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Simplifying the Form of Lie Groups Admitted by a Given Differential Equation

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Rather general results are obtained for determining the nature of infinitesimal generators which can be admitted by a given differential equation. Simple criteria are given to determine whether or not (1) the infinitesimals of independent variables can depend only on independent variables and (2) the infinitesimal of the dependent variable can depend at most linearly on the dependent variable. Many examples are given. A trivial consequence of this paper is that an admitted Lie group, for any linear partial differential equation of at least second order, must have infinitesimals of independent variables depending only on independent variables and the infinitesimal of the dependent variable depends at most linearly on the dependent variable. This latter result previously has only been proved for a second order linear PDE. © 1990 Academic Press, Inc.

1. INTRODUCTION

In the last century Sophus Lie developed the theory of continuous groups (Lie groups) of transformations. He showed that such transformations can be characterized in terms of infinitesimal generators. Moreover Lie gave an algorithm to find the infinitesimal generators admitted by a given differential equation (DE).

For any DE the admittance of a Lie group leads to the construction of a family of new solutions from any known solution. The admittance of a Lie group for any ordinary differential equation (ODE) leads constructively to a reduction of its order plus quadratures. If a partial differential equation (PDE) is invariant under a Lie group, one can construct special families of solutions called invariant solutions or similarity solutions. If a DE can be derived from a variational principle then admittance of a Lie group is a necessary condition in order to find conservation laws by means of Noether's theorem. For recent references on Lie's algorithm to find infinitesimal generators admitted by a given DE and various uses of Lie groups for DEs see [1–4, 7].

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To execute Lie's algorithm for finding the admitted infinitesimals of a given DE, one must solve a system of coupled PDEs called the determining equations. These determining equations are always linear and homogeneous in the infinitesimals and their derivatives and are generally overdetermined. Their solution places a severe limitation on the applicability of Lie group methods to a given DE. Recently [5] symbolic manipulation programs have been established to set up and solve the determining equations. These programs, though powerful, are not guaranteed to complete their task. The likelihood of these programs to solve explicitly the determining equations is greatly enhanced if one can significantly reduce the number of determining equations to be analyzed by reducing the number of given variables on which the infinitesimals depend.

In this paper, for an admitted group of a scalar DE, we give simple criteria to determine whether or not (1) the infinitesimals of independent variables can depend only on independent variables and (2) the infinitesimal of the dependent variable can depend at most linearly on the dependent variable. The main results are summarized in terms of eight theorems. A trivial consequence of these theorems is that for any linear PDE of at least second order, an admitted Lie group must have infinitesimals of independent variables depending only on independent variables and the infinitesimal of the dependent variable depending at most linearly on the dependent variable. Previously Ovsiannikov [6, Chap. 6] proved this latter result for a linear PDE of second order and in [3, p. 87] Ovsiannikov states that it holds for the "majority" of linear DEs.

2. Setting Up the Determining Equations

We consider a scalar nth order DE, $n \geqslant 2$ with $m \geqslant 1$ independent variables, of the form

$$f(x, u, \underset{1}{u}, \ldots, \underset{n}{u}) = 0 \tag{2.1}$$

with dependent variable u, independent variables $x = (x_1, x_2, \ldots, x_m)$; $\underset{k}{u}$ represents all kth order partial derivatives of u with respect to x, $k = 1, 2, \ldots, n$. We assume that (1.1) is linear in $\underset{n}{u}$ with the coefficients of components of $\underset{n}{u}$ depending only on (x, u). Hence without loss of generality (2.1) is assumed to be of the form $(n \geqslant 2)$

$$\frac{\partial^n u}{\partial x_1^n} = A_{i_1 i_2 \cdots i_n}(x, u) \frac{\partial^n u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} + F(x, u, \underset{1}{u}, \ldots, \underset{n-1}{u}), \tag{2.2}$$

where we adopt the convention of summation over a repeated index; $A_{i_1i_2\cdots i_n}(x, u)$ is symmetric in its indices with $A_{11\ldots1}=0$. Where convenient, we use the notation $u_{i_1i_2\cdots i_k} = \partial^k u/\partial x_{i_1}\,\partial x_{i_2}\cdots\partial x_{i_k}$, $k = 1, 2, \ldots$.

Consider a one-parameter (ε) Lie group of point transformations

$$x_i^* = X_i(x, u; \varepsilon) = x_i + \varepsilon\xi_i(x, u) + O(\varepsilon^2), \tag{2.3a}$$

$$u^* = U(x, u; \varepsilon) = u + \varepsilon\eta(x, u) + O(\varepsilon^2), \tag{2.3b}$$

with extensions

$$\begin{aligned} u_i^* &= U_i(x, u, \underset{1}{u}, \varepsilon) = u_i + \varepsilon\eta_i^{(1)}(x, u, \underset{1}{u}) + O(\varepsilon^2),\\ &\vdots \end{aligned} \tag{2.3c}$$

$$\begin{aligned} u^*_{i_1i_2\cdots i_k} &= U_{i_1i_2\cdots i_k}(x, u, \underset{1}{u}, \ldots, \underset{k}{u}; \varepsilon)\\ &= u_{i_1i_2\cdots i_k} + \varepsilon\eta^{(k)}_{i_1i_2\cdots i_k}(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) + O(\varepsilon^2), \end{aligned} \tag{2.3d}$$

$i = 1, 2, \ldots, m$; $i_l = 1, 2, \ldots, m$, $l = 1, 2, \ldots, k$ with $k = 1, 2, \ldots, n$.

In terms of the total derivative operators

$$D_i = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots + u_{ii_1i_2\cdots i_{n-1}}\frac{\partial}{\partial u_{i_1i_2\cdots i_{n-1}}},$$

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j, \qquad i = 1, 2, \ldots, m;$$

$$\eta^{(k)}_{i_1i_2\cdots i_k} = D_{i_k}\eta^{(k-1)}_{i_1i_2\cdots i_{k-1}} - (D_{i_k}\xi_j)u_{ji_1i_2\cdots i_{k-1}},$$

$$i_l = 1, 2, \ldots, m \quad \text{for} \quad l = 1, 2, \ldots, k \quad \text{with} \quad k = 2, 3, \ldots, n;$$

$$\mathbf{X} = \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u}$$

is the infinitesimal generator of the group of point transformations (2.3a, b); and

$$\mathbf{X}^{(k)} = \xi_i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial u} + \eta_i^{(1)}\frac{\partial}{\partial u_i} + \cdots + \eta^{(k)}_{i_1i_2\cdots i_k}\frac{\partial}{\partial u_{i_1i_2\cdots i_k}}$$

is the infinitesimal generator of the kth extension of (2.3a, b), $k = 1, 2, \ldots, n$.

The group of point transformations (2.3a, b) is admitted by (2.2) if and only if

$$\mathbf{X}^{(n)}\left(\frac{\partial^n u}{\partial x_1^n} - A_{i_1i_2\cdots i_n}u_{i_1i_2\cdots i_n} - F(x, u, \underset{1}{u}, \ldots, \underset{n-1}{u})\right) = 0 \tag{2.4}$$

when $\partial^n u/\partial x_1^n$ satisfies (2.2). The resulting expression is an identity in the components of $x, u, \underset{1}{u}, \ldots, \underset{n}{u}$ except for the component $\partial^n u/\partial x_1^n$ of $\underset{n}{u}$.

It is easy to see that (2.4) is equivalent to

$$\eta^{(n)}_{11\cdots 1} - A_{i_1 i_2 \cdots i_n}\eta^{(n)}_{i_1 i_2 \cdots i_n} - (\mathbf{X}A_{i_1 i_2 \cdots i_n})u_{i_1 i_2 \cdots i_n}$$
$$-\mathbf{X}^{(n-1)}F(x, u, \underset{1}{u}, ..., \underset{n-1}{u}) = 0 \tag{2.5}$$

after $\partial^n u/\partial x_1^n$ is replaced by the right-hand side of (2.2). Equation (2.5) is the equation from which the determining equations are derived.

3. Main Results

The main results of this paper can be summarized in terms of the following eight theorems which are proved in Section 4:

Theorem (I). *Suppose a Lie group of transformations* (2.3a, b) *is admitted by PDE* (2.2) *with* $n \geqslant 2$, $m \geqslant 2$. *Let*

$$A_{11\cdots 1k} = a_k, \qquad k = 2, 3, ..., m$$

and let

$$A_{11\cdots 1} = a_1 = -1.$$

Suppose the coefficients $\{A_{i_1 i_2 \cdots i_n}\}$ *do not satisfy*

$$A_{i_1 i_2 \cdots i_n} = (-1)^n a_{i_1} a_{i_2} \cdots a_{i_n}. \tag{3.1}$$

Then $\partial\xi_i/\partial u = 0$, $i = 1, 2, ..., m$.

If the coefficients $\{A_{i_1 i_2 \cdots i_n}\}$ *do satisfy* (3.1), *then*

$$\frac{\partial \xi_k}{\partial u} = -a_k \frac{\partial \xi_1}{\partial u}, \qquad k = 2, 3, ..., m.$$

Moreover in this case the set of components with nth order derivatives in PDE (2.2) *is reduced to the form*

$$a_{i_1} a_{i_2} \cdots a_{i_n} u_{i_1 i_2 \cdots i_n}. \tag{3.2}$$

Theorem (II). *Suppose PDE* (2.2) *is such that each coefficient* $A_{i_1 i_2 \cdots i_n}$ *depends only on* x, $n \geqslant 2$, $m \geqslant 2$. *If a Lie group of point transformations* (2.3a, b) *is admitted by* (2.2) *then* $\partial\xi_i/\partial u = 0$, $i = 1, 2, ..., m$, *if there does not exist some point transformation of* x *such that* (2.2) *is equivalent to*

$$\frac{\partial^n u}{\partial x_1^n} = G(x, u, \underset{1}{u}, ..., \underset{n-1}{u}) \tag{3.3}$$

for some function $G(x, \underset{1}{u}, ..., \underset{n-1}{u})$.

Moreover for a PDE of the form (3.3)

$$\frac{\partial \xi_k}{\partial u}=0, \qquad k=2, 3, ..., m. \tag{3.4}$$

THEOREM (III). *Suppose DE* (2.2) ($n \geqslant 3$, $m \geqslant 1$) *is such that* $F(x, u, \underset{1}{u}, ..., \underset{n-1}{u})$ *is linear in the components of* $\underset{n-1}{u}$ *with the coefficients of components of* $\underset{n-1}{u}$ *depending only on* $(x, u, \underset{1}{u})$. *If a Lie group of point transformations* (2.3a, b) *is admitted by* (2.2) *then* $\partial \xi_i/\partial u=0$, $i=1, 2, ..., m$.

THEOREM (IV). *Suppose DE* (2.2) ($n \geqslant 2$, $m \geqslant 1$) *is such that* $F(x, u, \underset{1}{u}, ..., \underset{n-1}{u})$ *is linear in the components of* $\underset{n-1}{u}$ *with the coefficients of components of* $\underset{n-1}{u}$ *depending only on* (x, u). *Suppose for any Lie group of point transformations* (2.3a, b) *admitted by* (2.2) *we have* $\partial \xi_i/\partial u=0$, $i=1, 2, ..., m$. *Then* $\partial^2 \eta/\partial u^2=0$.

THEOREM (V). *Suppose DE* (2.2) ($n \geqslant 3$, $m \geqslant 1$) *is such that* $F(x, u, \underset{1}{u}, ..., \underset{n-1}{u})$ *is linear in the components of* $\underset{n-1}{u}$ *with the coefficients of components of* $\underset{n-1}{u}$ *depending only on* (x, u). *If a Lie group of point transformations* (2.3a, b) *is admitted by* (2.2) *then*

$$\frac{\partial \xi_i}{\partial u}=0, \qquad i=1, 2, ..., m \qquad \textit{and} \qquad \frac{\partial^2 \eta}{\partial u^2}=0.$$

THEOREM (VI). *Suppose PDE* (2.1) ($n \geqslant 3$, $m \geqslant 2$) *is a linear PDE. If a Lie group of point transformations* (2.3a, b) *is admitted by* (2.1) *then*

$$\frac{\partial \xi_i}{\partial u}=0, \qquad i=1, 2, ..., m \qquad \textit{and} \qquad \frac{\partial^2 \eta}{\partial u^2}=0.$$

THEOREM (VII). *Suppose PDE* (2.1) ($n \geqslant 2$, $m \geqslant 2$) *is a linear PDE. If a Lie group of point transformations* (2.3a, b) *is admitted by* (2.1) *then*

$$\frac{\partial \xi_i}{\partial u}=0, \qquad i=1, 2, ..., m \qquad \textit{and} \qquad \frac{\partial^2 \eta}{\partial u^2}=0.$$

THEOREM (VIII). *Suppose* (2.1) *is a linear ODE* ($m=1$) *of order* $n \geqslant 3$. *If a Lie group of point transformations* (2.3a, b) *is admitted by* (2.1) *then setting* $\xi=\xi_1$, *we have*

$$\frac{\partial \xi}{\partial u}=0, \qquad \frac{\partial^2 \eta}{\partial u^2}=0.$$

4. PROOFS OF THE MAIN RESULTS

First we consider the proof of Theorem (I). It is easy to see that $\eta^{(k)}_{i_1i_2\cdots i_k}$ is linear in the components of $\underset{k}{u}$ if $k \geqslant 2$. Moreover in (2.5) the terms depending on products of the components of $\underset{1}{u}$ and $\underset{n}{u}$ are

$$\begin{gathered}-\frac{\partial \xi_j}{\partial u}\left[\frac{\partial^n u}{\partial x_1^n} u_j + n\,\frac{\partial^n u}{\partial x_1^{n-1}\,\partial x_j}\,u_1\right] \\ +\frac{\partial \xi_j}{\partial u} A_{i_1i_2\cdots i_n}[u_{i_1i_2\cdots i_n}u_j + nu_{ji_1i_3\cdots i_n}u_{i_1}],\end{gathered} \tag{4.1}$$

where $\partial^n u/\partial x_1^n$ is replaced by $A_{i_1i_2\cdots i_n}u_{i_1i_2\cdots i_n}$. Hence it must follow that

$$\frac{\partial \xi_j}{\partial u}\left[\frac{\partial^n u}{\partial x_1^{n-1}\,\partial x_j}\,u_1 - A_{i_1i_2\cdots i_n}u_{ji_2i_3\cdots i_n}u_{i_1}\right] \equiv 0, \tag{4.2}$$

where in (4.2)

$$\frac{\partial^n u}{\partial x_1^n} = A_{i_1i_2\cdots i_n}u_{i_1i_2\cdots i_n}. \tag{4.3}$$

Equation (4.2) is a polynomial form in the products of components of $\underset{1}{u}$ and $\underset{n}{u}$. Consequently the coefficients of like polynomial terms must equal zero after using the substitution (4.3).

Let

$$a_k = A_{11\cdots 1k}, \qquad k = 2, 3, ..., m.$$

Consider (4.2), (4.3). Setting to zero the coefficient of $u_1(\partial^n u/\partial x_1^{n-1}\partial x_k)$, $k \neq 1$, we get

$$\frac{\partial \xi_k}{\partial u} + na_k\frac{\partial \xi_1}{\partial u} - (n-1)a_k\frac{\partial \xi_1}{\partial u} = 0, \qquad k = 2, 3, ..., m.$$

Hence

$$\frac{\partial \xi_k}{\partial u} = -a_k\frac{\partial \xi_1}{\partial u}, \qquad k = 2, 3, ..., m. \tag{4.4}$$

It immediately follows that:

(i) if $\partial\xi_1/\partial u = 0$ then $\partial\xi_j/\partial u = 0$, $j = 1, 2, ..., m$;

(ii) if $a_k = 0$ then $\partial\xi_k/\partial u = 0$.

Let

$$a_{\sigma k} = A_{i_1 i_2 \cdots i_{n-\sigma} i_{n-\sigma+1} \cdots i_n},$$

where $i_1 = i_2 = \cdots = i_{n-\sigma} = 1$, $i_{n-\sigma+1} = i_{n-\sigma+2} = \cdots = i_n = k \neq 1$ so that $a_{1k} = a_k$.

Setting to zero the coefficient of $u_k(\partial^n u/\partial x_1^{n-\sigma}\, \partial x_k^\sigma)$ in (4.2), (4.3), $k \neq 1$, $1 \leqslant \sigma \leqslant n-1$, we get

$$\frac{n!}{(n-\sigma)!\,\sigma!} a_k a_{\sigma k} \frac{\partial \xi_1}{\partial u} + \frac{(n-1)!}{(n-1-\sigma)!\,\sigma!} a_{\sigma+1,k} \frac{\partial \xi_1}{\partial u}$$

$$+ \frac{(n-1)!}{(\sigma-1)!\,(n-\sigma)!} a_{\sigma k} \frac{\partial \xi_k}{\partial u} = 0, \qquad k = 2, 3, \ldots, m. \tag{4.5}$$

Equation (4.5) reduces to

$$[n a_k a_{\sigma k} + (n-\sigma) a_{\sigma+1,k}] \frac{\partial \xi_1}{\partial u} + \sigma a_{\sigma k} \frac{\partial \xi_k}{\partial u} = 0, \tag{4.6}$$

$k = 2, 3, \ldots, m$, $1 \leqslant \sigma \leqslant n-1$.

Substituting (4.4) into (4.6) we obtain

$$[a_k a_{\sigma k} + a_{\sigma+1,k}] \frac{\partial \xi_1}{\partial u} = 0, \qquad k = 2, 3, \ldots, m, \quad 1 \leqslant \sigma \leqslant n-1. \tag{4.7}$$

It immediately follows that if $\partial \xi_1/\partial u \neq 0$, then

$$a_{\sigma+1,k} = -a_k a_{\sigma k}, \qquad k = 2, \ldots, m, \quad 1 \leqslant \sigma \leqslant n-1.$$

Hence if $\partial \xi_1/\partial u \neq 0$ then it is necessary that

$$a_{\sigma k} = (-1)^{\sigma+1} (a_k)^\sigma, \qquad k = 2, \ldots, m, \quad 1 \leqslant \sigma \leqslant n-1. \tag{4.8}$$

Setting to zero the coefficient of $u_l(\partial^n u/\partial x_1^{n-1}\, \partial x_k)$, $k \neq l$, $k \neq 1$, $l \neq 1$, we get

$$n a_k a_l \frac{\partial \xi_1}{\partial u} + (n-1) A_{11\cdots 1kl} \frac{\partial \xi_1}{\partial u} + a_l \frac{\partial \xi_k}{\partial u} = 0. \tag{4.9}$$

Substituting (4.4) into (4.9) we find that

$$[A_{11\cdots 1kl} + a_k a_l] \frac{\partial \xi_1}{\partial u} = 0, \qquad k \neq l, k \neq 1, l \neq 1. \tag{4.10}$$

Hence if $\partial \xi_1/\partial u \neq 0$ then it is necessary that

$$A_{11\cdots 1kl} = -a_k a_l, \qquad k \neq l, k \neq 1, l \neq 1.$$

If we proceed inductively by successively setting to zero the coefficients of $u_l(\partial^n u/\partial x_1^{n-q}\,\partial x_{j_1}\,\partial x_{j_2}\cdots\partial x_{j_q})$, $l\neq 1$, $j_\alpha\neq 1$ for $\alpha=1,2,\dots,q$, with $q=1,2,\dots,n$, then we find that for $\partial\xi_1/\partial u\neq 0$ it is necessary that (3.1) holds. This completes the proof of Theorem (I).

Now consider Theorem (II). Suppose $a_k=a_k(x)$, $k=2,\dots,m$. Under a transformation of coordinates $y=y(x)$ the quantities $a_i(x)$ transform as components of a contravariant vector. Hence a system of coordinates can be found such that in it these components have values: $a_1=-1$, $a_k=0$, $k=2,3,\dots,m$. Then in terms of coordinates y the set of components with nth order derivatives will consist of only one component $u_{11\cdots1}$. Equations (3.4) follow from (4.4). Hence Theorem (II) is proved.

In considering Theorem (III), we focus on the form of $F(x,u,\underset{1}{u},\dots,\underset{n-1}{u})$ where $n\geqslant 3$, $m\geqslant 1$. Suppose $F(x,u,\underset{1}{u},\dots,\underset{n-1}{u})$ is linear in the components of $\underset{n-1}{u}$ with the coefficients of components of $\underset{n-1}{u}$ depending only on $(x,u,\underset{1}{u})$. Since $\eta^{(n-1)}_{i_1 i_2\cdots i_{n-1}}$ has no elements which are products of components of $\underset{2}{u}$ and $\underset{n-1}{u}$, the coefficient of $u_{11}(\partial^{n-1}u/\partial x_1^{n-1})$ in (2.5) is simply

$$N\frac{\partial\xi_1}{\partial u},$$

where

$$N=\begin{cases}3 & \text{if } n=3,\\ n(n+1)/2 & \text{if } n\geqslant 4.\end{cases}$$

Hence $\partial\xi_1/\partial u=0$. This yields the proof of Theorem (III).

In considering Theorem (IV) we further restrict $F(x,u,\underset{1}{u},\dots,\underset{n-1}{u})$ and assume that $F(x,u,\underset{1}{u},\dots,\underset{n-1}{u})$ is linear in the components of $\underset{n-1}{u}$ with coefficients of the components of $\underset{n-1}{u}$ depending only on (x,u). If $n\geqslant 3$, $m\geqslant 1$, then from Theorem (III) it follows that $\partial\xi_i/\partial u=0$, $i=1,2,\dots,m$. For $n=2$, $m\geqslant 1$ we assume that $\partial\xi_i/\partial u=0$, $i=1,2,\dots,m$. Then the coefficient of $u_1(\partial^{n-1}u/\partial x_1^{n-1})$ in (2.5) is simply

$$M\frac{\partial^2\eta}{\partial u^2},$$

where

$$M=\begin{cases}1 & \text{if } n=2,\\ n & \text{if } n\geqslant 3.\end{cases}$$

Hence $\partial^2\eta/\partial u^2=0$, yielding the proof of Theorem (IV).

Theorem (V) is an immediate consequence of Theorems (III) and (IV); Theorem (VI) follows immediately as a special case of Theorem (V).

Theorem (VI) combined with Ovsiannikov's result [6, Chap. 6] for the case $n = 2$, $m \geqslant 2$, yields the proof of Theorem (VII).

Theorem (VIII) also follows immediately as a special case of Theorem (V).

5. EXAMPLES

The following examples illustrate various aspects of Theorems (I)–(VIII).

(I) Consider a second order PDE of the form

$$a(x, y, u)\frac{\partial^2 u}{\partial x^2} + 2b(x, y, u)\frac{\partial^2 u}{\partial x\,\partial y} + c(x, y, u)\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \quad (5.1)$$

Suppose $b^2 \neq ac$; i.e., consider (5.1) where it is either hyperbolic or elliptic. From Theorem (I) it follows that (5.1) can only admit a Lie group of point transformations with infinitesimal generators of the form

$$\mathbf{X} = \xi_1(x, y)\frac{\partial}{\partial x} + \xi_2(x, y)\frac{\partial}{\partial y} + \eta(x, y, u)\frac{\partial}{\partial u}.$$

(II) Consider a second order PDE of the form

$$\begin{aligned} &a(x, y, u)\frac{\partial^2 u}{\partial x^2} + 2b(x, y, u)\frac{\partial^2 u}{\partial x\,\partial y} + c(x, y, u)\frac{\partial^2 u}{\partial y^2} \\ &\quad + d(x, y, u)\frac{\partial u}{\partial x} + e(x, y, u)\frac{\partial u}{\partial y} + f(x, y, u) = 0. \end{aligned} \quad (5.2)$$

Suppose $b^2 \neq ac$. From Theorems (I) and (IV) it follows that (5.2) can only admit a Lie group of point transformations with infinitesimal generators of the form

$$\mathbf{X} = \xi_1(x, y)\frac{\partial}{\partial x} + \xi_2(x, y)\frac{\partial}{\partial y} + [\alpha(x, y) + \beta(x, y)u]\frac{\partial}{\partial u}.$$

(III) Consider a third order DE of the form

$$\frac{\partial^3 u}{\partial x_1^3} = G(x, u, \underset{1}{u}), \quad (5.3)$$

where $x=(x_1, x_2, ..., x_m)$. From Theorems (III) and (IV) it follows that (5.3) can only admit a Lie group of point transformations with infinitesimal generators of the form

$$\mathbf{X}=\xi_i(x)\frac{\partial}{\partial x_i}+[\alpha(x)+\beta(x)u]\frac{\partial}{\partial u}.$$

(IV) Consider a third order ODE of the form

$$\frac{d^3u}{dx^3}=F\left(x, u, \frac{du}{dx}\right)\frac{d^2u}{dx^2}+G\left(x, u, \frac{du}{dx}\right). \tag{5.4}$$

From Theorem (III) it follows that (5.4) can only admit a Lie group of point transformations with infinitesimal generators of the form

$$\mathbf{X}=\xi(x)\frac{\partial}{\partial x}+\eta(x, u)\frac{\partial}{\partial u}.$$

(V) Consider a third order ODE of the form

$$\frac{d^3u}{dx^3}=F(x, u)\frac{d^2u}{dx^2}+G\left(x, u, \frac{du}{dx}\right). \tag{5.5}$$

From Theorems (III) and (IV) it follows that (5.5) can only admit a Lie group of point transformations with infinitesimal generators of the form

$$\mathbf{X}=\xi(x)\frac{\partial}{\partial x}+[\alpha(x)+\beta(x)u]\frac{\partial}{\partial u}.$$

The following example illustrates that a PDE of the form (3.3) can have $\partial\xi_1/\partial u\neq 0$:

(VI) The PDE

$$\frac{\partial^2 u}{\partial x_1^2}=\left(\frac{\partial u}{\partial x_1}\right)^2\frac{\partial u}{\partial x_2} \tag{5.6}$$

admits [4, p. 317]

$$\mathbf{X}=-x_1u\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial u}.$$

The following example shows that Theorem (VIII) does not hold in the case of a second order ODE:

(VII) The ODE

$$\frac{d^2u}{dx^2}=0 \tag{5.7}$$

admits [4, p. 106]

$$\mathbf{X}=xu\frac{\partial}{\partial x}+u^2\frac{\partial}{\partial u}.$$

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