Solving Norm Form Equations Via Lattice Basis Reduction

Michael A. Bennett

Dedicated to Wolfgang M. Schmidt on the occasion of his sixtieth birthday.

Abstract

The author uses irrationality and linear independence measures for certain algebraic numbers to derive explicit upper bounds for the solutions of related norm form equations. The Lenstra-Lenstra-Lovász lattice basis reduction algorithm is then utilized to show that the integer solutions to

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}+z)=\pm 1$$

(where $K=\mathbb{Q}(\sqrt[4]{N^4-1},\sqrt[4]{N^4+1})$) are given by $(x,y,z)=(0,0,\pm 1)$, $(\pm 1,0,\pm N)$ and $(0,\pm 1,\pm N)$, for $5\leq N\leq 100$.

1 Introduction

There has been a great deal of recent work published on techniques for finding the integer solutions of certain Diophantine equations. Most of the effective results in this area rely upon Baker's theory of linear forms in logarithms (for surveys of applications of this method to diophantine problems, the reader is directed to [19] and [22]). Via this approach, for instance, it is possible to find explicit upper bounds for the size of solutions to a given Thue equation

$$F(x,y)=m$$

where $F(x,y)\in\mathbb{Z}[x,y]$ is, say, an irreducible binary form (of degree ≥ 3) and m is a nonzero integer. Since these bounds are often extremely large, it is necessary to combine this with computational techniques from Diophantine approximation in order to fully determine all solutions (see e.g. [20], [23], [24], [25], [26] and [27]).

In this paper, we restrict our attention to norm form equations of the specific type

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}+z)=u \tag{1.1}$$

for integral u and $K=\mathbb{Q}(\sqrt[4]{N^4-1},\sqrt[4]{N^4+1})$. By a theorem of Schmidt [17], these equations have only finitely many solutions for each fixed $N\geq 2$. Further, from work of Győry and Papp [11] (see also Győry [10] and Kotov [12]), since

$$\left[\mathbb{Q}(\sqrt[4]{N^4-1},\sqrt[4]{N^4+1}):\mathbb{Q}(\sqrt[4]{N^4+1})\right]=4$$

and

$$\left[\mathbb{Q}(\sqrt[4]{N^4+1}):\mathbb{Q}\right]=4$$

it follows that we may find effective bounds for solutions to (1.1) through the theory of linear forms in logarithms. For additional results along these lines, see the papers of Gaál [7], [8], [9] and Sprindžuk [21].

In [1], Baker gave a technique for solving restricted classes of norm form equations without using linear forms in logarithms. Instead, he deduced effective lower bounds for the linear forms dividing the given norms via the method of Padé approximation to binomial functions. Fel'dman [6] also took this approach and showed how to bound solutions to

$$N_{K/\mathbb{Q}}(x_1\theta_1 + x_2\theta_2 + \cdots + x_m\theta_m) = f(x_1, x_2, \ldots, x_m)$$

where $K = \mathbb{Q}(\theta_1, \ldots, \theta_m)$, f is a polynomial in $x_1, \ldots, x_m$ and $\theta_1, \ldots, \theta_m$ are algebraic numbers satisfying certain approximation properties. Neither author, however, explicitly solved any particular norm form equations.

Here, we will follow Fel'dman's exposition closely, deriving, in Section 2, lower bounds for forms related to

$$|x\sqrt[4]{N^4-1} \ + \ y\sqrt[4]{N^4+1} \ + \ z|.$$

In Sections 3 and 4, we apply these to show, for instance, that solutions to (1.1) with $N \geq 20$ satisfy

$$\max\{|x|, |y|, |z|\} < 10^6 \ N^{5/2} \ |u|^{1/6.7}.$$

Through the use of the algorithm of Lenstra, Lenstra and Lovász for lattice basis reduction (see [13]), we are able to reduce these bounds and solve

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z) = \pm 1 \tag{1.2}$$

for $5 \leq N \leq 100$, finding that, in each case, all solutions are given by $(x, y, z) = (0, 0, \pm 1)$, $(\pm 1, 0, \pm N)$ and $(0, \pm 1, \pm N)$.

2 Some Diophantine Approximation Results

In [2], following work of Osgood [14], Fel'dman [5] and Rickert [16], we considered the problem of simultaneously approximating functions of the form

$$(1 + a_0 x)^{s/n}, \ldots, (1 + a_m x)^{s/n}$$

where the a_i's are distinct integers, $a_0 = 0, |a_i| < |x|^{-1}$ for $0 \leq i \leq m$ and s and n are positive, relatively prime integers with $s < n$. These approximations derive from the integral (see Rickert [16])

$$I_i(x) = \frac{1}{2\pi i} \int_\gamma \frac{(1+zx)^k(1+zx)^{s/n}}{(z-a_i)(A(z))^k} \, dz \ (0 \leq i \leq m) \tag{2.1}$$

where k is a positive integer,

$$A(z) = \prod_{i=0}^{m} (z - a_i)$$

and γ a closed, counter-clockwise contour enclosing the poles of the integrand. From the Residue theorem,

$$I_i(x) = \sum_{j=0}^{m} \ p_{ij}(x)(1 + a_j x)^{s/n} \ (0 \leq i \leq m)$$

where the $p_{ij}(x)$'s are polynomials in x with rational coefficients and degree at most k. By Lemma 3.3 of [16], we have

$$p_{ij}(x) = \sum \binom{k+s/n}{h_j} (1+a_j x)^{k-h_j} x^{h_j} \prod_{\substack{l=0 \\ l \neq j}}^{m} \binom{-k_{il}}{h_l} (a_j - a_l)^{-k_{il}-h_l} \tag{2.2}$$

where $\sum$ refers to summation over nonnegative $h_0, \ldots, h_m$ with $h_0 + \cdots + h_m = k + \delta_{ij} - 1$ for δ_{ij} the Kronecker delta. Bounding $|p_{ij}(1/N)|$ and $|I_i(1/N)|$ and finding, for eack k, rational C_k such that $C_k p_{ij}(1/N)$ is integral, we may deduce lower bounds for simultaneous rational approximation to the numbers

$$1, (1+a_1/N)^{s/n}, \ldots, (1+a_m/N)^{s/n}$$

through application of

Lemma 2.1 Let $\theta_1, \ldots \theta_m$ be arbitrary real numbers and $\theta_0 = 1$. Suppose there exist positive real numbers l, p, L and P $(L > 1)$ such that for each positive integer k, we can find integers p_{ijk} $(0 \leq i, j \leq m)$ with nonzero determinant,

$$|p_{ijk}| \leq pP^k \ (0 \leq i, j \leq m)$$

and

$$\left| \sum_{j=0}^{m} p_{ijk} \theta_j \right| \leq l L^{-k} \ (0 \leq i \leq m).$$

Then we may conclude that

$$\max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \ldots, \left| \theta_m - \frac{p_m}{q} \right| \right\} > c q^{-\lambda}$$

for all integers $p_1, \ldots, p_m$ and q, where

$$\lambda = 1 + \frac{\log(P)}{\log(L)}$$

and

$$c^{-1} = 2 \ m \ p \ P \ (\max(1, 2l))^{\lambda - 1}.$$

Proof: This is a slight modification of Lemma 2.1 of [16]. □

This result will allow us to derive lower bounds for the moduli of the linear factors of the norm forms in (1.1). We take $s = 1$ and $n = 4$ and consider separately the cases

$$(1) \ m = 1, \ a_1 = 2$$

and

$$(2) \ m = 2, \ a_1 = -1, \ a_2 = 1.$$

Our techniques for finding upper bounds for $|p_{ij}(1/N)|$ and $|I_i(1/N)|$ follow from those of Rickert [16] and, while not asymptotically sharp, are suitable for our purposes. We prove

Lemma 2.2 If $0 \le i, j \le m$, then
(i) If $m = 1$ and $a_1 = 2$, then

$$|p_{ij}(1/N)| \le 1.43 \left(\frac{N + \sqrt{2} + 1}{N} \right)^{1/4} \left(1 + \frac{\sqrt{2}+1}{N} \right)^{k}.$$

(ii) If $m = 2$, $a_1 = -1$ and $a_2 = 1$, then

$$|p_{ij}(1/N)| \le 1.55 \left(\frac{N\sqrt{3} + 2}{N\sqrt{3} - \sqrt{3}} \right)^{1/4} \left(\frac{3\sqrt{3}}{2} \left(1 + \frac{2}{N\sqrt{3}} \right) \right)^{k}.$$

Proof: (i) We note that $p_{ij}(1/N)(1 + a_j/N)^{1/4}$ is given by the same integral as (2.1), only with the contour changed so as to enclose only the pole corresponding to $z = a_j$, for $0 \le j \le m$. Since $a_0 = 0$ and $a_1 = 2$, it follows that the lemniscate

$$|z(z-2)| = 1$$

splits into two such contours, each, by numerical integration, of length less than 3.709. Further, on this lemniscate, we have

$$\sqrt{2} - 1 \le |z|, |z-2| \le \sqrt{2} + 1.$$

These inequalities, together with (2.1) imply that

$$|p_{ij}(1/N)| \le \frac{3.709}{2\pi} \left(\frac{1}{\sqrt{2}-1} \right) \left(1 + \frac{\sqrt{2}+1}{N} \right)^{1/4} \left(1 + \frac{\sqrt{2}+1}{N} \right)^{k}$$

which is less than

$$1.43 \left(\frac{N + \sqrt{2} + 1}{N} \right)^{1/4} \left(1 + \frac{\sqrt{2}+1}{N} \right)^{k}.$$

(ii) This is a special case of Lemma 2.2 in [3] which in turn follows from Rickert's Lemma 4.1 in [16]. □

For the integrals $I_i(1/N)$, we have

Lemma 2.3 If $0 \le i \le m$, then
(i) If $m = 1$ and $a_1 = 2$, then

$$|I_i(1/N)| \le \frac{5N}{8(N-2)} \left(4N(N-2)\right)^{-k}.$$

(ii) If $m = 2, a_1 = -1$ and $a_2 = 1$, then

$$|I_i(1/N)| \le \frac{135N}{512(N-1)} \left(\frac{27}{4}(N^3 - N) \right)^{-k}$$

Proof: (i) From (2.1), we may write

$$I_i(x) = \sum_{h=0}^{\infty} \binom{k+1/4}{h} x^h J_{ih} \ (0 \le i \le m) \tag{2.3}$$

where

$$J_{ih} = \frac{1}{2\pi i} \int_\gamma \frac{z^h}{(z-a_i)(A(z))^k} \, dz$$

vanishes for $h < (m+1)k$. By Lemma 3.2 of [16], if we let

$$J_i(x) = \sum_{h=0}^{\infty} x^h J_{ih} \ (0 \le i \le m)$$

then

$$J_i(x) = \frac{-1}{(1-a_i x)(A(1/x))^k} \ (0 \le i \le m) \tag{2.4}$$

provided $|x|^{-1} > |a_i|$, for $0 \le i \le m$. From (2.3), we have

$$|I_i(1/N)| \le \sum_{h=0}^{\infty} \left| \binom{k+1/4}{h} \right| |J_{ih}| N^{-h}$$

and so if $m = 1$ and $n = 4$, from

$$\left| \binom{k+1/4}{h} \right| \le \left| \binom{k+1/4}{2k} \right| \ (h \ge 2k),$$

we have

$$|I_i(1/N)| \le \left| \binom{k+1/4}{2k} \right| \sum_{h=0}^{\infty} |J_{ih}| N^{-h}. \tag{2.5}$$

Now, from (2.4),

$$J_0(x) = \frac{-x^{2k}}{(1-2x)^k}$$

whence $J_{0h} \le 0$ for all h. Also

$$J_1(x) = \frac{-x^{2k}}{(1-2x)^{k+1}},$$

so that $J_{1h} \le 0$ for all h. Thus (2.5) yields

$$|I_i(1/N)| \le \left| \binom{k+1/4}{2k} \right| |J_i(1/N)|$$

and the inequalities

$$\left| \binom{k+1/4}{2k} \right| \le \left(\frac{5}{8} \right) 4^{-k} \ (k \ge 1)$$

and

$$|J_i(1/N)| \le \left(\frac{N}{N-2} \right) (N(N-2))^{-k}$$

imply the result as stated.
(ii) The result here is essentially just Lemma 2.3 in [3].
□

We now turn our attention to determining, for each k, rational C_k such that $C_k p_{ij}(1/N)$ are integral for $0 \le i, j \le m$.

From [2], we have

Lemma 2.4 (i) If $m = 1$ and $a_1 = 2$, then

$$2^{4k}\ p_{ij}(x) \in \mathbb{Z}[x].$$

(ii) If $m = 2$, $a_1 = -1$ and $a_2 = 1$, then

$$2^{4k-1}\ p_{ij}(x) \in \mathbb{Z}[x].$$

Proof: See [2], Lemma 3.1 of [3] and Lemma 4.3 of [16] for details. □

The shape of the coefficients of the p_{ij}'s, as given in (2.2), suggests the presence of potentially large integer common factors. It is these factors that enable us to sharpen the work of Osgood, Fel'dman and Rickert and extend our results to a wider class of norm form equations. Define, for $\{x\} = x - [x]$ and $1 \le r < n$, $S(r, m, n, k)$ to be the set of all primes p satisfying $p > \sqrt{nk+1}, (p, nk) = 1$, $pr \equiv 1 \bmod n$ and $\left\{\frac{k-1}{p}\right\} > \max\left\{\frac{mn-r}{mn}, \frac{r}{n}\right\}$. If $m = 1$, we add the additional restriction that $(p, nk - n - 1) = 1$. We proved in [2] that

Lemma 2.5 If $p \in S(r, m, n, k)$, then

$$\operatorname{ord}_p\left(\binom{k+1/n}{h_0}\binom{k+h_1-1}{h_1}\cdots\binom{k+h_m-1}{h_m}\right) \ge 1$$

for all nonnegative integers $h_0, h_1 \ldots, h_m$ with sum equal to k or $k-1$.

Define $\Pi_1(k)$ to be the greatest common divisor of the coefficients of all the polynomials $2^{4k}p_{ij}(x)$ $(0 \le i, j \le 1)$ for $m = 1$ and $a_1 = 2$ and similarly define $\Pi_2(k)$ relative to $2^{3k-1}p_{ij}(x)$ $(0 \le i, j \le 2, m = 2, a_1 = -1, a_2 = 1)$. Then Lemma 2.5 implies that

Lemma 2.6 We have

$$(i)\ \Pi_1(k) > \frac{1}{143}(3/2)^k\ (k \ge 1)$$

$$(ii)\ \Pi_2(k) > \frac{1}{679}(4/3)^k\ (k \ge 1).$$

Proof: The proof of (ii) is given in [3] and depends upon recent estimates for primes in arithmetic progressions due to Ramaré and Rumely [15]. The first assertion follows from Lemma 3.3 of [3], where, via bounds upon the Chebyshev function

$$\theta(x) = \sum_{p \le x} \log p$$

from Schoenfeld [18], it is shown that

$$\Pi_1(k) > (3/2)^k$$

for all $k \ge 271$. Explicitly computing the coefficients of the $p_{ij}(x)$'s and their greatest common divisor, for $1 \le k \le 270$, yields (i). □

We separately consider

$$p_{ijk} = 2^{4k}\ N^k\ \Pi_1(k)^{-1}\ p_{ij}(1/N)\ (m=1, a_1=2)$$

and

$$p_{ijk} = 2^{4k-1}\ N^k\ \Pi_2(k)^{-1}\ p_{ij}(1/N)\ (m=2, a_1=-1, a_2=1).$$

Applying Lemma 2.1 and arguing as in [3], while noting that Lemma 3.4 of [16] ensures the nonvanishing of $\det(p_{ijk})$, we find

Theorem 2.7 (i) If p, q and N are positive integers with $N \geq 255$, then

$$\left| \sqrt[4]{1+\frac{2}{N}}\ -\ \frac{p}{q} \right| > \left(3.7 \times 10^7\ N\right)^{-1}\ q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log\left(\frac{32N+32(\sqrt{2}+1)}{3}\right)}{\log\left(\frac{3(N-2)}{8}\right)}.$$

(ii) If p_1, p_2, q and N are positive integers with $N \geq 256$, then

$$\max\left\{ \left| \sqrt[4]{1-\frac{1}{N}}\ -\ \frac{p_1}{q} \right|, \left| \sqrt[4]{1+\frac{1}{N}}\ -\ \frac{p_2}{q} \right| \right\} > \left(5.6 \times 10^6\ N\right)^{-1}\ q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(18\sqrt{3}N+36)}{\log\left(\frac{9}{16}(N^2-1)\right)}.$$

We remark that in Theorem 4.3 of [3], we derive a weaker version of (ii) subject to the condition $N \geq 4$.

3 A Class of Norm Form Equations

We now turn our attention to equation (1.1), where we suppose $N \geq 4$. It is straightforward to show that

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z) = \prod_{0 \leq s,t \leq 3} L_{s,t}$$

where

$$L_{s,t} = i^s x\sqrt[4]{N^4-1}\ +\ i^t y\sqrt[4]{N^4+1}\ +\ z, \tag{3.1}$$

so to solve (1.1) effectively, it will suffice to deduce suitable lower bounds for the linear forms $|L_{s,t}|$.

Throughout this section, we will assume that $xyz \neq 0$, dealing with the degenerate cases in section 4. Define $X = \max\{|x|, |y|, |z|\}$ and suppose that $X = |z|$. If L is any of the forms $L_{s,t}$ $(0 \leq s, t \leq 3)$, we associate to L the forms L_k $(1 \leq k \leq 3)$ defined by

$$L_k = i^k L\ +\ (1-i^k)z. \tag{3.2}$$

Since

$$\max\{|L|, |L_k|\} \geq \frac{1}{2}\left(|L_k - i^k L|\right) = \frac{1}{2}|1-i^k||z|$$

we conclude that the product of the three largest of $|L|, |L_1|, L_2|$ and $|L_3|$ is bounded below by $X^3/2$. Since this construction divides the forms $L_{s,t}$ $(0 \leq s, t \leq 3)$ into four disjoint groups of four forms each (say by taking L to be, successively, $L_{0,0}$, $L_{0,1}$,$L_{0,2}$ and $L_{0,3}$), we have that the product of the twelve largest of the $|L_{s,t}|$ is at least $X^{12}/16$.

Let the four smallest of the $|L_{s,t}|$ be those associated with the forms $L^{(i)}$ for $0 \leq i \leq 3$, where $|L^{(0)}|$ is minimal. If $\mathrm{Im}(L^{(0)}) \neq 0$, then since $xyz \neq 0$, considering real and imaginary parts yields

$$|L^{(0)}| \geq 1$$

whence

$$\left|N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z)\right| \geq \frac{1}{16}\ X^{12}. \tag{3.3}$$

If, however, $\mathrm{Im}(L^{(0)}) = 0$ and $|L^{(0)}| < 1$, then, without loss of generality, $L^{(0)} = L_{0,0}$ and $L^{(1)}, L^{(2)}$ and $L^{(3)}$ belong to the disjoint classes of forms associated via equation (3.2) to $L_{0,1}, L_{0,2}$ and $L_{0,3}$ respectively. Again considering real and imaginary parts, it follows that

$$|L^{(k)}| \geq |\mathrm{Im}(L^{(k)})| \geq \sqrt[4]{N^4-1}$$

for $k = 1$ or 3, while

$$|L^{(2)}| \geq \min\{2\ \sqrt[4]{N^4-1} - 1, X\} := m(N).$$

The last inequality follows from the fact that the real forms $L_{0,2}$ and $L_{2,0}$ differ from $L_{0,0}$ by $2y\sqrt[4]{N^4+1}$ and $2x\sqrt[4]{N^4-1}$ respectively. Thus, if $xyz \neq 0, X = |z|$ and $|L_{0,0}| < 1$, then

$$\left|N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z)\right| > \frac{m(N)}{16}\ |L_{0,0}|\ \sqrt{N^4-1}\ X^{12}. \tag{3.4}$$

It remains to bound $|L_{0,0}|$ from below. By Theorem 2.7 (ii), we have, if p_1, p_2, q and N are nonzero integers with $N \geq 4$, then

$$\max\left\{\left|\sqrt[4]{N^4-1}\ -\ \frac{p_1}{q}\right|, \left|\sqrt[4]{N^4+1}\ -\ \frac{p_2}{q}\right|\right\} > c_1^{-1} q^{-\lambda} \tag{3.5}$$

where

$$\lambda = 1 + \frac{\log(18\sqrt{3}N^4 + 36)}{\log(9(N^8-1)/16)}$$

and

$$c_1 = 5.6 \times 10^6\ N^{\lambda+3}.$$

To move from these lower bounds to ones for the related linear forms, we use a standard transference principle, namely

Lemma 3.1 Suppose that θ_1 and θ_2 are real numbers such that, if p_1, p_2 and q are any positive integers, then

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > c_1^{-1}\ q^{-\lambda}$$

where $c_1 > 0$ and $\lambda < 2$. We conclude that if x, y and z are integers, not all zero, and $X = \max\{|x|, |y|, |z|\}$, then

$$|\ x\ \theta_1\ +\ y\ \theta_2\ +\ z\ | > c_2^{-1}\ X^{-\lambda_1}$$

where $c_2 = c_1^{\frac{2}{2-\lambda}}\ 2^{\frac{2\lambda}{2-\lambda}}$ and $\lambda_1 = \frac{2\lambda-2}{2-\lambda}$.

Proof: This is just a special case of Theorem II, Chapter 5 of [4]. □

Applying this result in our situation yields

$$\left| x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z \right| > \left(c_1^{-2} 2^{-2\lambda} X^{-2\lambda+2}\right)^{\frac{1}{2-\lambda}}$$

where $X = \max\{|x|,|y|,|z|\}$. Thus, we may conclude, from (3.3) and (3.4), that if $xyz \neq 0$ and $X = |z|$, then

$$\left| N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z) \right| > \frac{m(N)}{16} \; (c_1 2^\lambda)^{\frac{-2}{2-\lambda}} \; \sqrt{N^4-1} \; X^{\frac{26-14\lambda}{2-\lambda}} \tag{3.6}$$

for c_1 and λ as in (3.5).

If, on the other hand, $X = |x|$ or $X = |y|$, we argue in a similar fashion, only with the linear forms $L_{s,t}$ of (3.1) divided into disjoint groups of four by associating to a given form L, the three forms

$$L_k = L + i^k x\sqrt[4]{N^4-1} \; (1 \leq k \leq 3)$$

or

$$L_k = L + i^k y\sqrt[4]{N^4+1} \; (1 \leq k \leq 3)$$

respectively. The lower bounds obtained for

$$|N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z)|$$

are in both cases at least as strong as (3.6) and hence (3.6) holds for any $xyz \neq 0$ with $X = \max\{|x|,|y|,|z|\}$. Since it is relatively easy to show by calculus that the functions

$$\left(\frac{1}{16}(c_1 2^\lambda)^{\frac{-2}{2-\lambda}} \sqrt{N^4-1} \; (2\sqrt[4]{N^4-1} - 1) \right)^{\frac{\lambda-2}{26-14\lambda}} N^{-5/2}$$

and

$$\left(\frac{1}{16}(c_1 2^\lambda)^{\frac{-2}{2-\lambda}} \sqrt{N^4-1} \right)^{\frac{\lambda-2}{28-15\lambda}} N^{-5/2}$$

are decreasing in N (for $N \geq 4$), by computing the various constants in (3.6), we attain

Theorem 3.2 If x, y, z, u and N are integers with $xyz \neq 0, N \geq 4$ and

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z) = u$$

then

$$\max\{|x|,|y|,|z|\} < c_2 N^{5/2} |u|^{1/c_3}$$

where we may take c_2 and c_3 as given in the following table :

N	c_2	c_3	N	c_2	c_3
4	1.1×10^{649}	0.2	13	2.4×10^7	7.6
5	1.4×10^{26}	3.8	14	1.2×10^7	7.7
6	1.1×10^{16}	5.2	15	6.7×10^6	7.8
7	3.1×10^{12}	6.0	16	4.2×10^6	7.8
8	4.6×10^{10}	6.5	17	2.7×10^6	7.9
9	3.4×10^9	6.8	18	1.9×10^6	8.0
10	5.6×10^8	7.1	19	1.4×10^6	8.0
11	1.5×10^8	7.3	≥ 20	10^6	8.1
12	5.3×10^7	7.4			

The functions $c_2(N)$ and $c_3(N)$ in the above tend to 0 and 10, respectively, as N tends to infinity.

Let us now consider equation (1.2) with $5 \leq N \leq 100$ and $xyz \neq 0$. The previous result asserts that we may bound solutions to these equations by

$$\max\{|x|, |y|, |z|\} \leq M$$

where M may be taken as follows :

$$\begin{array}{rl} N = 5 & M = 10^{28} \\ N = 6 & M = 10^{18} \\ N = 7 & M = 10^{15} \\ 8 \leq N \leq 100 & M = 10^{13} \end{array}$$

To reduce these upper bounds to a workable size, we apply the Lenstra-Lenstra-Lovász (L^3) algorithm, following closely the work of Tzanakis and de Weger in [24] (see also [25]).

If $xyz \neq 0$, then to have

$$\left|N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z)\right| = 1$$

we must have, without loss of generality,

$$|L_{0,0}| = \left|x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z\right| \leq X^{-12} \tag{4.1}$$

where

$$X = \max\{|x|, |y|, |z|\} \leq M.$$

Choose an integer c_0 such that $c_0 > M^3$ and consider the lattice Γ associated with the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \left[c_0\sqrt[4]{N^4-1}\right] & \left[c_0\sqrt[4]{N^4+1}\right] & c_0 \end{pmatrix}.$$

We apply the L^3 algorithm to find a reduced basis $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ of Γ and note that by Proposition (1.11) of [13], if $\mathbf{x} \neq \mathbf{0} \in \Gamma$, then

$$|\mathbf{x}| \geq \frac{1}{2}|\mathbf{b_1}|. \tag{4.2}$$

Since

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ \Lambda \end{pmatrix} \in \Gamma$$

where

$$\Lambda = x\left[c_0\sqrt[4]{N^4-1}\right] + y\left[c_0\sqrt[4]{N^4+1}\right] + zc_0,$$

we have

$$|\Lambda - c_0 L_{0,0}| < 2M.$$

Therefore

$$|\Lambda| < c_0\,|L_{0,0}| + 2M. \tag{4.3}$$

On the other hand, (4.2) implies that

$$x^2 + y^2 + \Lambda^2 \geq \frac{1}{4}|\mathbf{b_1}|^2$$

so that

$$|\Lambda| \geq \sqrt{\frac{1}{4}|\mathbf{b_1}|^2 - 2M^2} \tag{4.4}$$

provided $|\mathbf{b_1}| > \sqrt{2}M$. Together, (4.3) and (4.4) yield the inequality

$$|L_{0,0}| > \frac{1}{c_0}\left(\sqrt{\frac{1}{4}|\mathbf{b_1}|^2 - 2M^2} - 2M\right) := F(c_0, N, M)$$

as long as $|\mathbf{b_1}| > \sqrt{2}M$ and so from (4.1), we conclude that

$$X \leq C(c_0, N, M)$$

where

$$C(c_0, N, M) = \begin{cases} F(c_0, N, M)^{-1/12} & \text{if } |\mathbf{b_1}| > 2\sqrt{6}M \\ M & \text{otherwise.} \end{cases}$$

To explicitly perform the lattice basis reduction for specific choices of c_0, N and M, we utilize an existing implementation of the L^3 algorithm in Maple V. If we set $N = 5$ and apply this procedure, we find that

$$C(10^{87}, 5, 10^{28}) < 111250$$

so that in this case, if $xyz \neq 0$, then solutions to (1.2) satisfy

$$\max\{|x|, |y|, |z|\} \leq 111249.$$

Since

$$C(10^{18}, 5, 111249) < 14$$

and

$$C(10^9, 5, 100) < 4,$$

two further iterations reduce the above bound to

$$\max\{|x|, |y|, |z|\} \leq 3.$$

Similarly, the inequalities

$$C(10^{57}, 6, 10^{18}) < 1633,$$
$$C(10^{13}, 6, 1633) < 7,$$
$$C(10^9, 6, 100) < 4,$$
$$C(10^{48}, 7, 10^{15}) < 538$$

and

$$C(10^{11}, 7, 537) < 5$$

imply that

$$\max\{|x|, |y|, |z|\} \leq 3 \text{ or } 4$$

for solutions to (1.2) with $xyz \neq 0$ and $N = 6$ or 7, respectively. In all the remaining cases (i.e. $8 \leq N \leq 100$), we find that $C(10^{52}, N, 10^{13})$ is less than 1657 and are able to further reduce these bounds by computing $C(c_0, N, 1656)$ as follows :

$$\begin{array}{rcrc}
C(10^{13}, N, 1656) < 6 & N = 8 & C(10^{18}, N, 1656) < 13 & 49 \leq N \leq 57 \\
C(10^{13}, N, 1656) < 7 & 9 \leq N \leq 13 & C(10^{19}, N, 1656) < 14 & 58 \leq N \leq 65 \\
C(10^{14}, N, 1656) < 8 & 14 \leq N \leq 20 & C(10^{19}, N, 1656) < 15 & 66 \leq N \leq 73 \\
C(10^{15}, N, 1656) < 9 & 21 \leq N \leq 27 & C(10^{20}, N, 1656) < 16 & 74 \leq N \leq 82 \\
C(10^{16}, N, 1656) < 10 & 28 \leq N \leq 35 & C(10^{20}, N, 1656) < 17 & 83 \leq N \leq 91 \\
C(10^{17}, N, 1656) < 11 & 36 \leq N \leq 42 & C(10^{20}, N, 1656) < 18 & 92 \leq N \leq 100. \\
C(10^{17}, N, 1656) < 12 & 43 \leq N \leq 48 & &
\end{array}$$

Explicitly computing

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z)$$

for these remaining cases, we conclude

Theorem 3.3 If x, y, z and N are integers with $5 \leq N \leq 100$, then the equation

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z) = \pm 1$$

has no solutions with $xyz \neq 0$.

4 Degenerate Cases

We now turn our attention to the situation when at least one of x, y or z vanishes. To avoid trivialities, we assume that not all of x, y and z are zero, so that $X = \max\{|x|, |y|, |z|\}$ is at least 1.

If $x = 0$, then

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z) = \prod_{k=0}^{3} (i^k y\sqrt[4]{N^4+1} + z)^4. \tag{3.7}$$

Since

$$\left|\pm iy\sqrt[4]{N^4+1} + z\right| \geq \max\{|y|\sqrt[4]{N^4+1}, |z|\} \geq X$$

and

$$\max\{\left|y\sqrt[4]{N^4+1} + z\right|, \left|-y\sqrt[4]{N^4+1} + z\right|\} \geq \max\{|y|\sqrt[4]{N^4+1}, |z|\} \geq X,$$

it remains to bound the smaller of $\left|\pm y\sqrt[4]{N^4+1} + z\right|$.

By Theorem 2.7 (i), we have

$$\left|\pm y\sqrt[4]{N^4+1} + z\right| > (7.4 \times 10^7 N^4)^{-1} X^{-\lambda_1+1}$$

for

$$\lambda_1 = 1 + \frac{\log(32(2N^4 + \sqrt{2} + 1)/3)}{\log(3(N^4-1)/4)},$$

where the inequality follows from treating the cases $|z| \geq |y|N$ and $|z| < |y|N$ separately. Thus (3.7) implies that

$$\left|N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1} + y\sqrt[4]{N^4+1} + z)\right| > (7.4 \times 10^7 N^4)^{-4} X^{16-4\lambda_1}. \tag{3.8}$$

The case when $y=0$ is similar. We use Theorem 2.7 (i), supposing that $z \neq 0$, to deduce a bound for

$$\left|\pm x\sqrt[4]{N^4-1}+z\right| = \left(\frac{|z|}{N}\sqrt[4]{N^4-1}\right)\left|\sqrt[4]{1+\frac{1}{N^4-1}} \pm \frac{xN}{z}\right|$$

and find that

$$\left|N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}+z)\right| > (7.4\times 10^7 N^4)^{-4} X^{16-4\lambda_2} \tag{3.9}$$

where

$$\lambda_2 = 1+\frac{\log(32(2N^4+\sqrt{2}-1)/3)}{\log(3(N^4-2)/4)}.$$

If, however, $z=0$, then

$$\left|N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}+z)\right| = \prod_{k=0}^{3} |i^k x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}|^4$$

and we have

$$\left|\pm ix\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}\right| \geq \sqrt[4]{N^4-1}\ X$$

and

$$\max\left\{\left|-x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}\right|, \left|x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}\right|\right\} \geq \sqrt[4]{N^4-1}\ X.$$

Further, Theorem 2.7 (i) gives

$$\left|\pm x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}\right| > (3.7\times 10^7 N^3)^{-1} X^{-\lambda_3+1}$$

with

$$\lambda_3 = 1+\frac{\log(32(N^4+\sqrt{2})/3)}{\log(3(N^4-3)/8)}.$$

It follows that in this situation, we have

$$\left|N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}+z)\right| > (3.8\times 10^7)^{-4} X^{16-4\lambda_3} \tag{3.10}$$

and so, combining (3.8), (3.9) and (3.10), computing

$$(7.4\times 10^7 N^4)^{\frac{4}{16-4\lambda_2}}$$

and noting that this quantity decreases as N increases yields

Theorem 4.1 If x, y, z, u and N are integers with $xyz=0$, $N \geq 4$ and

$$N_{K/\mathbb{Q}}(x\sqrt[4]{N^4-1}+y\sqrt[4]{N^4+1}+z) = u$$

then

$$\max\{|x|, |y|, |z|\} < c_2 N^{5/2} |u|^{1/c_3}$$

for c_2 and c_3 as follows :

N	c_2	c_3	N	c_2	c_3
4	1118042	5.0	11	54227	6.4
5	383474	5.5	12	47158	6.5
6	205615	5.8	13	41769	6.5
7	134945	6.0	14	37532	6.6
8	98951	6.1	15	34119	6.6
9	77699	6.2	16	31311	6.6
10	63868	6.3	≥ 17	28963	6.7

Here, the functions $c_2(N)$ and $c_3(N)$ tend to 0 and 8, respectively, as $N \to \infty$.

We once again set out to reduce the above bounds upon solutions to (1.2), via the L^3 algorithm. If $x = 0$, from our previous remarks we have, without loss of generality,

$$\left| y \sqrt[4]{N^4+1} + z \right| \leq X^{-3}$$

where $X = \max\{|y|, |z|\} \leq M$. Choose $c_0 > M^2$ and consider Γ derived from

$$A = \begin{pmatrix} 1 & 0 \\ \left[c_0 \sqrt[4]{N^4+1}\right] & c_0 \end{pmatrix}.$$

As before, we find a reduced basis $\mathbf{b_1}, \mathbf{b_2}$ for Γ with (by Proposition (1.11) of [13])

$$|\mathbf{x}| \geq \frac{1}{\sqrt{2}} |\mathbf{b_1}|$$

for all $\mathbf{x} \neq \mathbf{0} \in \Gamma$. Arguing as previously, we find that

$$X \leq C_1(c_0, N, M)$$

where

$$C_1(c_0, N, M) = \begin{cases} F_1(c_0, N, M)^{-1/3} & \text{if } |\mathbf{b_1}| > 2M \\ M & \text{otherwise} \end{cases}$$

and

$$F_1(c_0, N, M) = \frac{1}{c_0} \left(\sqrt{\frac{1}{2} |\mathbf{b_1}|^2 - M^2} - M \right).$$

For each N between 5 and 100, we note that Theorem 4.1 implies the bound $X \leq 10^{10}$. We find that

$$C_1(10^{26}, N, 10^{10}) < 152780, \; 5 \leq N \leq 100$$

and

$$C_1(10^{16}, N, 152779) < 3563, \; 5 \leq N \leq 100.$$

Since we also have

$$C_1(10^{11}, N, 3562) < \begin{cases} 221 & 5 \leq N \leq 39 \\ 2N & 40 \leq N \leq 100, \end{cases}$$

$$C_1(10^{9}, N, 220) < \begin{cases} 73 & 5 \leq N \leq 17 \\ 2N & 18 \leq N \leq 39, \end{cases}$$

$$C_1(10^7, N, 72) < \begin{cases} 19 & 5 \leq N \leq 8 \\ 2N & 9 \leq N \leq 17 \end{cases}$$

and

$$C_1(300000, N, 18) < 2N$$

for $5 \leq N \leq 8$, it follows that solutions to (1.2) with $x = 0$ satisfy

$$\max\{|y|, |z|\} < 2N.$$

We handle the cases when $y = 0$ or $z = 0$ in a similar fashion, considering lattices generated by

$$A = \begin{pmatrix} 1 & 0 \\ \left[c_0 \sqrt[4]{N^4 - 1}\right] & c_0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 & 0 \\ \left[c_0 \sqrt[4]{N^4 - 1}\right] & \left[c_0 \sqrt[4]{N^4 + 1}\right] \end{pmatrix}$$

respectively. In either situation, we again find an explicit pair of functions $C_2(c_0, N, M)$ and $C_3(c_0, N, M)$ with

$$\max\{|x|, |y|, |z|\} \leq \begin{cases} C_2(c_0, N, M) & \text{if } y = 0 \\ C_3(c_0, N, M) & \text{if } z = 0 \end{cases}$$

and through application of the L^3 algorithm as above, find in all cases that solutions to

$$N_{K/\mathbb{Q}}(x \sqrt[4]{N^4 - 1} + y \sqrt[4]{N^4 + 1} + z) = \pm 1$$

with $5 \leq N \leq 100$ and $xyz = 0$ satisfy

$$\max\{|x|, |y|, |z|\} < 2N.$$

To conclude, we are only left to check the remaining small values for x, y and z. To do this, we once again explicitly compute

$$N_{K/\mathbb{Q}}(x \sqrt[4]{N^4 - 1} + y \sqrt[4]{N^4 + 1} + z)$$

and see whether (1.2) is satisfied. Performing this computation, in conjunction with Theorem 3.3, yields

Theorem 4.2 If x, y, z and N are integers with $5 \leq N \leq 100$, then the equation

$$N_{K/\mathbb{Q}}(x \sqrt[4]{N^4 - 1} + y \sqrt[4]{N^4 + 1} + z) = \pm 1$$

has only the ten integral solutions given by $(x, y, z) = (0, 0, \pm 1)$,$(\pm 1, 0, \pm N)$ and $(0, \pm 1, \pm N)$.

5 Concluding Remarks

In the results of the previous section, we have omitted the case $N = 4$ due to computational constraints. If $N \leq 3$, we are unable to produce any bound upon supposed solutions to (1.1) by this method. Since, as mentioned in the introduction, it is possible to derive explicit bounds via linear forms in logarithms, for $N \geq 2$, it would be of interest to see how difficult such a computation would be to carry out and how the results would compare to those given in Sections 3 and 4.

6 Acknowledgements

The author would like to thank Professor Cameron Stewart for helpful discussions on the content of this paper and the referee for suggesting improvements in the exposition.

References

[1] A. Baker. Simultaneous rational approximations to certain algebraic numbers. *Proc. Cambridge Philos. Soc.* **63** (1967), 693-702.

[2] M. Bennett. Simultaneous rational approximation to binomial functions. *Trans. Amer. Math. Soc.*, to appear.

[3] M. Bennett. Simultaneous approximation to pairs of algebraic numbers. to appear.

[4] J.W.S. Cassels. An Introduction to Diophantine Approximation. Cambridge University Press, Cambridge, 1957.

[5] N.I. Fel'dman. An estimate of an incomplete linear form in several algebraic variables. *Math. Notes* **7** (1970), 343-349.

[6] N.I. Fel'dman. Effective bounds for the number of solutions of certain diophantine equations. *Math. Notes* **8** (1970), 674-679.

[7] I. Gaál. Norm form equations with several dominating variables and explicit lower bounds for inhomogeneous linear forms with algebraic coefficients. *Studia Sci. Math. Hungar.***19** (1984), 399-411.

[8] I. Gaál. Norm form equations with several dominating variables and explicit lower bounds for inhomogeneous linear forms with algebraic coefficients II. *Studia Sci. Math. Hungar.***20** (1985), 333-344.

[9] I. Gaál. On the resolution of inhomogeneous norm form equations in two dominating variables. *Math. Comp.* **51**, no.183 (1988), 359-373.

[10] K. Győry. On S-integral solutions to norm form, disriminant form and index form equations. *Studia Sci. Math. Hungar.***16** (1981), 149-161.

[11] K. Győry and Z.Z. Papp. Effective estimates for the integer solutions of norm form and discriminant form equations. *Publ. Math. Debrecen* **25** (1978), 311-325.

[12] S.V. Kotov. Effective bounds for the values of the solutions of a class of Diophantine equations of norm form type. *Mat. Zametki* **33** (1983), 801-806.

[13] A.K. Lenstra, H.W. Lenstra and L. Lovász. Factoring polynomials with rational coefficients. *Math. Ann.* **261** (1982), 515-534.

[14] C.F. Osgood. The simultaneous diophantine approximation of certain kth roots. *Proc. Cambridge Philos. Soc.* **67** (1970), 75-86.

[15] O. Ramaré and R. Rumely. Primes in arithmetic progressions. University of Georgia preprint no. 14.

[16] J.H. Rickert. Simultaneous rational approximations and related diophantine equations. *Proc. Cambridge Philos. Soc.* **113** (1993), 461-472.

[17] W. Schmidt. Linearformen mit algebraischen Koeffizienten II. *Math. Ann.* **191** (1971), 1-20.

[18] L. Schoenfeld. Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$ II *Math. Comp.* **30** (1976), 337-360.

[19] T.N. Shorey and R. Tijdeman. Exponential Diophantine Equations. Cambridge University Press, Oxford, 1986.

[20] N.P. Smart. The solution of triangularly connected decomposable form equations. *Math. Comp.* **64** (1995), 819-840.

[21] V.G. Sprindžuk. Classical Diophantine Equations. Springer Verlag, Lecture Notes in Mathematics 1559, 1993.

[22] R. Tijdeman. On the Gel'fond-Baker Method and it's Application. In volume 28 of Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, R.I., 1976, 241-268.

[23] N. Tzanakis. Explicit solution to a class of quartic Thue equations. *Acta Arith.* **LXIV 3** (1993), 271-283.

[24] N. Tzanakis and B.M.M. de Weger. On the practical solution of the Thue equation. *J. Number Theory* **31** (1989), 99-132.

[25] N. Tzanakis and B.M.M. de Weger. How to explicitly solve a Thue-Mahler equation. *Comp. Math.* **84** (1992), 223-288.

[26] B.M.M. de Weger. Solving exponential diophantine equations using lattice basis reduction algorithms. *J. Number Theory* **26** (1987), 325-367.

[27] B.M.M. de Weger. A Thue equation with quadratic integers as variables. *Math. Comp.* **64** (1995), 855-861.

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
E-mail address : mabennet@math.lsa.umich.edu