{h_l}{p}\right\} \le 1 - \left\{\frac{k}{p}\right\}. \ (1 \le l \le m) \tag{4.4}$$

¿From (4.3), we may therefore write

$$\sum_{l=1}^{m} \left\{\frac{h_l}{p}\right\} \le m\left(1 - \left\{\frac{k}{p}\right\}\right) < m\left(\frac{r}{nm} - \frac{1}{p}\right) = \frac{r}{n} - \frac{m}{p}. \tag{4.5}$$

Hence $\sum_{l=0}^{m} h_l = k$ or $k-1$ in conjunction with

$$\left\{\frac{k-1}{p}\right\} > \frac{r}{n} > \sum_{l=1}^{m} \left\{\frac{h_l}{p}\right\}$$

yields

$$\left\{\frac{k-h_0}{p}\right\} \le \sum_{l=1}^{m} \left\{\frac{h_l}{p}\right\} + \frac{1}{p}. \tag{4.6}$$

We wish to show that $\operatorname{ord}_p \binom{k+s/n}{h_0} \ge 1$. Since $p \not| \, n$, we have

$$\operatorname{ord}_p \binom{k+s/n}{h_0} = \operatorname{ord}_p \left(\frac{(nk+s)(n(k-1)+s)\cdots(n(k-h_0)+n+s)}{h_0!}\right)$$

and so by a result of Chudnovsky [6, Lemma 4.5] (recalling that $p > \sqrt{nk+s}$)

$$\operatorname{ord}_p \binom{k+s/n}{h_0} = \left\{\frac{k-\theta-h_0}{p}\right\} + \left\{\frac{h_0}{p}\right\} - \left\{\frac{k-\theta}{p}\right\} \tag{4.7}$$

where $\theta = \dfrac{pr-s}{n}$. We thus have that $\mathrm{ord}_p \begin{pmatrix} k+s/n \\ h_0 \end{pmatrix} \geq 1$ exactly when $\left\{\dfrac{h_0}{p}\right\} > \left\{\dfrac{k-\theta}{p}\right\}$ or, equivalently, when

$$\left\{\frac{k-h_0}{p}\right\} < \frac{r}{n} - \frac{s}{pn}. \tag{4.8}$$

Now, combining (4.5) and (4.6) gives

$$\left\{\frac{k-h_0}{p}\right\} \leq \frac{r}{n} - \frac{(m-1)}{p} \tag{4.9}$$

which for $m \geq 2$ implies $\left\{\dfrac{k-h_0}{p}\right\} \leq \dfrac{r}{n} - \dfrac{1}{p} < \dfrac{r}{n} - \dfrac{s}{pn}$, as desired (since $1 \leq s < n$). It remains to consider the case $m = 1$. If we have strict inequality in (4.4), then

$$\left\{\frac{h_1}{p}\right\} \leq 1 - \left\{\frac{k}{p}\right\} - \frac{1}{p} < \frac{r}{n} - \frac{2}{p}$$

whence

$$\left\{\frac{k-h_0}{p}\right\} \leq \left\{\frac{h_1}{p}\right\} + \frac{1}{p} < \frac{r}{n} - \frac{1}{p}$$

which again implies (4.8). Similarly, we attain the inequality (4.8) if $h_0 + h_1 = k$. If $\left\{\dfrac{h_0}{p}\right\} < \left\{\dfrac{k-\theta}{p}\right\}$, then

$$\left\{\frac{k-h_0}{p}\right\} \geq \frac{r}{n} - \frac{s}{pn} + \frac{1}{p} > \frac{r}{n}$$

contradicting (4.9). To complete the proof of the lemma, we have only to show that the simultaneous equalities

$$\left\{\frac{h_1}{p}\right\} = 1 - \left\{\frac{k}{p}\right\}$$

$$\left\{\frac{h_0}{p}\right\} = \left\{\frac{k-\theta}{p}\right\}$$

and

$$h_0 + h_1 = k - 1$$

produce a contradiction. Well, the first of these implies that $p|k + h_1$, while the second yields $p|n(k - h_0) + s$. But these together with $h_0 + h_1 = k - 1$ imply that p divides $nk - s - n$, contradicting our initial assumptions.

■

Now if by $\pi(x, n, s)$ we denote the number of primes $p \le x$ in the arithmetic progression $bn + s$, then, analogous to the standard prime number theorem, we have

$$\pi(x, n, s) = \frac{x}{\phi(n)\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

It follows that if $\beta > \alpha > 0$, then

$$\lim_{k\to\infty} \frac{1}{k} \log \prod p = \frac{\beta - \alpha}{\phi(n)} \tag{4.10}$$

where the product is over all primes p in the interval $\alpha k < p < \beta k$ and satisfying $p \equiv a \pmod{n}$. The inequality $\left\{\frac{k-1}{p}\right\} > \max\left(\frac{nm - r}{nm}, \frac{r}{n}\right)$ defines a collection of open intervals for primes p in $S(r)$, of the form

$$\left(\frac{k}{l+1}, \min\left(\frac{nmk}{(l+1)nm - r}\ ,\ \frac{nk}{ln + r}\right) + O\left(\sqrt{k}\right)\right),\ l = 0, 1, 2, \ldots$$

where the shape of the error term follows from the assumption that $p > \sqrt{nk + s}$. From (4.10), we may therefore write

$$\lim_{k\to\infty} \frac{1}{k} \log \prod_{p\in S(r)} p = \frac{1}{\phi(n)} \sum_{l=0}^{\infty} \left(\min\left(\frac{nm}{(l+1)nm - r}\ ,\ \frac{n}{ln + r}\right) - \frac{1}{l+1}\right).$$

Since, from Bateman and Erdélyi [4], the function $\psi(z) = \frac{d \log \Gamma(z)}{dz}$ satisfies

$$\psi(\beta) - \psi(\alpha) = \sum_{l=0}^{\infty} \left(\frac{1}{l+\alpha} - \frac{1}{l+\beta}\right)$$

we may conclude that

$$\lim_{k\to\infty}\frac{1}{k}\log\prod_{p\in S(r)}p=\frac{1}{\phi(n)}\left(\psi(1)-\psi\left(\max\left\{\frac{nm-r}{nm},\frac{r}{n}\right\}\right)\right)$$

whence, recalling the equality $\psi(1)=-\gamma$,

$$\lim_{k\to\infty}\frac{1}{k}\log\prod_{\substack{1\le r<n\\(r,n)=1}}\prod_{p\in S(r)}p=-\gamma-\frac{1}{\phi(n)}\sum_{\substack{1\le r<n\\(r,n)=1}}\psi\left(\max\left\{\frac{nm-r}{nm},\frac{r}{n}\right\}\right). \tag{4.11}$$

We thus have

Lemma 4.2 *For each k, there is a rational number C_k such that*

$$C_kP_{uv}\left(\frac{1}{N}\right)\in\mathbb{Z}\ for\ all\ 0\le u,v\le m$$

and

$$\limsup_{k\to\infty}\frac{1}{k}\log|C_k|\le\log(c_4N/c_8)$$

for c_4 and c_8 as in §1.

Proof. If we define $\mu_{n,r}=\prod_{p|n}p^{[r/(p-1)]}$, then by Chudnovsky [6, Lemma 4.1], we have

$$\mu_{n,r}\cdot n^r\binom{k+s/n}{r}\in\mathbb{Z} \tag{4.12}$$

for $k\ge r\ge 0$. Let

$$\alpha=\max_{0\le l<v\le m}\mathrm{ord}_p(a_l-a_v)$$

and

$$\beta_v(h_0,\dots h_m)=\mu_{n,h_v}\cdot n^{h_v}\prod_{\substack{0\le l\le m\\ l\neq v}}(a_l-a_v)^{h_l}$$

where $\sum_{0\le l\le m}h_l=k$ or $k-1$.

If p is prime and p does not divide n, then

$$\begin{aligned}
&\operatorname{ord}_p(\beta_v(h_0, \dots h_m)) = \operatorname{ord}_p \prod_{\substack{0 \le l \le m \\ l \ne v}} (a_l - a_v)^{h_l} \\
&= \sum_{\substack{0 \le l \le m \\ l \ne v}} h_l \operatorname{ord}_p(a_l - a_v) \le \sum_{0 \le l \le m} \alpha h_l \le \alpha k.
\end{aligned}$$

On the other hand, if $p|n$, then

$$\begin{aligned}
&\operatorname{ord}_p\left(\beta_v(h_0, \dots h_m)\right) \le \left[\frac{h_v}{p-1}\right] + h_v \operatorname{ord}_p n + \sum_{\substack{0 \le l \le m \\ l \ne v}} h_l \operatorname{ord}_p(a_l - a_v) \\
&\le h_v \left(\operatorname{ord}_p n + \frac{1}{p-1}\right) + \sum_{\substack{0 \le l \le m \\ l \ne v}} \alpha h_l \\
&\le \max\left(\operatorname{ord}_p n + \frac{1}{p-1}, \alpha\right) \sum_{0 \le l \le m} h_l \\
&\le \max\left(\operatorname{ord}_p n + \frac{1}{p-1}, \alpha\right) k.
\end{aligned}$$

Since $h_l \le k$ for $0 \le l \le m$, (1.1) and (1.2) yield that

$$\beta_v(h_0, \dots, h_m)^{-1} \cdot (c_2 \cdot c_3)^k \tag{4.13}$$

is an integer for all $0 \le v \le m$ and nonnegative integers $h_0, \dots h_m$ with sum $k-1$ or k.

¿From (4.1), (4.2), (4.12), (4.13) and Lemma 4.1, we therefore have that

$$(c_4 \cdot N)^k \cdot c_2 \cdot k \left(\prod_{\substack{1 \le r \le n \\ (r,n)=1}} \prod_{p \in S(r)} p \right)^{-1} P_{uv}\left(\frac{1}{N}\right)$$

is also an integer for $0 \le u, v \le m$ (using the fact that $P_{uv}(x)$ has degree at most k). The lemma then follows from equality (4.11).

∎

5. Proof of Theorem 1.1

To apply the results of the previous sections, we first state a lemma that connects our approximating polynomials with bounds of the form (0.1). Let $\theta_0, \theta_1, \ldots \theta_m$ be distinct real numbers with $\theta_r = 1$ for some r. Then

Lemma 5.1 *Suppose there are positive real numbers P and Q such that, for $\epsilon > 0$ and each positive integer k greater than an effective constant k_0, we can find integers P_{uvk} $(0 \leq u, v \leq m)$ with nonzero determinant,*

$$\frac{1}{k} \log |P_{uvk}| \leq P + \epsilon \ (0 \leq u, v \leq m)$$

and

$$\frac{1}{k} \log \left| \sum_{v=0}^{m} P_{uvk} \ \theta_v \right| \leq -Q + \epsilon \ (0 \leq u \leq m).$$

Then (0.1) *holds for any* $\lambda > 1 + \dfrac{P}{Q}$ and $q \geq q_0(\theta_0, \ldots \theta_m, \lambda)$ *effectively computable.*

Proof. This follows directly from Lemma 2.1 in [14].

∎

To complete the proof of Theorem 1.1, we write

$$\theta_l = \left(1 + \frac{a_l}{N}\right)^{s/n} \quad (0 \leq l \leq m)$$

and

$$P_{uvk} = C_k P_{uv} \left(\frac{1}{N}\right) \ (0 \leq u, v \leq m)$$

where C_k is as in Lemma 4.2. Then from Lemmas 3.1 and 4.2, we may take

$$P = \log \left(\frac{c_4}{c_7 \cdot c_8}\right)$$

and

$$Q = \log \left(\frac{c_5 \cdot c_8}{c_4}\right)$$

in Lemma 5.1. If $c_7 \cdot c_8 < c_4 < c_5 \cdot c_8$, then both of these quantities are positive and, recalling that $\det_{0 \le u,v \le m} (P_{uv}(1/N))$ is nonzero, we can utilize this last lemma to obtain the desired result. Since we can make Lemma 5.1 effective, the same is true for Theorem 1.1.

■

6. Proof of Theorem 1.2

Let $x_0, \ldots x_m, a_0, \ldots a_m, s, n, N$ and X be as in §1, C_k and P_{uvk} as in §4 and §5 and write

$$L = \sum_{u=0}^{m} x_u \cdot \left(1 + \frac{a_u}{N}\right)^{s/n}.$$

Then the fact that $\det_{0 \le u,v \le m} (P_{uv}(1/N)) \neq 0$ implies that we can find m of the $C_k I_u(1/N)$ $(0 \le u \le m)$, say $C_k I_1(1/N), \ldots C_k I_m(1/N)$ with

$$L, C_k I_1\left(\frac{1}{N}\right), \ldots C_k I_m\left(\frac{1}{N}\right) \tag{6.1}$$

independent forms in the numbers $(1+a_0/N)^{s/n}, \ldots (1+a_m/N)^{s/n}$. Following Fel'dman [9], we consider the determinant

$$\Delta_k = \begin{vmatrix} x_0 & x_1 & \ldots & x_m \\ P_{10k} & P_{11k} & \ldots & P_{1mk} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ P_{m0k} & P_{m1k} & \ldots & P_{mmk} \end{vmatrix} = L \cdot \Delta_{k,0} + \sum_{l=1}^{m} C_k I_l\left(\frac{1}{N}\right) \Delta_{k,l} \tag{6.2}$$

where $\Delta_{k,l}$ $(0 \le l \le m)$ are the cofactors of the elements of the first column of Δ_k. Since the P_{uvk}'s and x_v's are integers, the independence of the forms in (6.1) ensures that $|\Delta_k| \ge 1$ and thus (6.2) implies that

$$|L| \cdot |\Delta_{k,0}| + \sum_{l=1}^{m} |C_k I_l(1/N)| \cdot |\Delta_{k,l}| \ge 1. \tag{6.3}$$

Define, for $\sigma \in S_m$ (i.e. σ a permutation on $\{1, 2, \dots m\}$) and $v = 1, 2, \dots m$,

$$P(\sigma) = P_{\sigma(1)1k}\ P_{\sigma(2)2k} \dots P_{\sigma(m)mk}$$

and

$$P_v(\sigma) = P_{\sigma(1)1k} \dots P_{\sigma(v-1)(v-1)k}\ P_{\sigma(v+1)(v+1)k} \dots P_{\sigma(m)mk}.$$

Since

$$\Delta_{k,0} = \begin{vmatrix} P_{11k} & \dots & P_{1mk} \\ . & & . \\ . & & . \\ . & & . \\ . & & . \\ P_{m1k} & \dots & P_{mmk} \end{vmatrix}$$

we have that

$$|\Delta_{k,0}| \leq m! \cdot \max\ \{|P(\sigma)| : \sigma \in S_m\}$$

and thus Lemmas 3.1 and 4.2 yield

$$\limsup_{k\to\infty} \frac{1}{k} \log |\Delta_{k,0}| \leq m \log\left(\frac{c_4}{c_8}\right) - \sum_{v=1}^{m} \log(c_6(v)). \tag{6.4}$$

Similarly, if $1 \leq l \leq m$, then $\Delta_{k,l}$ satisfies

$$|\Delta_{k,l}| \leq (m-1)! \cdot \sum_{v=1}^{m} |x_v| \cdot \max\{|P_v(\sigma)| : \sigma \in S_m\}$$

and Lemmas 3.2 and 4.2 imply (for $1 \leq v, l \leq m$) that

$$\limsup_{k\to\infty} \frac{1}{k} \log \left(C_k I_l \left(\frac{1}{N}\right) \max\{|P_v(\sigma)| : \sigma \in S_m\} \right)$$

$$\leq m \log\left(\frac{c_4}{c_8}\right) - \log(c_5) - \min_{1\leq l\leq m} \sum_{\substack{v=1 \\ v\neq l}}^{m} \log(c_6(v)). \tag{6.5}$$

Let us denote the right hand side of (6.4) by D_1 and the right hand side of (6.5) by D_2. Given $\epsilon_1 > 0$, from (6.4) we can find a $k_0 = k_0(\epsilon_1)$ for which

$$|\Delta_{k,0}| \leq \frac{1}{2} \cdot e^{(D_2+\epsilon_1)k} \tag{6.6}$$

for $k \geq k_0$. Inequality (6.5) implies that for $X \geq X_0(\epsilon_1)$, we may find a $k = k(X)$ with both

$$(6.7) \qquad m! \cdot \sum_{l=1}^{m} |C_k I_l \left(\frac{1}{N}\right)| \cdot \max \{|P_v(\sigma)| : \sigma \in S_m, 1 \leq v \leq m\} \leq \frac{1}{2X}$$

and

$$(6.8) \qquad X \leq e^{-(D_1-\epsilon_1)k}.$$

Now (6.3) and (6.7) yield $|L| \cdot |\Delta_{k,0}| \geq 1/2$ whence

$$|L| \geq \frac{1}{2|\Delta_{k,0}|} \geq e^{-(D_2+\epsilon_1)k}$$

via (6.6). Applying (6.8), then, gives

$$|L| \geq X^{\frac{D_2+\epsilon_1}{D_1-\epsilon_1}} \geq X^{D_2/D_1+\epsilon}$$

for suitably small ϵ_1 relative to ϵ and $X \geq X_0(\epsilon)$. This completes the proof of Theorem 1.2.

∎

7. The Case of a Single Binomial Function

In the following analysis, we specialize Theorem 1.1 by fixing $m = 1$, and writing $a_1 = a$ (the ordering of the a_u's, as given in §1, is unimportant here). Further, take $n \geq 3$ and define

$$c_9 = \prod_{p|n} p^{\min\{ord_p a, ord_p n + \frac{1}{p-1}\}}$$

and

$$c_{10} = \exp\left(\frac{1}{\phi(n)} \sum_{\substack{1 \leq r < n/2 \\ (r,n)=1}} \left(\psi\left(\frac{n-r}{n}\right) - \psi\left(\frac{r}{n}\right)\right)\right).$$

We then obtain the following result of Chudnovsky [6, Theorem 5.3] as a corollary to Theorem 1.1:

Theorem 7.1 *If a, N, s and n are integers with $N > |a|, 1 \leq s < n$, $(s, n) = 1, \epsilon > 0$ and $\left(\sqrt{N+a} - \sqrt{N}\right)^2 \cdot c_{10} < c_9$, then*

$$\left| \left(1 + \frac{a}{N}\right)^{s/n} - \frac{p}{q} \right| > q^{-\lambda-\epsilon} \tag{7.1}$$

for p and q integers with $q \geq q_0(\epsilon, a, N, s, n)$ and

$$\lambda = 1 - \frac{\log\left\{\left(\sqrt{N+a} + \sqrt{N}\right)^2 \cdot c_{10}/c_9\right\}}{\log\left\{\left(\sqrt{N+a} - \sqrt{N}\right)^2 \cdot c_{10}/c_9\right\}} .$$

Proof. From (1.1) and (1.2), we have that $c_1 = c_2 = a$. Since $A(z) = z(z-a)$, the zeros of polynomial (1.4) are given by

$$z_0 = -N - \sqrt{N^2 + aN}$$

and

$$z_1 = -N + \sqrt{N^2 + aN} .$$

It follows that

$$c_5 = 2N + a + 2\sqrt{N^2 + aN} = \left(\sqrt{N+a} + \sqrt{N}\right)^2$$

and

$$c_7 = \left|2N + a - 2\sqrt{N^2 + aN}\right| = \left(\sqrt{N+a} - \sqrt{N}\right)^2$$

and so we may apply Theorem 1.1 to deduce a bound of the form (7.1) with

$$\begin{aligned}
\lambda &= 1 + \frac{\log(c_4/c_7 \cdot c_8)}{\log(c_5 \cdot c_8/c_4)} \\
&= 1 + \frac{\log\left(a^2 \cdot c_3/(\sqrt{N+a} - \sqrt{N}\ )^2 \cdot c_8\right)}{\log\left(\left(\sqrt{N+a} + \sqrt{N}\right)^2 \cdot c_8/a^2 \cdot c_3\right)} \\
&= 1 - \frac{\log\left(\left(\sqrt{N+a} + \sqrt{N}\right)^2 \cdot c_3/c_8\right)}{\log\left((\sqrt{N+a} - \sqrt{N}\ )^2 \cdot c_3/c_8\right)}
\end{aligned}$$

provided

$$\left(\sqrt{N+a}-\sqrt{N}\right)^2 \cdot c_3 < c_8 < \left(\sqrt{N+a}+\sqrt{N}\right)^2 \cdot c_3.$$

It remains to show that $c_3/c_8 = c_{10}/c_9$ or, equivalently, that

$$\log(c_8 \cdot c_{10}) = \log(c_3 \cdot c_9). \tag{7.2}$$

Recalling the various definitions of our constants, the left hand side of equation (7.2) becomes

$$\frac{2}{\phi(n)} \sum_{\substack{1\le r<n/2 \\ (r,n)=1}} \left(\psi(1)-\psi\left(\frac{n-r}{n}\right)\right) + \frac{1}{\phi(n)} \sum_{\substack{1\le r<n/2 \\ (r,n)=1}} \left(\psi\left(\frac{n-r}{n}\right)-\psi\left(\frac{r}{n}\right)\right)$$

$$= \frac{1}{\phi(n)} \sum_{\substack{1\le r<n/2 \\ (r,n)=1}} \left(2\psi(1)-\psi\left(\frac{r}{n}\right)-\psi\left(\frac{n-r}{n}\right)\right). \tag{7.3}$$

In order to express $\psi(z)$ with rational arguments as a finite combination of elementary functions, we appeal to a result of Gauss (see e.g. [4]):

$$\begin{aligned}\psi\left(\frac{r}{n}\right) &= -\gamma - \log n - \frac{1}{2}\pi\cot\left(\frac{\pi r}{n}\right) \\ &\quad + \sideset{}{'}\sum_{1\le k\le n/2} \cos(2\pi kr/n)\log(2-2\cos(2\pi k/n)).\end{aligned}$$

Here the prime over the summation indicates that only half the value associated with $k = n/2$ is applied to the sum. The finite sum in (7.3) thus may be written as (again recalling that $\psi(1) = -\gamma$)

$$\log n - \frac{1}{\phi(n)} \sum_{\substack{1\le r<n \\ (r,n)=1}} \sideset{}{'}\sum_{1\le k\le n/2} \cos(2\pi kr/n)\log(2-2\cos(2\pi k/n)). \tag{7.4}$$

Now, if we let $q(n) = \prod_{p|n} p$, then

$$\sum_{\substack{1\le r<n/2 \\ (r,n)=1}} \cos\left(\frac{2\pi kr}{n}\right) = \phi^*\left(\frac{(n,k)}{n}q(n)\right)\cdot\mu\left(\frac{n}{(n,k)}\right)\cdot\frac{n}{q(n)} \tag{7.5}$$

where ϕ^* is the Euler totient for integer arguments, zero otherwise, and μ is the Möbius function. To see this, express the above sum in terms of primitive $n/(n,k)$th roots of unity. Also, if $d > 2$ is a fixed divisor of n,

$$\prod_{\substack{1\le k<n/2\\(n,k)=n/d}} \left(2-2\cos\left(\frac{2k\pi}{n}\right)\right) = \prod_{\substack{1\le j<d/2\\(j,d)=1}} \left(2-2\cos\left(\frac{2j\pi}{d}\right)\right) = \prod_{\substack{1\le j<d\\(j,d)=1}} (1-e^{2\pi ij/d}).$$

If d is prime, then this is the product of the roots of the polynomial

$$\frac{(x+1)^d-1}{x} = x^{d-1} + dx^{d-2} + \cdots + d$$

and hence equal to d. If, however, d has at least two distinct prime factors, then it is not difficult to show (see e.g. Washington [19, Prop. 2.8]) that

$$\prod_{\substack{1\le j<d\\(j,d)=1}} (1-e^{2\pi ij/d}) = 1.$$

Collecting these facts together with (7.5) yields

$$\begin{aligned}
&\sum_{\substack{1\le r<n\\(r,n)=1}} \sum_{1\le k<n/2} \cos\left(\frac{2\pi kr}{n}\right) \log(2-2\cos(2\pi k/n))\\
&= \sum_{\substack{2<d\le n\\d|n}} \phi^*(q(n)/d)\cdot\mu(d)\cdot\frac{n}{q(n)}\cdot\log\left(\prod_{\substack{1\le k<n/2\\(n,k)=n/d}} \left(2-2\cos\left(\frac{2k\pi}{n}\right)\right)\right)\\
&= -\sum_{\substack{p|n\\p>2}} \phi(q(n)/p)\frac{n}{q(n)}\log(p)\\
&= -\sum_{\substack{p|n\\p>2}} \frac{\phi(n)}{p-1}\cdot\log(p)
\end{aligned}$$

where these last two sums are over p prime.

This, with (7.3) and (7.4), implies that

$$\log(c_8\cdot c_{10}) = \log n + \sum_{p|n} \frac{\log p}{p-1} = \log(n\mu_n)$$

where $\mu_n = \prod_{p|n} p^{1/(p-1)}$ (the contribution from $p = 2$, for even n, is obtained by taking $k = n/2$ in (7.4)). Now $c_3 \cdot c_9$ satisfies

$$\begin{aligned} c_3 \cdot c_9 &= \prod_{p|n} p^{\max\{ord_p(n/a) \ + \frac{1}{p-1}, 0\} + \min\{ord_p(a), ord_p(n) + \frac{1}{p-1}\}} \\ &= \prod_{p|n} p^{ord_p(n) + \frac{1}{p-1}} = n\mu_n \end{aligned}$$

whence equality (7.2) holds, finishing the proof.

∎

Application of this result with $n = 3, s = 1, a = 3$ and $N = 125$ enables one to obtain the bound (0.2) while other choices lead to a variety of examples, including those discussed in [6] and [12].

8. Some Applications

As the previous section indicates, it is possible to obtain versions of Theorem 1.1 (and Theorem 1.2, for that matter), for certain m, which are more explicit than that stated in §1. Though we will not explore these aspects here, we note that Rickert's Theorem of [15] (compare to Theorem 1.1 with $n = m = 2$ and $a_u \in \{-1, 0, 1\}$) essentially follows from the fact that the zeros z_0, z_1 and z_2 of polynomial (1.4) are well approximated by $-3N/2, -1/\sqrt{3}$ and $1/\sqrt{3}$, respectively.

In the case $n = m = 2$, we obtain nontrivial bounds from Theorem 1.1 whenever $\lambda < 2$. For larger values of n or m, however, we require not only that $\lambda < n$, but also that λ is smaller than that induced by any proper subset of the related numbers θ_i in (0.1) ($1 \leq i \leq m$). By way of example, direct application of Theorem 1.1 with $n = 3, m = 2, N = 8$, and $a_i \in \{-2, -1, 0\}$ yields

$$\max\left\{ \left| \sqrt[3]{6} - \frac{p_1}{q} \right|, \left| \sqrt[3]{7} - \frac{p_2}{q} \right| \right\} > q^{-2.66985} \tag{8.1}$$

for $q \geq q_0$ effective. If we instead consider $n = 3, m = 1, N = 101847558$ and $a_1 = 5$, we find (see Chudnovsky [6, Table 1])

$$\left| \sqrt[3]{6} - \frac{p}{q} \right| > q^{-2.32056}$$

for $q \geq q_1$ effectively computable, so that (8.1) is in fact weaker than the latter result.

In Table 1, we collect examples of bounds of the form (0.1) for pairs of square roots and cube roots of fairly small integers. We find these examples by considering integers "close" to squares (or cubes) which themselves possess "large" square (or cubic) factors. For instance, we obtain a simultaneous measure for $(\sqrt{3}, \sqrt{5})$ because $123^2 = 3 \cdot 71^2 + 6 = 5 \cdot 55^2 + 4$. It is not difficult to show that if Theorem 1.1 provides a bound for $(\theta_1, \theta_2) = \left(\sqrt{A}, \sqrt{B}\right)$, then it yields like bounds for $\left(\sqrt{A}, \sqrt{AB}\right)$ and $\left(\sqrt{B}, \sqrt{AB}\right)$ with the same value for λ. For example, we have, taking $N = 49$, and $a_i \in \{-1, 0, 1\}$,

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right| \right\} > q^{-1.79155} \ (q \geq q_0)$$

while $N = 100$ and $a_i \in \{-4, -2, 0\}$ gives

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{6} - \frac{p_2}{q} \right| \right\} > q^{-1.79155} \ (q \geq q_1)$$

and $N = 144$ and $a_i \in \{0, 3, 6\}$ yields

$$\max \left\{ \left| \sqrt{3} - \frac{p_1}{q} \right|, \left| \sqrt{6} - \frac{p_2}{q} \right| \right\} > q^{-1.79155} \ (q \geq q_2).$$

We will restrict ourselves in Table 1 to listing only examples which are "primitive" in this sense (to avoid duplication). Analogous results to this hold for other values of n and m and it is of interest to note that while the measure produced from Theorem 1.1 (taking $N = 8$ and $a_i \in \{0, 1, 2\}$) for $(\theta_1, \theta_2) = \left(\sqrt[3]{9}, \sqrt[3]{10}\right)$ is weaker than that obtained for $\sqrt[3]{10}$ alone, the same is not known to be the case for the related pair $\left(\sqrt[3]{3}, \sqrt[3]{30}\right)$.

For larger values of m, a couple of interesting examples are

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right|, \left| \sqrt{47} - \frac{p_3}{q} \right|, \left| \sqrt{51} - \frac{p_4}{q} \right| \right\} > q^{-1.67429}$$

for $q \geq q_0$, and

$$\max \left\{ \left| \sqrt[3]{25} - \frac{p_1}{q} \right|, \left| \sqrt[3]{26} - \frac{p_2}{q} \right|, \left| \sqrt[3]{28} - \frac{p_3}{q} \right|, \left| \sqrt[3]{29} - \frac{p_4}{q} \right| \right\} > q^{-1.86545}$$

for $q \geq q_1$, where both q_0 and q_1 are effective. These follow by taking $N = 49, a_u \in \{-2, -1, 0, 1, 2\}, n = 2$ and $N = 27, a_u \in \{-2, -1, 0, 1, 2\}, n = 3$, respectively, in Theorem 1.1.

It appears to be rather harder to find "attractive" examples of the utility of Theorem 1.2. Part of the difficulty is that if $\theta_1, \ldots \theta_m$ are elements of a real field of degree $m + 1$ over $\mathbb{Q}$ with $1, \theta_1, \ldots \theta_m$ linearly independent over $\mathbb{Q}$, then

$$|x_0 + x_1\theta_1 + \cdots + x_m\theta_m| > X^{-m}$$

for sufficiently large $X = \max_{0 \leq j \leq m} |x_i|$, via consideration of norms (see e.g. Cassels [5, Chapter V]). We are therefore only interested in bounds upon linear forms which sharpen this. One such result follows from Theorem 1.2 with $n = m = 2, N = 816^2$ and $a_i \in \{0, 2, 4\}$, whence

$$|x + y\sqrt{2} + z\sqrt{985}| > X^{-2.89514}$$

for x, y and z integers, $X = \max\{|x|, |y|, |z|\}$ and $X \geq X_0$ effectively computable. We may also attain this bound via transference from Theorem 1.1 (from which one obtains $\lambda = 1.59144$ as a simultaneous irrationality measure for $(\sqrt{2}, \sqrt{985}))$, with direct application of Theorem 1.2 providing an improvement only noticeable in the sixth decimal place of λ_1. In general, Theorem 1.2 represents only a slight sharpening of the corresponding transference result if the a_i's are "evenly" distributed, but is more effective otherwise.

The problem of solving simultaneous Pell's equations is discussed in [14], motivated by the desire to find elliptic curves over $\mathbb{Q}(\sqrt{\ntrianglelefteq})$ with good reduction away from the prime 2, and in [18], related to solutions of Thue equations. Results obtained from Theorem 1.1 may be applied to this problem. For instance, to solve

$$x^2 - 3z^2 = u, \ y^2 - 5z^2 = v \tag{8.2}$$

we use the bound

$$\max\left\{\left|\sqrt{3} - \frac{p_1}{q}\right|, \left|\sqrt{5} - \frac{p_2}{q}\right|\right\} > q^{-1.82227}$$

$(q \geq q_0)$ from Table 1. Arguing as in [15], one may deduce that

$$\max\{|x|, |y|, |z|\} \leq k_0(\max\{|u|, |v|\})^{5.62652}$$

for an effectively computable absolute constant k_0. For fixed u and v, this enables us to find all solutions to (8.2). The examples considered by Rickert [15] are the pair of simultaneous equations

$$x^2 - 2z^2 = u, \ y^2 - 3z^2 = v$$

and

$$x^2 - 2z^2 = u, \ y^2 - 58z^2 = v.$$

For these, we obtain

$$\max\{|x|, |y|, |z|\} \le k_1 (\max\{|u|, |v|\})^{4.79732}$$

and

$$\max\{|x|, |y|, |z|\} \le k_2 (\max\{|u|, |v|\})^{2.67294}$$

respectively, where k_1 and k_2 are effective. Similar results for simultaneous Pell-type equations (of greater degree) may also be readily produced.

9. Norm Form Equations

In [10], Fel'dman considers norm form equations

$$N_{K/\mathbb{Q}}(x_0 w_0 + \cdots + x_m w_m) = f(x_0, \ldots x_m)$$

where $w_0, \ldots w_m$ are certain algebraic numbers, f a polynomial of small degree and $K = \mathbb{Q}(w_0, \ldots w_m)$. To solve these, he essentially utilizes a version of Theorem 1.2 to simultaneously approximate the w_i's. We will show that a slight refinement of his techniques, utilizing both Theorems 1.1 and 1.2, enables one to effectively solve the equation (u constant)

$$N_{K/\mathbb{Q}}\left(x + y\sqrt[4]{M^4 - 1} + z\sqrt[4]{M^4 + 1}\right) = u \tag{9.1}$$

for all $M \ge 6$. Similarly, we find all solutions to

$$N_{K/\mathbb{Q}}\left(\ x + y\sqrt[6]{M^6 - 1} + z\sqrt[6]{M^6 + 1}\right) = u \tag{9.2}$$

for $M \ge 4$.

Let us concentrate on equation (9.1); the argument is essentially unchanged for (9.2). We will require the observation that Theorems 1.1 and 1.2

remain valid if $p_0, \dots p_m, q$ and $x_0, \dots x_m$ are taken to be integers in a fixed imaginary quadratic field (see Fel'dman [9]). Let

$$L = x + y\sqrt[4]{M^4 - 1} + z\sqrt[4]{M^4 + 1}$$

for x, y and z integers, not all zero, and let $L_{s,t}$ $(0 \le s, t \le 3)$ be the conjugates of L associated with the extension $K = \mathbb{Q}(\sqrt[4]{M^4 - 1}, \sqrt[4]{M^4 + 1})$, as given by

$$L_{s,t} = x + y\sqrt[4]{M^4 - 1}\ i^s + z\sqrt[4]{M^4 + 1}\ i^t. \tag{9.3}$$

Reasoning as in [10, §3], all but four of these conjugates satisfy

$$|L_{s,t}| \ge c_0 X \tag{9.4}$$

for c_0 effective, $X = \max\{|x|, |y|, |z|\}$. Let L_i $(1 \le i \le 4)$ be the remaining conjugates, ordered so that

$$|L_1| \le |L_2| \le |L_3| \le |L_4|. \tag{9.5}$$

We apply Theorem 1.2 (with $x, y, z \in \mathbb{Z}[i]$) to estimate the smallest conjugate, finding that

$$|L_1| > X^{-6.35740} \tag{9.6}$$

for $X \ge X_0$ and $M \ge 6$. It follows from (9.3) that

$$|L_2| \ge \frac{1}{2}|L_2 - L_1| = \frac{1}{2}|y\sqrt[4]{M^4 - 1}\ (i^{s_1} - i^{s_2}) + z\sqrt[4]{M^4 + 1}\ (i^{t_1} - i^{t_2})|. \tag{9.7}$$

Applying Theorem 1.1 (more specifically, Theorem 7.1 with $p, q \in \mathbb{Z}[i]$), we have

$$\left| \sqrt[4]{1 + \frac{2}{M^4 - 1}} - \frac{p}{q} \right| > |q|^{-2.49951}$$

for $|q| \ge q_0$ and $M \ge 6$. Thus (9.7) implies

$$|L_2| > X^{-1.49952} \tag{9.8}$$

for $X \ge X_1$ and $M \ge 6$. Inequalities (9.4), (9.5), (9.6) and (9.8) combine to yield

$$|N_{K/\mathbb{Q}}(x + y\sqrt[4]{M^4 - 1} + z\sqrt[4]{M^4 + 1})| = |\prod_{0 \le s,t \le 3} L_{s,t}| > X^{1.14405}$$

for $X \geq X_2$ and $M \geq 6$, which enables one to solve (9.1) as desired.

If we replace the constant u in equation (9.1) or (9.2) by a polynomial in $\mathbb{Z}[x, y, z]$ of fixed degree, then the preceding argument may permit solution of the resulting equation, provided M is large enough. For example, to solve

$$N_{K/\mathbb{Q}}(x + y\sqrt[4]{M^4 - 1} + z\sqrt[4]{M^4 + 1}) = f(x, y, z)$$

we require only that M be bounded as follows.

degree f	$M \geq$
0	6
1	6
2	8
3	10
4	16
5	41
6	777

In essence, in this situation, we are dealing with a Diophantine inequality of the form

$$\left|N_{K/\mathbb{Q}}(x + y\sqrt[4]{M^4 - 1} + z\sqrt[4]{M^4 + 1})\right| < X^\delta$$

which we may solve provided $\delta < 7$.

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Table 1: Cases of Theorem 1.1 for $m = 2$:

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > q^{-\lambda}$$

for $q \geq q_0(\theta_1, \theta_2, \lambda)$.

θ_1, θ_2	$N, a_i \neq 0$	λ	θ_1, θ_2	$N, a_i \neq 0$	λ
$\sqrt{2}, \sqrt{3}$	$7^2, -1, 1$	1.79155	$\sqrt{10}, \sqrt{11}$	$3^2, 1, 2$	1.98873
$\sqrt{2}, \sqrt{7}$	$82^2, 3, 4$	1.88908	$\sqrt{10}, \sqrt{22}$	$136^2, -6, 6$	1.76160
$\sqrt{2}, \sqrt{11}$	$10^2, -2, -1$	1.74788	$\sqrt{10}, \sqrt{31}$	$22^2, 6, 12$	1.99858
$\sqrt{2}, \sqrt{17}$	$4^2, 1, 2$	1.89813	$\sqrt{10}, \sqrt{38}$	$6^2, 2, 4$	1.88331
$\sqrt{2}, \sqrt{23}$	$24^2, -1, 2$	1.97500	$\sqrt{14}, \sqrt{15}$	$4^2, -2, -1$	1.91627
$\sqrt{2}, \sqrt{29}$	$140^2, 2, 4$	1.62588	$\sqrt{15}, \sqrt{17}$	$4^2, -1, 1$	1.90671
$\sqrt{2}, \sqrt{51}$	$7^2, 1, 2$	1.79008	$\sqrt{19}, \sqrt{22}$	$61^2, -3, 3$	1.79694
$\sqrt{3}, \sqrt{5}$	$123^2, -6, -4$	1.82227	$\sqrt{21}, \sqrt{23}$	$5^2, -4, -2$	1.96101
$\sqrt{3}, \sqrt{23}$	$5^2, -2, 2$	1.94595	$\sqrt{23}, \sqrt{47}$	$48^2, -4, -1$	1.95362
$\sqrt{3}, \sqrt{26}$	$5^2, 1, 2$	1.84700	$\sqrt{34}, \sqrt{35}$	$6^2, -2, -1$	1.81853
$\sqrt{3}, \sqrt{29}$	$5^2, 2, 4$	1.93291	$\sqrt{34}, \sqrt{38}$	$6^2, -2, 2$	1.89037
$\sqrt{3}, \sqrt{35}$	$71^2, -1, 2$	1.83947	$\sqrt{35}, \sqrt{37}$	$6^2, -1, 1$	1.81607
$\sqrt{3}, \sqrt{47}$	$7^2, -2, -1$	1.79307	$\sqrt{37}, \sqrt{38}$	$6^2, 1, 2$	1.81372
$\sqrt{5}, \sqrt{11}$	$199^2, -1, 4$	1.90617	$\sqrt[3]{3}, \sqrt[3]{21}$	$3^3, -6, -3$	2.57831
$\sqrt{5}, \sqrt{47}$	$7^2, -4, -2$	1.85791	$\sqrt[3]{3}, \sqrt[3]{30}$	$3^3, -3, 3$	2.50876
$\sqrt{5}, \sqrt{79}$	$9^2, -2, -1$	1.75964	$\sqrt[3]{5}, \sqrt[3]{9}$	$3^3, -3, -2$	2.52360
$\sqrt{5}, \sqrt{82}$	$9^2, -1, 1$	1.75893	$\sqrt[3]{7}, \sqrt[3]{9}$	$2^3, -1, 1$	2.57831
$\sqrt{5}, \sqrt{93}$	$29^2, -4, 4$	1.82863	$\sqrt[3]{22}, \sqrt[3]{43}$	$14^3, 6, 8$	2.22145
$\sqrt{6}, \sqrt{23}$	$5^2, -2, -1$	1.85558	$\sqrt[3]{23}, \sqrt[3]{25}$	$3^3, -4, -2$	2.78961
$\sqrt{6}, \sqrt{26}$	$5^2, -1, 1$	1.85115	$\sqrt[3]{25}, \sqrt[3]{26}$	$3^3, -2, -1$	2.13916
$\sqrt{6}, \sqrt{59}$	$169^2, -5, 5$	1.79760	$\sqrt[3]{25}, \sqrt[3]{29}$	$3^3, -2, 2$	2.73786
$\sqrt{7}, \sqrt{38}$	$37^2, -1, 3$	1.99312	$\sqrt[3]{26}, \sqrt[3]{28}$	$3^3, -1, 1$	2.13097
$\sqrt{7}, \sqrt{55}$	$37^2, 3, 6$	1.84857	$\sqrt[3]{28}, \sqrt[3]{29}$	$3^3, 1, 2$	2.12328
$\sqrt{7}, \sqrt{62}$	$8^2, -2, -1$	1.77425	$\sqrt[3]{29}, \sqrt[3]{30}$	$3^3, 2, 3$	2.48614
$\sqrt{7}, \sqrt{65}$	$8^2, -1, 1$	1.77324	$\sqrt[3]{29}, \sqrt[3]{31}$	$3^3, 2, 4$	2.69341
$\sqrt{7}, \sqrt{83}$	$82^2, -1, 3$	1.88907	$\sqrt[3]{30}, \sqrt[3]{33}$	$3^3, 3, 6$	2.45379

REFERENCES

1. A. Baker, Rational approximations to certain algebraic numbers. *Proc. London Math. Soc.* (3) **14** (1964), 385-398.

2. A. Baker, Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers. *Quart. J. Math. Oxford Ser.* (2) **15** (1964), 375-383.

3. A. Baker, Simultaneous rational approximations to certain algebraic numbers. *Proc. Cambridge Philos. Soc.* **63** (1967), 693-702.

4. H. Bateman and A. Erdélyi, Higher Transcendental Functions, McGraw-Hill, 1953, v.1.

5. J.W.S. Cassels, An Introduction to Diophantine Approximation, Cambridge University Press, Cambridge, 1957.

6. G.V. Chudnovsky, On the method of Thue-Siegel, *Ann. of Math.,* II Ser. **117** (1983), 325-382.

7. J. Dieudonné, Calcul infinitésimal, Hermann, Paris, 1968.

8 D. Easton, Effective irrationality measures for certain algebraic numbers. *Math. Comp.* **46**, no. 174 (1986), 613-622.

9. N.I. Fel'dman, An estimate of an incomplete linear form in several algebraic variables. *Math. Notes* **7** (1970), 343-349.

10. N.I. Fel'dman, Effective bounds for the number of solutions of certain diophantine equations. *Math. Notes* **8** (1970), 674-679.

11. M. Hata, Rational approximation to π and some other numbers. *Acta Arith.* **63** (1993), 335-349.

12. A. Heimonen, T. Matala-aho and K. Väänänen, An application of Jacobi type polynomials to irrationality measures, *Bull. Austral. Math. Soc.* **50** (1994), 225-243.

13. C.F. Osgood, The simultaneous diophantine approximation of certain kth roots. *Proc. Cambridge Philos. Soc.* **67** (1970), 75-86.

14. R.G.E. Pinch, Simultaneous Pellian equations, *Proc. Cambridge Philos. Soc.* **103** (1988), 35-46.

15. J.H. Rickert, Simultaneous rational approximations and related diophantine equations. *Proc. Cambridge Philos. Soc.* **113** (1993), 461-472.

16. K.R. Roth. Rational approximations to algebraic numbers. *Mathematika* **2** (1955), 1-20: corrigendum, *ibid.* **168.**

17. W.M. Schmidt, Simultaneous approximation to algebraic numbers by rationals. *Acta Math.* **125** (1970), 189-201.

18. N. Tzanakis, Explicit solution of a class of quartic Thue equations, *Acta Arith.* **LXIV 3** (1993), 271-283.

19. L. Washington, Introduction to Cyclotomic fields, Springer-Verlag, GTM83, 1982.

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