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# SIMILARITY SOLUTIONS OF THE ONE-DIMENSIONAL FOKKER–PLANCK EQUATION

G. W. BLUMAN
Department of Mathematics, University of British Columbia, Vancouver, Canada

**Abstract**—Group theoretic methods are used to construct new exact solutions for the one-dimensional Fokker–Planck equation corresponding to a class of non-linear forcing functions $f(x)$. An important sub-class, corresponding to $f(x) = \alpha/x + \beta x$, $\alpha < 1$, $\beta > 0$ is shown to lead to stable solutions. A discussion is given on how generalized similarity methods could be applied to higher dimensional systems.

## 1. INTRODUCTION

WE consider the stochastic differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + f(x) = n(t) \tag{1}$$

where $n(t)$ is stationary Gaussian white noise

$$\left.\begin{aligned} &\langle n(t)\rangle = 0 \\ &\langle n(t_1)\, n(t_2)\rangle = D\delta(t_2 - t_1) \\ \text{and}\quad &x(0) = x_0 \end{aligned}\right\} \tag{2}$$

The output $x(t)$ of (1) is a stationary Markov process and is completely specified by finding the transitional probability density $p(x, t/x_0) \geqslant 0$ which satisfies the Fokker–Planck equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}[f(x)\,p], \tag{3a}$$

with the initial condition

$$p(x, 0/x_0) = \delta(x - x_0) \tag{4}$$

$x = r$ is a reflecting boundary for process (1), (2) if

$$\lim_{x\to r}\left[D\frac{\partial p}{\partial x} + f(x)\,p\right] = 0.$$

If $r_1$ and $r_2$ are reflecting boundaries, $r_1 < r_2$, and $x_0 \in R = (r_1; r_2)$, then

$$\int_{r_1}^{r_2} p(x, t)\,\mathrm{d}x = 1 \tag{5}$$

We limit our discussion to those processes for which $f(x)$ is odd, i.e.

$$f(x) = -f(-x) \tag{6}$$

The group properties of (3a), (4) will be studied by the methods discussed in Bluman and Cole [1] and Bluman [2], in the sense that we search for those $f(x)$ for which at least a one-parameter Lie group of transformations leaves invariant (3a) and (4). Finding the invariants of the group transformations and assuming a unique solution $p(x, t/x_0)$, we are then able to reduce (3a) to a linear ordinary differential equation, which can be solved in terms of tabulated special functions. Hence an explicit, closed form similarity solution is obtained.

Closed form solutions have been obtained for the case $f(x) = k \operatorname{sgn} x$, $-\infty < x < \infty$, by Caughey and Dienes [3], and for $f(x)$ piecewise linear by Atkinson and Caughey [4].

For simplification of notation we replace $f(x)/D$ by $f(x)$ and $Dt$ by $t$ in (3a) and (4) and assume that $x_0 > 0$. Hence we are led to the system

$$\left.\begin{aligned} &\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}[f(x)\,p] = \frac{\partial p}{\partial t} \\ &p(x, 0/x_0) = \delta(x - x_0) \end{aligned}\right\} \tag{3b}$$

We will show that a Lie group of transformations leaves invariant (3b) for a three-parameter family of functions $f(x)$ satisfying (6). It turns out that a two parameter subfamily of these functions, namely,

$$f(x) = \frac{\alpha}{x} + \beta x, \qquad \beta > 0, \qquad -\infty < \alpha < 1, \tag{7}$$

leads to stable solutions, i.e. $\lim_{t\to\infty} \int_{r_1}^{r_2} x^2 p(x, t/x_0)\,dx < \infty$. The generated solutions are defined for $x \in R = (0; \infty)$, i.e. $r = 0$ is a reflecting boundary.

## 2. DERIVATION OF THE GROUP OF THE FOKKER–PLANCK EQUATION

Let

$$\left.\begin{aligned} x' &= x'(x, t, p; \varepsilon) \\ t' &= t'(x, t, p; \varepsilon) \\ p' &= p'(x, t, p; \varepsilon) \end{aligned}\right\} \tag{8}$$

be a one-parameter group of transformations leaving system (3b) invariant such that $\varepsilon = 0$ corresponds to the identity transformation, i.e. if $p = P(x, t)$ is the unique solution to (3b), then setting $p = P(x, t)$ in (8), $(x', t', p')$ satisfy:

$$\left.\begin{aligned} &\frac{\partial^2 p'}{\partial x'^2} + \frac{\partial}{\partial x'}[f(x')\,p'] = \frac{\partial p'}{\partial t'} \\ &p'[x, 0, P(x, 0); \varepsilon] = \delta\{x'[x, 0, P(x, 0); \varepsilon] - x_0\} \\ &t'[x_0, 0, P(x_0, 0); \varepsilon] = 0 \\ &x'[x_0, 0, P(x_0, 0); \varepsilon] = x_0 \end{aligned}\right\} \tag{9}$$

Uniqueness of the solution to (3b) implies that

$$P(x', t') = p'[x, t, P(x, t); \varepsilon] \tag{10}$$

From (10) we deduce that system (3b) has a similarity solution, i.e. we can reduce (3b) to an ordinary differential equation. It turns out that for an equation of the form (3b), $p'$ depends linearly on $p$; $x'$ and $t'$ are both independent of $p$.

As shown in [1] and [2], only the local behaviour of the Lie group (8) is needed to obtain the corresponding similarity solution. Expanding (8) about the identity $\varepsilon = 0$, we get:

$$\left.\begin{aligned} x' &= x + \varepsilon X(x, t) + O(\varepsilon^2) \\ t' &= t + \varepsilon T(x, t) + O(\varepsilon^2) \\ p' &= p + \varepsilon g(x, t)\, p + O(\varepsilon^2) \end{aligned}\right\} \tag{11}$$

The $O(\varepsilon)$ terms in the expansion of (10) about $\varepsilon = 0$ lead to the invariant surface condition

$$X(x, t)\frac{\partial P}{\partial x} + T(x, t)\frac{\partial P}{\partial t} = g(x, t)\, P \tag{12}$$

which relates the solution $P(x, t)$ to the infinitesimals $\{X, T, g\}$. The corresponding characteristic equations are:

$$\frac{\mathrm{d}x}{X(x, t)} = \frac{\mathrm{d}t}{T(x, t)} = \frac{\mathrm{d}P}{g(x, t)\, P} \tag{13}$$

Solving (13), we get two constants of integration. The similarity variable

$$\eta(x, t) = \text{const.} \tag{14}$$

is the integral of the first equality in (13).

$$P(x, t) = F(\eta)\, G(x, t) \tag{15}$$

is the integral of the second equality in (13) where the dependence of $G$ on $x$ and $t$ is known explicitly and $F(\eta)$ is some arbitrary function of $\eta$. Substitution of (15) into (3b) leads to a second order linear ordinary differential equation, satisfied by the new dependent variable $F(\eta)$. This differential equation has two linearly independent solutions. However the correct combination can be determined by judicious use of the source condition in (3b).

If, perchance, a two parameter Lie group of transformations leaves (3b) invariant and if the invariants corresponding to one of the parameters differ from those of the other, then we obtain two distinct functional forms (15) for the solution $p(x, t/x_0) = P(x, t)$. We label the parameters by subscripts 1 and 2, and let $\{T_i, X_i, g_i\}$, $i = 1, 2$ be the corresponding infinitesimals. Parameters 1 and 2 lead to the same invariants if there exists some function $\lambda(x, t)$ such that

$$X_2\frac{\partial}{\partial x} + T_2\frac{\partial}{\partial t} - g_2 p\frac{\partial}{\partial p} = \lambda(x, t)\left[X_1\frac{\partial}{\partial x} + T_1\frac{\partial}{\partial t} - g_1 p\frac{\partial}{\partial p}\right] \tag{16}$$

Let $\eta_i$, $F_i(\eta_i)$ and $G_i(x, t)$, $i = 1, 2$, be the similarity variables and functional forms corresponding to the respective parameters of the Lie group. Then

$$P(x, t) = F_1(\eta_1)\, G_1(x, t) = F_2(\eta_2)\, G_2(x, t) \tag{17}$$

Say $\eta_1 \neq fn(\eta_2)$, as is usually the case, then choosing $\eta_1$ and $\eta_2$ as the new independent variables instead of $x$ and $t$, we can, in principle, solve the functional equation (17). The resulting solution will be of the form

$$P(x, t) = AH(\eta_1)\, G_1(x, t) \tag{18}$$

where the dependence of $H$ on $\eta_1$ is known explicitly and $A$ is some constant determined by the source condition in (3b).

We note that invariance under a two-parameter Lie group reduces a partial differential equation in two independent variables, to a functional equation of the form (16). Hence, no further use is made of the given partial differential equation, as in the case of invariance under a one-parameter group. This fact is especially important for extension to systems of partial differential equations having $n \geqslant 3$ independent variables. If an $m$ parameter group leaves invariant a system of partial differential equations and the associated boundary and initial conditions, and if the invariants of the respective parameters are functionally independent of each other, then the number of variables can be reduced by $m$. (see Ovsjannikov [5] for some discussion of this).

So far in our discussion we have assumed that some Lie group (8) leaves invariant (3b). We now turn our attention to the problem of finding such a group and the corresponding forcing function $f(x)$. From (12) to (15) we see that it is not the global group (8) but the infinitesimal generators $\{X, T, g\}$ corresponding to a particular $f(x)$ which are needed. Using the methods mentioned in [1], [2], [5], or Müller and Matschat [6], we find that:

$$\left.\begin{aligned} T(x, t) &= T(t) \\ X(x, t) &= xT'(t) + A(t) \\ g(x, t) &= B(t) - \frac{xf(x)\,T'(t)}{4} - \frac{f(x)\,A(t)}{2} - \frac{xA'(t)}{2} - \frac{x^2T''(t)}{8} \end{aligned}\right\} \tag{19}$$

where $A(t)$, $B(t)$, $T(t)$, and $f(x)$ satisfy the equation

$$N_1(x, t) + N_2(x, t) = 0 \tag{20}$$

with

$$N_1(x, t) = T'(t)\left[\frac{f^2(x)}{4} + \frac{xf(x)f'(x)}{4} - \frac{f'(x)}{2} - \frac{xf''(x)}{4}\right] + \frac{T''(t)}{4} + T'''(t)\left[-\frac{x^2}{8}\right] + B'(t) \tag{21}$$

$$N_2(x, t) = A(t)\left[\frac{f(x)f'(x)}{2} - \frac{f''(x)}{2}\right] + A''(t)\left[-\frac{x}{2}\right] \tag{22}$$

Since $f(x)$ is odd,

$$N_1(x, t) = N_2(x, t) = 0 \tag{23}$$

We now consider two possible cases:

*Case* 1. $T(t) \neq 0$, $A(t) = 0$

From (21), we see that

$$[f^2(x) + xf(x)f'(x) - 2f'(x) - xf''(x)]''' = 0 \tag{24}$$

The solution of (6), (23), (24) is

$$2f'(x) - f^2(x) + \beta^2 x^2 - \gamma + \frac{16\nu^2 - 1}{x^2} = 0 \tag{25}$$

$$\left.\begin{aligned} T'''(t) &= 4\beta^2 T'(t) \\ B'(t) &= \frac{\gamma T'(t)}{4} - \frac{T''(t)}{4} \end{aligned}\right\} \tag{26}$$

where $\beta$, $\gamma$, $\nu$ are constants of integration.

Invariance of the initial source further restricts $\{X, T, g\}$:

$$\left.\begin{aligned} T(0) &= 0 \\ X(x_0, 0) &= 0 \\ g(x_0, 0) &= -X_x(x_0, 0) \end{aligned}\right\} \tag{27}$$

Substitution of (19) into (27) leads to the initial conditions

$$\left.\begin{aligned} T(0) &= 0 \\ T'(0) &= 0 \\ B(0) - x_0^2 \frac{T''(0)}{8} &= 0 \end{aligned}\right\} \tag{28}$$

After solving (26) subject to the initial conditions (28) and making the appropriate substitution, we find that

$$\left.\begin{aligned} T &= 4\sinh^2 \beta t \\ X &= 2\beta x \sinh 2\beta t \\ g &= \gamma \sinh^2 \beta t - \beta[1 + xf(x)] \sinh 2\beta t - \beta^2 x^2 \cosh 2\beta t + x_0^2 \beta^2 \end{aligned}\right\} \tag{29}$$

*Case* 2. $A(t) \neq 0$, $T(t) = 0$

From (22), we see that here

$$[f(x)f'(x) - f''(x)]'' = 0 \tag{30}$$

The solution of (6), (23), (30) is:

$$2f'(x) - f^2(x) + \beta^2 x^2 - \gamma = 0 \tag{31}$$

$$A''(t) = \beta^2 A(t) \tag{32}$$

Using the source condition (4) and then making the necessary substitutions, we find that

$$\left.\begin{aligned} T &= 0 \\ X &= 2\sinh \beta t \\ g &= \beta(x_0 - x\cosh \beta t) - f(x)\sinh \beta t \end{aligned}\right\} \tag{33}$$

Note that (31) is a special case of (25) where $\nu = \pm\frac{1}{4}$. Hence for $f(x)$ satisfying (31), a two parameter Lie group of transformations leaves invariant the corresponding Fokker–Planck equation. For the respective group parameters, (29) and (33) are the corresponding infinitesimals.

### 3. CONSTRUCTION OF SIMILARITY SOLUTIONS

The most general similarity solution of the one-dimensional Fokker–Planck equation corresponds to a forcing function $f(x)$ satisfying (25). Solving the ordinary differential equation which corresponds to the first equality of (13), we are led to the similarity variable

$$\eta(x, t) = \frac{x}{\sqrt{(T(t))}} \tag{34}$$

where $T(t) = 4\sinh^2 \beta t$. The use of (34) in the second equality of (13) leads to the following functional form for the solution:

$$p(x, t/x_0) = F(\eta)\, C(t) \cdot \exp\left[-(\mu(t)\, x^2 + \tfrac{1}{2}\textstyle\int f(x)\,\mathrm{d}x)\right]$$

where

$$\left.\begin{aligned} C(t) &= \frac{e^{\gamma t/4}}{T^{\frac{1}{4}}} \cdot \exp\left[\frac{x_0^2 \beta}{2(1 - e^{2\beta t})}\right] \\ \mu(t) &= \frac{\beta}{4} \coth \beta t \\ T(t) &= 4\sinh^2 \beta t \end{aligned}\right\} \tag{35}$$

Substitution of (35) into (3b) yields a second order linear ordinary differential for $F(\eta)'$ whose general solution can be expressed in terms of Modified Bessel Functions:

$$F(\eta) = \begin{cases} \eta^{\frac{1}{2}}[A_1 I_{2\nu}(\kappa\eta) + A_2 I_{-2\nu}(\kappa\eta)] & \text{for} \quad x > 0 \\ |\eta|^{\frac{1}{2}}[B_1 K_{2\nu}(\kappa|\eta|) + B_2 I_{2\nu}(\kappa|\eta|)] & \text{for} \quad x < 0 \end{cases} \tag{36}$$

where $\kappa = \beta x_0$ and $A_1$, $A_2$, $B_1$, and $B_2$ are arbitrary constants to be determined by boundary and continuity conditions.

As $t \to 0$, $\eta \to +\infty$. As $z \to +\infty$, (see [7], 7.23),

$$K_{2\nu}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\,[1 + O(z^{-1})] \tag{37}$$

$$I_{2\nu}(z) = \left(\frac{1}{2\pi z}\right)^{\frac{1}{2}} e^{z}\,[1 + O(z^{-1})] \tag{38}$$

Hence in order to have a source only at $x = x_0$, [i.e. satisfy the initial condition of (3b)], we must set $B_2 = 0$ (otherwise we would generate sources at $x = \pm x_0$)

Let

$$f(x) = -\frac{2V'(x)}{V(x)} \tag{39}$$

After substituting (39) into (25), we find that $V(x)$ satisfies

$$4V'' + \left\{ \gamma - \beta^2 x^2 - \frac{(16\nu^2 - 1)}{x^2} \right\} V = 0. \tag{40}$$

The solution of (40) leading to a reliable probability distribution is

$$V(x) = (\tfrac{1}{2}\beta x^2)^{\frac{1}{4}+\nu} e^{-\beta x^2/4} M(a, b, \tfrac{1}{2}\beta x^2) \tag{41}$$

where $M(a, b, z)$ denotes Kummer's hypergeometric function of the first kind (see [8], Chapter 13), and

$$a = \tfrac{1}{2} + \nu - \gamma/8\beta \tag{42}$$

$$b = 1 + 2\nu$$

with $\nu > -\frac{1}{2}, a \geqslant 0$. The properties of $M(a, b, z)$ are well known:

As $z \to 0$,

$$M(a, b, z) = 1 + \frac{a}{b} z + O(z^2) \tag{43}$$

As $z \to +\infty$,

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}[1 + O(z^{-1})] \tag{44}$$

$$M(O, b, z) \equiv 1 \tag{45}$$

Using (43) and (44) we can show that if $a \neq 0$

$$\text{(i)} \lim_{x\to\infty} \frac{f(x)}{x} = -\beta \tag{46}$$

$$\text{(ii)} \lim_{x\to\infty} x f(x) = -(4\nu + 1) \tag{47}$$

In order that $\int_{r_1}^{r_2} p(x, t/d_0)\, dx = \text{const.}$, $\partial p/\partial x + fp$ must be a continuous function of $x$ for $x \in R = (r_1 ; r_2)$, $t > 0$. This restriction combined with the requirement of the continuity of $\partial p/\partial x$ and $p$ for $x \in R$, leads to two cases:

*Case* 1. $R = (-\infty ; \infty)$ corresponding to $\nu = -\frac{1}{4}$ (to be discussed in Section 4).

*Case* 2. $R = (0; \infty)$, i.e. $x = 0$ is a reflecting barrier, corresponding to $\nu > -\frac{1}{2}$. In this case we find that $B_1 = A_2 = 0$ and that

$$p(x, t/x_0) = A_1 \eta^{\frac{1}{2}} I_{2\nu}(\kappa\eta)\, C(t)\, V(x)\, e^{-\mu(t) x^2} \tag{48}$$

where $x > 0, \nu > -\frac{1}{2}, a \geqslant 0$.

Imposition of the source condition leads to

$$A_1 = \frac{\beta x_0^{\frac{1}{2}} (\frac{1}{2}\beta x_0^2)^{-\frac{1}{4}-\nu}}{M(a, b, \frac{1}{2}\beta x_0^2)} \tag{49}$$

Since $\int_0^\infty p(x, t/x_0)\,\mathrm{d}x = 1$, from the group properties of (3a) as a bonus we are able to compute the following 5 parameter $(\beta, x_0, t, v, a)$ definite integral:

$$\int_0^\infty x^{2v+1} \exp\left[-\frac{\frac{1}{2}\beta x^2}{(1-\mathrm{e}^{-2\beta t})}\right] I_{2v}\left(\frac{\beta x x_0}{2\sinh\beta t}\right) M(a, 1+2v, \tfrac{1}{2}\beta x^2)\,\mathrm{d}x$$
$$= x_0^{2v} \exp\left[(1+2v-2a)\beta t\right]\sqrt{\left(\frac{2\sinh\beta t}{\beta}\right)} \exp\left[\frac{\frac{1}{2}\beta x_0^2}{\mathrm{e}^{2\beta t}-1}\right] M(a, 1+2v, \tfrac{1}{2}\beta x_0^2) \qquad (50)$$

where $v > -\frac{1}{2}$, $a \geqslant 0$

Next we compute

$$\langle x^2\rangle = \int_0^\infty x^2 p(x, t/x_0)\,\mathrm{d}x \qquad (51)$$

Let

$$\xi = \eta\kappa, \qquad \zeta = \tfrac{1}{2}\beta x_0^2, \qquad M' = \frac{\mathrm{d}M}{\mathrm{d}\zeta}(a, b, \zeta) \qquad (52)$$

Then

$$\frac{\partial p}{\partial x_0} = l(x_0, t)\,p + \frac{\beta\eta I_2'(\xi)}{I_{2v}(\xi)}\,p \qquad (53)$$

where

$$l(x_0, t) = -\frac{2v}{x_0} - \beta x_0\frac{M'}{M} + \frac{\beta x_0}{1-\mathrm{e}^{2\beta t}} \qquad (54)$$

$\langle \partial p/\partial x_0\rangle = \partial/\partial x_0\,\langle p\rangle$ implies that

$$l(x_0, t) = -\beta\left\langle \eta\frac{I_{2v}'(\xi)}{I_{2v}(\xi)}\right\rangle \qquad (55)$$

$$\frac{\partial^2 p}{\partial x_0^2} = \left[l^2 + \frac{\partial l}{\partial x_0} + 2l\beta\eta\frac{I_{2v}'(\xi)}{I_{2v}(\xi)} + \beta^2\eta^2\frac{I_{2v}''(\xi)}{I_{2v}(\xi)}\right]p \qquad (56)$$

$I_{2v}(\xi)$, $\quad M = M(a, b, \zeta)$ satisfy:

$$\xi^2 I_{2v}'' = (\xi^2 + 4v^2)\,I_{2v} + \xi I_{2v}' \qquad (57)$$

$$\zeta M'' = (\zeta - b)\,M' + aM \qquad (58)$$

$$\zeta M' = -aM + aM(a+1, b, \zeta) \qquad (59)$$

Using (55), (57), (58), (59), (54) and $\langle \partial^2 p/\partial x_0^2 \rangle = 0$, we find that

$$\langle x^2 \rangle = \frac{4aM(a+1, b, \tfrac{1}{2}\beta x_0^2)}{M(a, b, \tfrac{1}{2}\beta x_0^2)} \sinh 2\beta t + \frac{\tfrac{1}{2}\gamma(1 - e^{-2\beta t})}{\beta^2} + x_0^2 e^{-2\beta t} \tag{60}$$

We see that $\langle x^2 \rangle$ is bounded iff $a = 0$, i.e.

$$\gamma = 4\beta(1 + 2\nu) \tag{61}$$

and $\beta > 0$.

This important special case will be considered in Section 5.

## 4. INVARIANCE OF A CLASS OF FOKKER–PLANCK EQUATIONS UNDER A TWO-PARAMETER GROUP OF TRANSFORMATIONS

We now consider the case $\nu = -\tfrac{1}{4}$ where there are no reflecting boundaries, i.e. $R = (-\infty; \infty)$. Here

$$V(x) = e^{-\beta x^2/4} M(a, \tfrac{1}{2}, \tfrac{1}{2}\beta x_0^2), \qquad a = \tfrac{1}{4} - \frac{\gamma}{8\beta} \tag{62}$$

i.e. a special case of (41). In Section 3, solution (49) was obtained by substituting the similarity form (15) into the original partial differential equation and solving the resulting ordinary differential equation for $F(\eta)$. Since for $V(x)$ of the form (62) a two-parameter group leaves (3b) invariant, no further use has to be made of the original partial differential equation.

In Section 3 we showed that (29) leads to a solution of the form

$$p(x, t/x_0) = F_1(\eta_1)\, G_1(x, t)$$

where

$$\eta_1 = \frac{\tfrac{1}{2}x}{\sinh \beta t}$$

and

$$G_1(x, t) = \frac{e^{\gamma t/4}}{\sqrt{(\sinh \beta t)}} \exp\left[\frac{\tfrac{1}{2}\beta x_0^2}{1 - e^{2\beta t}}\right] \exp\left[-(\beta x^2 \coth \beta t)/4\right] V(x) \tag{63}$$

Similarly we can show that (33) leads to a solution of the form

$$p(x, t/x_0) = F_2(\eta_2)\, G_2(x, t)$$

where

$$\eta_2 = t$$

and

$$G_2(x, t) = V(x) \exp\left[-(\beta x^2 \coth \beta t)/4\right] \exp\left[\tfrac{1}{2}\beta x x_0/\sinh \beta t\right] \tag{64}$$

Uniqueness of the solution implies that

$$(63) \equiv (64)$$

Hence

$$F_2(t) = D\,e^{\gamma t/4} \exp\left[\tfrac{1}{2}\beta x_0^2/(1 - e^{2\beta t})\right]/\sqrt{(\sinh \beta t)} \tag{65}$$

with the constant $D$ determined by the source condition in (3b). This implies that

$$D = \frac{1}{2}\sqrt{\left(\frac{\beta}{\pi}\right)}\frac{1}{M(a, \frac{1}{2}, \frac{1}{2}\beta x_0^2)} \tag{66}$$

The formula for $\langle x^2 \rangle$ is a special case of (60) with $b = \frac{1}{2}$, $a = \frac{1}{4} - \gamma/8\beta$. Note that the bounded case $a = 0$ corresponds to the well-known Brownian motion where $f(x) = \beta x$, $R = (-\infty; \infty)$, and

$$p(x, t/x_0) = \frac{1}{2}\sqrt{\left(\frac{\beta}{\pi}\right)}\frac{e^{\frac{1}{2}\beta t}}{(\sinh \beta t)^{\frac{1}{2}}} \exp\left[-(\beta(1 + \coth \beta t)/4)\{x - x_0\,e^{-\beta t}\}^2\right]$$

$$\langle x^2 \rangle = 1/\beta + (x_0^2 - 1/\beta)\,e^{-2\beta t} \tag{67}$$

## 5. STABLE SOLUTIONS FOR $R = (0; \infty)$

In Section 3 we showed that the generated solutions to the Fokker–Planck equation are stable iff $\gamma = 4\beta(1 + 2\nu)$ and $\beta > 0$. These correspond to the two-parameter subfamily of forcing functions

$$f(x) = \frac{\alpha}{x} + \beta x, \qquad \alpha < 1, \qquad \beta > 0. \tag{68}$$

This corresponds to setting in (41)

$$\left.\begin{aligned} &a = 0, \qquad b = \tfrac{1}{2} - \tfrac{1}{2}\alpha, \qquad \gamma = 2(1 - \alpha)\,\beta, \text{ and hence} \\ &V(x) = (\tfrac{1}{2}\beta x^2)^{-\alpha/4}\,e^{-\beta x^2/4} \end{aligned}\right\} \tag{69}$$

The transitional probability density is

$$p(x, t/x_0) = \beta x_0^{\frac{1}{2}}(x/x_0)^{-\frac{1}{2}\alpha}\,\eta^{\frac{1}{2}} I_{-(\frac{1}{2}+\frac{1}{2}\alpha)}(\kappa\eta)\,C(t) \exp\left[-\mu(t)x^2 - \beta x^2/4\right] x \geqslant 0. \tag{70}$$

The corresponding second moment is

$$\langle x^2 \rangle = (1/\beta - \alpha/\beta) + \left[x_0^2 - (1/\beta - \alpha/\beta)\right] e^{-2\beta t} \tag{71}$$

Note that

$$\lim_{t\to\infty} \langle x^2 \rangle = (1 - \alpha)/\beta. \tag{72}$$

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**Résumé**—On utilise des méthodes de la théorie des groupes pour construire de nouvelles solutions exactes de l'équation de Fokker–Planck à une dimension correspondant à une classe de fonctions du second membre non linéaires $f(x)$. On montre qu'une sous classe importante correspondant à $f(x) = \alpha/x + \beta x$, $\alpha < 1$, $\beta > 0$, conduit à des solutions stables. On discute comment on pourrait appliquer des méthodes de similitude généralisées à des systèmes de dimensions plus élevées.

**Zusammenfassung**—Methoden der Gruppentheorie werden benützt, um neue exakte Lösungen für die eindimensionale Fokker–Planck Gleichung, die einer Klasse von nichtlinearen Druckfunktionen $f(x)$ entspricht, zu bestimmen. Es wird gezeigt, dass eine wichtige Untergruppe, für die $f(x) = \alpha/x + \beta x, \alpha < 1, \beta > 0$, zu stabilen Lösungen führt. Es wird diskutiert, in welcher Weise verallgemeinerte Ähnlichkeitsmethoden für höherdimensionale Systeme angewendet werden könnten.

**Аннотация**—Применяются методы теории групп с целью построения новых точных решений одномерного уравнения Фоккер–Планка, соответствующих классу нелинейных вынуждающих функций $f(x)$. Показывается, что важный подкласс соответствующий $f(x) = \alpha/x+\beta x$ $\alpha < 1$, $\beta > 0$ ведёт к устойчивым решениям. Приводится дискуссию по поводу того, как обобщенные методы подобия могут иримeняться в случае систем высшей размерности.