Explicit Lower Bounds for Rational Approximation to Algebraic Numbers

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Abstract

In this paper, we apply Padé approximation methods to derive completely explicit measures of irrationality for certain classes of algebraic numbers. Our approach is similar to that taken previously by G.V. Chudnovsky but has some fundamental advantages with regards to determining implicit constants. Our general results may be applied to produce specific bounds of the flavour of

$$\left|\sqrt[3]{2}-\frac{p}{q}\right|>\frac{1}{4}\;q^{-2.45}\quad\text{and}\quad\left|\sqrt[7]{5}-\frac{p}{q}\right|>\frac{1}{4}\;q^{-4.43}$$

which we show to hold for any nonzero integers p and q. Further examples are tabulated and applications to Diophantine equations are briefly discussed as are other topics of related interest.

1 Introduction

Lower bounds for rational approximation to algebraic numbers have been studied for many years, both for their intrinsic interest and for their applications to Diophantine equations. At one end of the spectrum, we have the weak bound due to Liouville who showed that if θ is an algebraic number of degree n, then

$$\left|\theta-\frac{p}{q}\right|>c(\theta)\;q^{-n}$$

for nonzero integers p and q. At the other end, we have the famous inequality of Roth [19] to the effect that, for $\epsilon>0$,

$$\left|\theta-\frac{p}{q}\right|>c(\theta,\epsilon)\;q^{-2-\epsilon}$$

again for any nonzero p and q. Unfortunately, while the first of these bounds is effective in that the constant $c(\theta)$ is computable from Liouville's proof, Roth's theorem is not. Efforts to produce effective improvements upon Liouville's bound have centred about three lines of attack: the theory of linear forms in logarithms (see e.g. [3], [4] and [13]), the Thue principle (see [8] and [9]) and the method of Baker-Siegel, utilizing rational function approximation (see [1], [2], [5], [6], [11], [12], [14] and [17]). While the first of these techniques is indisputably more general, we restrict our attention in this paper to the last, which is characterized by attractive and strong bounds in special settings. In [1] and [2], via Padé approximation to the binomial function $(1-z)^{\nu}$, Baker was able to obtain effective improvements upon Liouville's theorem for restricted classes of algebraic numbers. In particular, he showed that

$$\left|\sqrt[3]{2}-\frac{p}{q}\right|>10^{-6}q^{-2.955}\tag{1}$$

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for all positive integers p and q. This implies that solutions (x, y) to the Diophantine equation

$$x^3 - 2y^3 = u$$

satisfy

$$\max\{|x|, |y|\} \leq \left(3 \cdot 10^5 \cdot |u|\right)^{23}.$$

These results were subsequently sharpened by Chudnovsky [11], who improved inequality (1) to

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > c\ q^{-2.42971} \tag{2}$$

for p and q positive integers and c some effectively computable constant. The value of c, however, was left undetermined and, for technical reasons, is in fact extremely difficult to compute. In [6], though, the author used a variation upon Chudnovsky's approach to produce a completely explicit version of (2), of close to the same strength, namely that

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{1}{4}\ q^{-2.47} \tag{3}$$

for integers p and q ($q \neq 0$). This implies that

$$\left| x^3 - 2y^3 \right| \geq \sqrt{x}$$

for all nonnegative integers x and y. The essential ingredient of this approach, as compared to that of [11] and [12], is that certain required estimates for primes in arithmetic progressions may be replaced by sharper estimates over all primes in given intervals. While, in the latter paper, we considered only examples of cubic irrationalities, here we turn our attention to algebraic numbers of higher degree. Defining, for positive integers a and n,

$$\kappa(a, n) = \prod_{p|n} p^{\max\left\{ord_p(n/a) + \frac{1}{p-1}, 0\right\}}$$

where the product is over prime p, we prove

Theorem 1.1 For integer n, define the constant $c_1(n)$ by

n	$c_1(n)$	n	$c_1(n)$	n	$c_1(n)$	n	$c_1(n)$	n	$c_1(n)$
3	2.03	11	1.67	23	1.53	41	1.45	59	1.40
4	1.62	13	1.65	29	1.51	43	1.43	61	1.39
5	1.84	17	1.58	31	1.51	47	1.44	67	1.38
7	1.76	19	1.56	37	1.46	53	1.40	71	1.36

Suppose that a, N, s and n are integers with $1 \leq a < N$, $1 \leq s < n/2$, $(s, n) = 1$ and n occuring in the above table. If, further, we have that

$$\left(\sqrt{N} + \sqrt{N + a}\right)^{2(n-2)} > a^{2(n-1)} \left(\frac{\kappa(a, n)}{c_1(n)}\right)^n,$$

then we can conclude that

$$\left| \left(1 + \frac{a}{N}\right)^{s/n} - \frac{p}{q} \right| > N^{-1}\ (10^{10}\ q)^{-\lambda}$$

with

$$\lambda = 1 + \frac{\log\left(\frac{\kappa(a,n)}{c_1(n)}\left(\sqrt{N}+\sqrt{N+a}\right)^2\right)}{\log\left(\frac{c_1(n)}{a^2\kappa(a,n)}\left(\sqrt{N}+\sqrt{N+a}\right)^2\right)}.$$

Through application of the above theorem, we may derive explicit irrationality measures for certain $\sqrt[n]{m}$ with $m \in \mathbb{Z}$. For algebraic numbers of this form, generating distinct number fields and satisfying $2 \le m \le 50$, we have

Corollary 1.2 If p and q are positive integers, then we have

$$\left|\theta - \frac{p}{q}\right| > c(\theta)\ q^{-\lambda(\theta)}$$

where we may take $\theta, c(\theta)$ and $\lambda(\theta)$ as follows:

θ	$c(\theta)$	$\lambda(\theta)$	θ	$c(\theta)$	$\lambda(\theta)$	θ	$c(\theta)$	$\lambda(\theta)$
$\sqrt[4]{5}$	0.03	2.77	$\sqrt[5]{18}$	0.21	4.29	$\sqrt[7]{10}$	0.38	5.17
$\sqrt[4]{14}$	0.06	3.78	$\sqrt[5]{22}$	0.14	4.91	$\sqrt[7]{11}$	0.40	3.34
$\sqrt[4]{15}$	0.03	3.27	$\sqrt[5]{28}$	0.05	3.41	$\sqrt[7]{12}$	0.42	3.88
$\sqrt[4]{17}$	0.03	3.24	$\sqrt[5]{30}$	0.02	3.04	$\sqrt[7]{13}$	0.44	4.91
$\sqrt[4]{18}$	0.05	3.67	$\sqrt[5]{31}$	0.01	2.83	$\sqrt[7]{17}$	0.03	5.20
$\sqrt[4]{37}$	0.33	3.34	$\sqrt[5]{33}$	0.01	2.82	$\sqrt[7]{23}$	0.43	6.03
$\sqrt[4]{39}$	0.005	2.52	$\sqrt[5]{34}$	0.02	3.02	$\sqrt[7]{45}$	0.27	5.10
$\sqrt[5]{3}$	0.24	3.61	$\sqrt[5]{37}$	0.05	3.48	$\sqrt[11]{48}$	0.42	5.05
$\sqrt[5]{6}$	0.43	3.33	$\sqrt[5]{39}$	0.08	2.91	$\sqrt[13]{6}$	0.14	4.22
$\sqrt[5]{10}$	0.41	3.92	$\sqrt[5]{40}$	0.09	3.90	$\sqrt[13]{20}$	0.25	5.87
$\sqrt[5]{11}$	0.38	4.23	$\sqrt[5]{42}$	0.11	4.19	$\sqrt[17]{50}$	0.25	6.96
$\sqrt[5]{15}$	0.28	4.27	$\sqrt[7]{5}$	0.25	4.43			

For a forthcoming paper [7] on Diophantine equations, we require a specialization of Theorem 1.1 to the cases where $a = s = 1$, valid for all $n \ge 3$. Defining

$$\mu_n = \prod_{p \mid n} p^{\frac{1}{p-1}},$$

we have

Theorem 1.3 If n and N are positive integers with $n \ge 3$ and

$$\left(\sqrt{N}+\sqrt{N+1}\right)^{2(n-2)} > (n\mu_n)^n$$

then

$$\left|\sqrt[n]{1+\frac{1}{N}} - \frac{p}{q}\right| > (8\ n\ \mu_n\ N)^{-1}\ q^{-\lambda}$$

with

$$\lambda = 1 + \frac{\log\left(n\mu_n\left(\sqrt{N}+\sqrt{N+1}\right)^2\right)}{\log\left(\frac{1}{n\mu_n}\left(\sqrt{N}+\sqrt{N+1}\right)^2\right)}.$$

Before proceeding, we note that the restriction in Theorem 1.1 that $1 \leq s < n/2$ is unimportant since a nontrivial irrationality measure for $(1 + a/N)^{s/n}$ implies one for $(1 + a/N)^{1-s/n}$ (the latter being a rational multiple of the reciprocal of the former). We similarly exclude nonprime $n \geq 6$ from Theorem 1.1, since an inequality of the form

$$\left| \left(1 + \frac{a}{N}\right)^{s/n} - \frac{p^k}{q^k} \right| < c_1 \; (q^k)^{-\lambda}$$

with $\lambda < n$, implies

$$\left| \left(1 + \frac{a}{N}\right)^{s/nk} - \frac{p}{q} \right| < c_2 \; q^{-k\lambda}.$$

For this reason, in Corollary 1.2, we do not list irrationality measures for numbers of the form (e.g.) $\sqrt[6]{m}$. Applying our machinery directly with $n = 6$ fails to yield new examples with $m \leq 100$ (though the related measures are on occasion improved). Bounds for rational approximation to certain $\sqrt[3]{m}$ are given in [6] and will not be repeated here, though we mention that Theorem 1.1 slightly sharpens the corresponding result in [6]. This implies, for instance, an improvement of (3) to

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{1}{4} \; q^{-2.45}$$

for all positive p and q.

2 Technical Preliminaries

To deduce the aforementioned results, we appeal to the following lemma, obtained from the proof of Lemma 2.1 of [17]:

Lemma 2.1 Suppose θ is real and that there exist positive real numbers c, d, C and D ($D > 1$) such that for each positive integer k with $k \geq k_0$ ($k_0 \in \mathbb{N}$), we can find integers p_{lmk} ($0 \leq l, m \leq 1$) with nonzero determinant,

$$|p_{lmk}| \leq c \; C^k \quad (0 \leq l, m \leq 1)$$

and

$$|p_{l0k} \; + \; p_{l1k}\theta| \leq d \; D^{-k} \quad (0 \leq l \leq 1).$$

Then, if t is any real number with $t > 1$, we may conclude that

$$\left| \theta - \frac{p}{q} \right| > \left(t \; c \; C \left(\max\left\{ 1, \frac{td}{t-1} \right\} \right)^{\frac{\log(C)}{\log(D)}} \right)^{-1} q^{-1-\frac{\log(C)}{\log(D)}}$$

for all positive integers p and q with $q \geq D^{k_0 - 1}$.

As in [6], to apply this result we require a sequence of good rational approximations to numbers of the form $(1 + a/N)^{s/n}$. To find these, we construct the diagonal Padé approximants to $(1 + ax)^{s/n}$, from the contour integral

$$I_l(x) = \frac{1}{2\pi i} \int_\gamma \frac{(1 + zx)^{k+s/n}}{(z - la)(z(z-a))^k} \; dz \quad (0 \leq l \leq 1)$$

where $|x| < 1/a$ and γ is a closed counter-clockwise contour enclosing 0 and a. Lemma 3.3 of [17] implies, then, that

$$I_l(x) = p_{l0}(x) \ + \ (1+ax)^{s/n} \ p_{l1}(x) \quad (0 \leq l \leq 1)$$

where

$$p_{lm}(x) = a^{-2k} \sum_{r=0}^{k_{lm}} (-1)^{mr+k_{lm}} \binom{k+\frac{s}{n}}{r} \binom{2k-r-1}{k_{lm}-r} (ax)^r \ (1+axm)^{k-r}$$

with $k_{lm} = k - 1 + \delta_{lm}$ and δ_{lm} the Kronecker delta $(0 \leq l, m \leq 1)$. We prove

Lemma 2.2 Suppose that a, N, s and n are positive integers satisfying the hypotheses of either Theorem 1.1 or Theorem 1.3 and that $0 \leq l, m \leq 1$ and $k \geq 1$. Then

$$(i) \ \ |I_l(1/N)| \leq \frac{5.2s}{n} \left(N \left(\sqrt{N} + \sqrt{N+a} \right)^2 \right)^{-k}$$

and

$$(ii) \ \ |p_{lm}(1/N)| \leq 1.3 \left(\frac{\left(\sqrt{N} + \sqrt{N+a} \right)^2}{a^2 N} \right)^k .$$

Proof: (i) For $n = 3$, the result follows from Lemma 2.1 of [6], upon noting that

$$\left(\sqrt{N} + \sqrt{N+a} \right)^2 > a^4 \ \left(\frac{\kappa(a,3)}{2.03} \right)^3$$

implies that $N \geq 4a$ (since N and a are integral). We may therefore assume that $n \geq 4$. Suppose that a, n and N satisfy

$$\left(\sqrt{N} + \sqrt{N+a} \right)^{2(n-2)} > a^{2(n-1)} \ \left(\frac{\kappa(a,n)}{c} \right)^n$$

where we will take $c = c_1(n)$ or $c = 1$ as appropriate. Fixing $\delta = N/a$, this therefore yields

$$2\delta + 1 + 2\sqrt{\delta(\delta+1)} > \left(\frac{a \ \kappa(a,n)}{c} \right)^{\frac{n}{n-2}} \geq \left(\frac{n\mu_n}{c} \right)^{\frac{n}{n-2}} . \tag{4}$$

From the proof of Lemma 3.2 of [5],

$$|I_l(1/N)| = \frac{|1 - e^{2\pi i s/n}|}{2\pi} \ N^{-2k} \ \int_0^\infty \frac{x^{k+s/n} dx}{(x+1+al/N)\left((x+1)(x+1+a/N)\right)^k}$$

whence, for $k \geq 4$, we can utilize the inequalities

$$|1 - e^{2\pi i s/n}| < \frac{2\pi s}{n}$$

and

$$\frac{x^{s/n}}{x+1} < 1$$

(where the latter is valid provided $x > 0$) to deduce that

$$|I_l(1/N)| < \frac{s}{n}\ N^{-2k} \int_0^\infty \left(\frac{x}{(x+1)(x+1+a/N)}\right)^k dx.$$

Write this last integral as

$$\int_0^4 \left(\frac{x}{(x+1)(x+1+a/N)}\right)^k dx \ + \ \int_4^\infty \left(\frac{x}{(x+1)(x+1+a/N)}\right)^k dx.$$

By calculus, the first of these is bounded by

$$4\ N^k \ \left(\sqrt{N}+\sqrt{N+a}\right)^{-2k}$$

while the second is bounded above by

$$\int_4^\infty \left(\frac{x}{(x+1)^2}\right)^k dx < 5^{-k}$$

for $k \geq 4$. From (4), we find that $\delta > 2$ for each $n \geq 4$ with either choice of $c = c_1(n)$ or $c = 1$. This implies that

$$\frac{\left(\sqrt{N}+\sqrt{N+a}\right)^2}{N} < 5$$

in all cases under consideration, so that

$$\int_0^\infty \left(\frac{x}{(x+1)(x+1+a/N)}\right)^k dx < 5 \left(\frac{\left(\sqrt{N}+\sqrt{N+a}\right)^2}{N}\right)^{-k}$$

which gives the desired result.

If $k = 1$, then we note that

$$\int_0^\infty \frac{x^{1+s/n}dx}{(x+1)^2(x+1+a/N)} < \int_0^\infty \frac{x^{1+s/n}dx}{(x+1)^3}$$

which equals

$$\frac{n\Gamma(1-\frac{s}{n})\Gamma(3+\frac{s}{n})}{2(2n+s)}.$$

Since $1 \leq s < n/2$, this is less than $3\pi/8$ and hence less than 1.1781. Now, from (4), we have $\delta > 5$ for $n = 4$ and $n \geq 23$ (assuming $c = c_1(n)$) and for all $n \geq 4$ (with $c = 1$). This implies that

$$\frac{\left(\sqrt{N}+\sqrt{N+a}\right)^2}{N} < 4.4$$

and hence that

$$|I_l(1/N)| < \frac{5.2s}{n}\left(N\left(\sqrt{N}+\sqrt{N+a}\right)^2\right)^{-1}. \qquad (5)$$

For $n = 5, 7, 11, 13$ and 17, we compute

$$\int_0^\infty \frac{x^{1+s/n}dx}{(x+1)^3}$$

for each $1 \leq s < n/2$, find explicit values for $|1 - e^{2\pi i s/n}|$ and use (4) to derive lower bounds upon the related δ's. In all cases, we verify that (5) is satisfied.

Finally, if $2 \leq k \leq 3$, we utilize the inequality

$$\int_0^\infty \frac{x^{k+s/n}dx}{(x+1)\left((x+1)\left(x+1+a/N\right)\right)^k} < \int_0^1 \frac{x^k dx}{(x+1)^{2k+1}} + \int_1^\infty \frac{x^{k+1/2}dx}{(x+1)^{2k+1}}$$

and check that the right hand side is bounded above by 5.2×5^{-k} in both cases, whence

$$|I_l(1/N)| < \frac{5.2s}{n}\left(N\left(\sqrt{N}+\sqrt{N+a}\right)^2\right)^{-k}$$

for all k, as desired.

(ii) Again, we may assume (via Lemma 2.3 of [6]) that $n \geq 4$. Repeating the arguments from that lemma, we have that

$$|p_{lm}(1/N)| \leq \sqrt{\frac{N+a}{N}}\left(2 - \sqrt{\frac{N}{N+a}}\right)^{s/n}\left(\frac{\left(\sqrt{N}+\sqrt{N+a}\right)^2}{a^2 N}\right)^k.$$

From (4), in all cases except when $n = 5, 7$ or 11 and $c = c_1(n)$, we have that $\delta > 3$, whence

$$\sqrt{\frac{N+a}{N}}\left(2 - \sqrt{\frac{N}{N+a}}\right)^{s/n} < \sqrt{\frac{N+a}{N}}\left(2 - \sqrt{\frac{N}{N+a}}\right)^{1/2} < 1.3.$$

If $n = 5, 7$ or 11 and $c = c_1(n)$, then we have δ bounded below by $11/5, 16/7$ or $31/11$, respectively. It follows that

$$\sqrt{\frac{N+a}{N}}\left(2 - \sqrt{\frac{N}{N+a}}\right)^{s/n} < \sqrt{\frac{N+a}{N}}\left(2 - \sqrt{\frac{N}{N+a}}\right)^{(n-1)/2n} < 1.3$$

in each remaining case, completing the proof. □

3 Coefficients of the $p_{lm}(1/N)$

As in [5], [6], [11], [12] and [14], we refine more classical estimates through a careful analysis of the common factors present in the numerators of the (rational) coefficients of the polynomials $p_{lm}(x)$. Lemma 4.2 of [11] implies that if n, s, r, j and k are nonnegative integers with $1 \leq s < n$ and $k \geq 1$, then

$$n^j \prod_{p|n} p^{\left[\frac{j}{p-1}\right]} \binom{k+s/n}{r}$$

is an integer for each $0 \le r \le j$ (where $[x]$ denotes the greatest integer not greater than x). We may thus define

$$G(j) = \gcd \left\{ n^j \prod_{p|n} p^{\left[\frac{j}{p-1}\right]} \binom{k+s/n}{r} \binom{2k-r-1}{j-r} \; : \; r = 0, 1, \ldots j \right\}$$

and take

$$G_k = \gcd \{ G(k), G(k-1) \},$$

suppressing dependence on s and n. We prove

Lemma 3.1 For n and $c_1(n)$ appearing in Theorem 1.1 and k and s positive integers with $1 \le s < n/2$, we have

$$G_k > e^{-16} \; c_1(n)^k$$

except for $n = 4$, where

$$G_k > e^{-20} \; c_1(3)^k.$$

To obtain this result, we require

Lemma 3.2 Suppose that $k \ge 1$ and let s and n be integers with $n \ge 3$, $1 \le s < n/2$ and $(s, n) = 1$. If p is a prime with $(p, n) = 1$ and q is chosen such that $1 \le q < n/2$ and

$$pq \equiv \pm s \pmod{n}$$

then if

$$p \in \bigcup_{j=1}^{[\sqrt{k/n}]} \left[\frac{k+1}{j}, \frac{nk-n-1}{nj-q} \right], \tag{6}$$

we have that p divides G_k.

Proof: (Of Lemma 3.2) This is essentially just a slight generalization of Lemma 3.1 of [6]. Condition (6) implies that $(p, k) = 1$, $p > \sqrt{nk+s}$ and $\left\{ \frac{k-1}{p} \right\} > \frac{n-q}{n}$ which together are sufficient to guarantee that one of $\text{ord}_p \binom{k+s/n}{r}$ or $\text{ord}_p \binom{2k-r-1}{j-r}$ is positive for $j = k$ and $k-1$ and $0 \le r \le j$. □

For $k \ge 10^6$, Lemma 3.1 will follow from application of Chebyshev-like estimates for primes, either in arithmetic progressions (due to Ramaré-Rumely [16]) or otherwise (due to Rosser-Schoenfeld [18] and Schoenfeld [20]), to the sets described in Lemma 3.2. The reader is directed to [6] for a more detailed account in the case when $n = 3$. We illustrate the proof for $n = 5$.

Suppose $n = 5$, $s = 1$ or 2 and $k \ge 10^6$. Lemma 3.2 implies that if $p \equiv \pm s \pmod 5$ and $\frac{k+1}{j} \le p \le \frac{5k-6}{5j-1}$ for some positive integer $j \le \sqrt{k/5}$, then p divide G_k. Similarly, if $p \equiv \pm 2s \pmod 5$ and $\frac{k+1}{j} \le p \le \frac{5k-6}{5j-2}$ for some $j \le \sqrt{k/5}$, then p divides G_k. We apply Corollary 2* and the closing remarks of [20] to derive a lower bound for the product of the primes in

$$I_1 = \bigcup_{j=1}^{[\sqrt{k/5}]} \left[\frac{k+1}{j}, \frac{5k-6}{5j-1} \right]$$

and then use Theorems 1 and 2 of [16] to do likewise for the primes $p \equiv \pm 2s \pmod 5$ in

$$I_2 = \bigcup_{j=1}^{[\sqrt{k/5}]} \left[\frac{5k-6}{5j-1}, \frac{5k-6}{5j-2} \right].$$

Regarding the first of these, if we further assume that $k < 10^9$, then we can utilize the fact that

$$\theta(x) = \sum_{p \le x} \log p < x$$

for $x < 10^{11}$, to conclude (applying Corollary 2* of [20] to the intervals $\left[\frac{k+1}{j}, \frac{5k-6}{5j-1}\right]$ for $1 \le j \le 24$) that

$$\sum_{p \in I_1} \log p > 0.36496k - 4.96911.$$

Since $k < 10^9$, we may apply the results of Table 2 of [16] to deduce that

$$\sum_{\substack{p \in I_2 \\ p \equiv \pm 2s \ (mod\ 5)}} \log p > 0.24925k - 0.29912$$

whence

$$\log G_k > 0.61421k - 5.26833 > \log(1.84)k.$$

If we have $k \ge 10^9$, then we utilize the inequality $\theta(x) < 1.000081x$ (valid for $x > 0$) and Theorem 1 of [16], in conjunction with the previously mentioned results, to reach the same conclusion.

To deal with the values of $k < 10^6$, we argue as in the proof of Lemma 4.2 of [6]. Fixing k and s, we explicitly compute the product of the primes in I_1, together with the primes $\equiv \pm 2s \pmod 5$ in I_2. If we denote this product by P_k, we find that

$$P_k / 1.84^k > e^{-16} \tag{7}$$

for all $1 \le k < 10^6$ and $1 \le s \le 2$ with exactly 154 exceptions, the largest being with $k = 623$ and $s = 2$. The arguments of Lemma 4.2 of [6] allow us to reduce this calculation from roughly two million values for k to a few thousand. For the remaining pairs (k, s) which fail to satisfy (7), we compute G_k directly from the definition, verifying

$$G_k / 1.84^k > e^{-16}$$

in all cases (with $G_k/1.84^k$ minimal for $k = 199$ and $s = 2$).

We argue similarly for the other values of n, only with e^{-16} replaced by e^{-20} if $n = 4$. In all cases, we apply the estimates of [16] and [20] to deduce bounds for $k \ge 10^6$ and then optimize the inequality for smaller k. All these computations were performed using Pari on Sparc IPC, Sparc 20 and Deck Alpha machines. In Table 1, we list the total number and largest exceptions to the inequality

$$P_k / c_1(n)^k > c(n) \tag{8}$$

for $1 \le s < n/2$, where P_k is defined analogously to above and $c(n)$ is e^{-20} if $n = 4$ and e^{-16} otherwise. We also include the values of k and s which minimize $G_k/c_1(n)^k$.

Table 1

n	exceptions to (8)	minimal $\log(G_k/c_1(n)^k)$
3	617 $(k=3702, s=1)$	-13.36 $(k=369, s=1)$
4	352 $(k=3868, s=1)$	-19.42 $(k=606, s=1)$
5	154 $(k=623, s=2)$	-12.65 $(k=199, s=2)$
7	121 $(k=850, s=3)$	-13.38 $(k=293, s=2)$
11	111 $(k=467, s=2)$	-12.43 $(k=104, s=5)$
13	97 $(k=467, s=4)$	-12.06 $(k=257, s=2)$
17	37 $(k=464, s=3)$	-8.75 $(k=28, s=3)$
19	57 $(k=242, s=6)$	-15.95 $(k=74, s=6)$
23	42 $(k=363, s=11)$	-12.12 $(k=68, s=10)$
29	93 $(k=368, s=6)$	-10.64 $(k=62, s=13)$
31	169 $(k=285, s=6)$	-11.08 $(k=161, s=10)$
37	96 $(k=305, s=18)$	-10.55 $(k=33, s=17)$
41	111 $(k=317, s=9)$	-9.63 $(k=152, s=7)$
43	76 $(k=171, s=14)$	-9.41 $(k=104, s=6)$
47	215 $(k=286, s=15)$	-8.61 $(k=37, s=23)$
53	222 $(k=211, s=15)$	-10.06 $(k=45, s=6)$
59	444 $(k=235, s=11)$	-11.48 $(k=75, s=26)$
61	459 $(k=362, s=13)$	-7.64 $(k=45, s=8)$
67	637 $(k=284, s=2)$	-7.41 $(k=37, s=19)$
71	671 $(k=283, s=19,34)$	-7.46 $(k=36, s=19)$

4 Proof of Theorems 1.1 and 1.3

Define, for integral $k>0, 0 \leq l,m \leq 1$ and a, N, s and n as in the hypotheses of either Theorem 1.1 or 1.3,

$$p_{lmk} = N^k \ a^{2k} \ \left(\prod_{p \mid n} p^{\max\left\{ [\frac{k}{p-1}] + k ord_p(n/a), 0 \right\}} \right) \ G_k^{-1} \ p_{lm}(1/N).$$

The results of the previous section, then, show that each p_{lmk} is integral and we may apply Lemma 2.1 (Lemma 3.4 of [17] implies the nonvanishing of $\det(p_{lmk})$ for each positive integer k). To prove Theorem 1.1, we take $k_0 = 1$ and $t = 3$ in Lemma 2.1, whence by Lemmas 2.2 and 3.1, we have, for $n \neq 4$, that

$$\left| \left(1 + \frac{a}{N}\right)^{s/n} - \frac{p}{q} \right| > c^{-1} \left(\frac{3.9(n-1)}{n} \ e^{16} \ q \right)^{-\lambda}$$

where

$$c = \frac{n \ \kappa(a,n)}{(n-1) c_1(n)} \ \left(\sqrt{N} + \sqrt{N+a} \right)^2$$

with λ and $c_1(n)$ as in Theorem 1.1. The bound $N/a \geq 11/5$ derived from (4) and valid for all n and the fact that $\lambda > 2$, then, implies the stated result. For $n = 4$ (whence $s = 1$), we argue similarly only with the term $\frac{3.9(n-1)}{n}$ replaced by 1.95 and e^{16} replaced by e^{20}. It should be noted at this juncture that for specific n in the range under consideration here, the constant 10^{10} appearing in the statement of Theorem 1.1 may be reduced through application of the precise minima given in Table 1. For instance, with $n = 3$, we may replace 10^{10} by 3.23×10^6.

To prove Theorem 1.3 for $n \geq 14$, we utilize Lemma 2.1 with $k_0 = 1$ and $t = \frac{n}{n-5.2}$ and set $a = s = 1$. Bounding G_k by the trivial estimate that $G_k \geq 1$ and noting that the constant 1.3 in Lemma 2.2 (ii) may in fact be replaced by

$$\sqrt{\frac{N+1}{N}} \left(2 - \sqrt{\frac{N}{N+1}}\right)^{1/n} \tag{9}$$

yields Theorem 1.3 for these n, since

$$\kappa(1,n) = n\mu_n.$$

To deal with the remaining values of n requires a more detailed analysis. Specifically, we need a sharpened version of Lemma 2.2 (i). Suppose that $k \geq 2$. We show that

$$|I_l(1/N)| \leq c \left(N\left(\sqrt{N} + \sqrt{N+1}\right)^2\right)^{-k} \tag{10}$$

where we may take $c = 0.475$ if $n = 3$ and $c = 0.374$ if $4 \leq n \leq 13$. To prove this, we note, as in the proof of Lemma 2.2, that

$$|I_l(1/N)| \leq \frac{|1 - e^{2\pi i/n}|}{2\pi} N^{-2k} \int_0^\infty \frac{x^{k+1/n} dx}{(x+1)\left((x+1)(x+1+1/N)\right)^k}. \tag{11}$$

Explicitly computing

$$\frac{|1 - e^{2\pi i/n}|}{2\pi} \left(\frac{\left(\sqrt{N} + \sqrt{N+1}\right)^2}{N}\right)^k \int_0^\infty \frac{x^{k+1/n} dx}{(x+1)^{2k+1}}$$

for $3 \leq n \leq 13$, N minimal satisfying the hypotheses of Theorem 1.3 and $2 \leq k \leq 25$, we find that this quantity is bounded above by either 0.475 (if $n = 3$) or 0.374 (if $4 \leq n \leq 13$) in all cases, which proves (10) for $2 \leq k \leq 25$. To handle $k \geq 26$, we first use (11) to deduce the bound

$$|I_l(1/N)| \leq \frac{|1 - e^{2\pi i/n}|}{2\pi} \frac{(n-1)^{\frac{n-1}{n}}}{n} N^{-2k} \int_0^\infty \left(\frac{x}{(x+1)(x+1+1/N)}\right)^k dx$$

and evaluate this last integral in a similar fashion to Lemma 2.2, noting that it is bounded by the sum of

$$\int_0^{1/2} \left(\frac{x}{(x+1)^2}\right)^k dx,$$

$$\int_{1/2}^2 \left(\frac{x}{(x+1)(x+1+1/N)}\right)^k dx$$

and

$$\int_2^\infty \left(\frac{x}{(x+1)^2}\right)^k dx.$$

This implies that

$$\int_0^\infty \left(\frac{x}{(x+1)(x+1+1/N)}\right)^k dx$$

is bounded above by

$$\frac{3}{2}\left(\frac{N}{\left(\sqrt{N}+\sqrt{N+1}\right)^2}\right)^k + \frac{1}{2}\left(\frac{2}{9}\right)^k + \sum_{r=2}^{\infty}\left(\frac{r}{(r+1)^2}\right)^k.$$

Since the hypotheses of Theorem 1.3 for the values of n in question force $N \geq 6$, we have

$$\frac{\left(\sqrt{N}+\sqrt{N+1}\right)^2}{N} \leq 4.327 \tag{12}$$

so that $k \geq 26$ yields

$$\int_0^\infty \left(\frac{x}{(x+1)(x+1+1/N)}\right)^k dx \leq 2.5 \left(\frac{N}{\left(\sqrt{N}+\sqrt{N+1}\right)^2}\right)^k.$$

Since further

$$\frac{|1-e^{2\pi i/n}|}{2\pi} \frac{(n-1)^{\frac{n-1}{n}}}{n} \leq 0.146$$

for $3 \leq n \leq 13$, we have (10) as desired for these k also (with $c = 0.475$ for $n = 3$ or $c = 0.374$ for $4 \leq n \leq 13$). Replacing 1.3 in Lemma 2.2 (ii) by (9), this yields Theorem 1.3 for $3 \leq n \leq 13$ with

$$q \geq \frac{\left(\sqrt{N}+\sqrt{N+1}\right)^2}{n\mu_n}$$

through application of Lemma 2.1 with $t = 1.905$ ($n = 3$) or $t = 1.598$ ($4 \leq n \leq 13$) and $k_0 = 2$ (where, again, we utilize the lower bounds upon N derived from the hypotheses of Theorem 1.3).

To complete the proof of Theorem 1.3, we need show that the desired inequality holds for all

$$q < \frac{\left(\sqrt{N}+\sqrt{N+1}\right)^2}{n\mu_n} \tag{13}$$

with $3 \leq n \leq 13$ and N as previously. If $p = q = 1$, then the result obtains from the mean value theorem, so we may assume that $p > q \geq 1$. If q satisfies (13), then

$$\frac{p}{q} \geq \frac{q+1}{q} > 1 + \frac{n\mu_n}{\left(\sqrt{N}+\sqrt{N+1}\right)^2}$$

and the mean value theorem yields

$$\sqrt[n]{1+\frac{1}{N}} < 1 + \frac{1}{nN}$$

so that

$$\left| \sqrt[n]{1+\frac{1}{N}} \;-\; \frac{p}{q} \right| > \frac{n\mu_n}{\left(\sqrt{N}+\sqrt{N+1}\right)^2} - \frac{1}{nN}.$$

By (12), we therefore have

$$\left| \sqrt[n]{1+\frac{1}{N}} - \frac{p}{q} \right| > \frac{n}{4.237N} - \frac{1}{nN} > \frac{1}{3N}$$

which easily implies Theorem 1.3.

5 Proof of Corollary 1.2

We will illustrate the proof of Corollary 1.2 with the example $\theta = \sqrt[4]{5}$. Here we take $N = 80$ and $a = 1$ in Theorem 1.1 and may thus conclude that

$$\left| \sqrt[4]{81/80} \;-\; \frac{3q}{2p} \right| > 80^{-1} \;\left(2 \times 10^{10}\; p\right)^{-\lambda}$$

for

$$\lambda = 1 + \frac{\log\left(\frac{8}{1.62}(4\sqrt{5}+9)^2\right)}{\log\left(\frac{1.62}{8}(4\sqrt{5}+9)^2\right)} = 2.76457\cdots$$

and any integers p and q (with $q \neq 0$). It follows that

$$\left| \sqrt[4]{5} \;-\; \frac{p}{q} \right| > \sqrt[4]{5}\; \left(\frac{p}{q}\right)^{1-\lambda} \; 120^{-1} \;\left(2 \times 10^{10}\; q\right)^{-\lambda}$$

whence

$$\left| \sqrt[4]{5} \;-\; \frac{p}{q} \right| > \left(4.906267 \times 10^{30}\right)^{-1} \; q^{-2.764575}$$

holds for all nonzero p and q. This implies that

$$\left| \sqrt[4]{5} \;-\; \frac{p}{q} \right| > 0.03\; q^{-2.77} \tag{14}$$

provided $q \geq 10^{5377}$. To deal with q satisfying $1 \leq q < 10^{5377}$, we note that a rational p/q which fails to satisfy (14) must be a convergent in the continued fraction expansion to $\sqrt[4]{5}$, say p_i/q_i. We therefore have that

$$\left| \sqrt[4]{5} \;-\; \frac{p_i}{q_i} \right| > \frac{1}{(a_{i+1}+2)\; q_i^2}$$

where a_{i+1} is the $(i+1)$st partial quotient in this continued fraction expansion (see e.g. [15]). This implies that

$$a_{i+1} > 33 q_i^{0.77} - 2. \tag{15}$$

Checking that the first forty convergents to $\sqrt[4]{5}$ satisfy (14) and noting that $q_{41} > 10^{21}$, implies, from (15), that we need only show that $a_j < 10^{17}$ for $j < 12000$ (since we verify that $q_{12000} > 10^{6000}$). Computing the desired partial quotients via Pari (gp), we find that the largest of the first 12000 partial quotients is equal to 4057, enabling us to conclude as desired.

Similar arguments apply for our other examples. The bounds in Corollary 1.2 follow from Theorem 1.1 with the choices of a, N, s and n noted in the following table (Table 2), provided, in each case, $q < 10^{6000}$. Computing the first 12000 partial quotients for each θ and the related convergents, we list the largest partial quotients in this range. In every case, the 12000th convergent in the continued fraction expansion to $\sqrt[n]{m}$ has denominator exceeding 10^{6000} and analogous bounds to (15) enable us to conclude as in Corollary 1.2.

Table 2

θ	s, n, a, N	$\max a_i$	θ	s, n, a, N	$\max a_i$
$\sqrt[4]{5}$	$1, 4, 1, 80$	4057	$\sqrt[5]{34}$	$1, 5, 1, 16$	61217
$\sqrt[4]{14}$	$1, 4, 1, 7$	16353	$\sqrt[5]{37}$	$1, 5, 5, 32$	91267
$\sqrt[4]{15}$	$1, 4, 1, 15$	38166	$\sqrt[5]{39}$	$2, 5, 10, 3^7$	37108
$\sqrt[4]{17}$	$1, 4, 1, 16$	5301	$\sqrt[5]{40}$	$1, 5, 1, 4$	6720
$\sqrt[4]{18}$	$1, 4, 1, 8$	284610	$\sqrt[5]{42}$	$1, 5, 5, 16$	23743
$\sqrt[4]{37}$	$1, 4, 28, 15^4$	48324	$\sqrt[7]{5}$	$2, 7, 3, 125$	58199
$\sqrt[4]{39}$	$1, 4, 1, 624$	8611	$\sqrt[7]{10}$	$3, 7, 7, 25$	180016
$\sqrt[5]{3}$	$2, 5, 5, 27$	21171	$\sqrt[7]{11}$	$3, 7, 7, 121$	76362
$\sqrt[5]{6}$	$2, 5, 1, 8$	16724	$\sqrt[7]{12}$	$3, 7, 1, 8$	6793
$\sqrt[5]{10}$	$2, 5, 3, 125$	8835	$\sqrt[7]{13}$	$2, 7, 10, 3^7$	31257
$\sqrt[5]{11}$	$1, 5, 122, 21^5$	266282	$\sqrt[7]{17}$	$1, 7, 11, 2176$	9333
$\sqrt[5]{15}$	$2, 5, 2, 25$	64273	$\sqrt[7]{23}$	$2, 7, 95, 23^4$	47970
$\sqrt[5]{18}$	$2, 5, 1, 3$	19358	$\sqrt[7]{45}$	$3, 7, 2, 25$	10659
$\sqrt[5]{22}$	$1, 5, 5, 11$	295784	$\sqrt[11]{48}$	$5, 11, 1, 8$	26741
$\sqrt[5]{28}$	$1, 5, 1, 7$	7117	$\sqrt[13]{6}$	$5, 13, 13, 243$	66846
$\sqrt[5]{30}$	$1, 5, 1, 15$	7812	$\sqrt[13]{20}$	$4, 13, 3, 125$	8300
$\sqrt[5]{31}$	$1, 5, 1, 31$	45545	$\sqrt[17]{50}$	$5, 17, 3, 125$	4130182
$\sqrt[5]{33}$	$1, 5, 1, 32$	22143			

6 Some Diophantine Consequences

In [7], the author (joint with B.M.M. de Weger) will apply these techniques, together with lower bounds for linear forms in logarithms of algebraic numbers, to the problem of bounding the number of solutions to the binomial Thue equation $|ax^n - by^n| = 1$. For special cases of this and related equations, Corollary 1.2, via the factorization

$$x^n - my^n = (x - \sqrt[n]{m}y)(x^{n-1} + \sqrt[n]{m}x^{n-2}y + \cdots + m^{\frac{n-1}{n}}y^{n-1})$$

provides a means for explicitly determining all solutions. By way of example, for the equations considered in §5 of [11], we have

Corollary 6.1 If x and y are integers, then

$$\left|x^5 - 3y^5\right| \geq (\max\{|x|, |y|\})^{1.34}$$

$$\left|x^7 - 5y^7\right| \geq (\max\{|x|, |y|\})^{2.57}$$

$$\left|x^7 - 13y^7\right| \geq (\max\{|x|,|y|\})^{2.09}$$

and

$$\left|x^7 - 17y^7\right| \geq (\max\{|x|,|y|\})^{1.80}.$$

It is easy to derive analogous bounds for the other examples in Corollary 1.2. In general (see [4] and [6] for the cubic situation), when the Padé approximation machinery of Theorem 1.1 may be applied to deduce an irrationality measure for a particular $\sqrt[n]{m}$, the resulting measure will almost certainly be stronger than that obtained via the theory of linear forms in logarithms. On the other hand, the latter theory applies for all algebraic numbers while, arguing as in [6], if one defines $N_n(x)$ to be the number of positive integers $\leq x$ for which Theorem 1.1 may be applied to produce a nontrivial effective measure of irrationality, one has only that

$$N_n(x) \gg x^{\frac{5n-8}{2n(n-1)}}.$$

The arithmetic "flukes" that enable one to appeal to Theorem 1.1 correspond to exceptional convergents in the continued fraction expansion to $\sqrt[n]{m}$. As noted in [10], these in turn are related to extreme examples in the *abc*-conjecture.

7 Concluding Remarks

The fundamental improvement over [1] and [2] that one finds in [6], [11], [12], [14] and this paper obtains from lower bounds upon the quantity $G_k = G_k(n)$ defined in Section 3 (or related quantities in [11] and [12]). While asymptotically, our approach and that of [11] are equivalent (see [5]), for approximation to algebraic numbers of degrees $3, 4$ and 6, we are able to avoid reference to primes in arithmetic progressions, leading to sharper explicit results. In fact (again, see [5]), for any $n \geq 3$ one can show that

$$\lim_{k\to\infty} \frac{1}{k}\log(G_k(n)) = -\gamma - \frac{2}{\phi(n)} \sum_{\substack{1\leq r<n/2 \\ (r,n)=1}} \Psi\left(\frac{n-r}{n}\right)$$

where γ is the Euler-Mascheroni constant, $\phi(n)$ is Euler's totient function and $\Psi(z)$ is the derivative of the logarithm of $\Gamma(z)$. For $n = 3$, then, we have

$$\lim_{k\to\infty} G_k(3)^{1/k} = 3\sqrt{3}e^{-\frac{\pi}{2\sqrt{3}}} = 2.09807\cdots$$

which compares quite well with our lower bound given in Theorem 1.1 of $c_1(3) = 2.03$. Denoting $\lim_{k\to\infty} G_k(n)^{1/k}$ by $\alpha(n)$, via Euler-Maclaurin summation, one can show that

$$\lim_{n\to\infty} \alpha(n) = \pi/e^{\gamma} = 1.76387\cdots$$

and even that $\alpha(n) > \pi/e^{\gamma}$ if p is prime. A comparison to Theorem 1.1 shows that for larger n (where we utilize the bounds for primes in arithmetic progressions due to Ramaré and Rumely [16]), we are increasingly unable to approach the asymptotic results in strength. Improvements in these latter estimates would therefore be of clear interest in this regard.

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