On the remarkable nonlinear diffusion equation $(\partial/\partial x)[a(u+b)^{-2}(\partial u/\partial x)] - (\partial u/\partial t) = 0$

George Bluman and Sukeyuki Kumei
Department of Mathematics and Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, British Columbia, Canada V6T1W5

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We study the invariance properties (in the sense of Lie–Bäcklund groups) of the nonlinear diffusion equation $(\partial/\partial x)[C(u)(\partial u/\partial x)] - (\partial u/\partial t) = 0$. We show that an infinite number of one-parameter Lie–Bäcklund groups are admitted if and only if the conductivity $C(u) = a(u+b)^{-2}$. In this special case a one-to-one transformation maps such an equation into the linear diffusion equation with constant conductivity, $(\partial^2\bar{u}/\partial\bar{x}^2) - (\partial\bar{u}/\partial\bar{t}) = 0$. We show some interesting properties of this mapping for the solution of boundary value problems.

1. INTRODUCTION

In recent years nonlinear diffusion processes described by the partial differential equation (p.d.e)

$$\frac{\partial}{\partial x}\left[C(u)\frac{\partial u}{\partial x}\right] - \frac{\partial u}{\partial t} = 0, \qquad (1)$$

with a variable conductivity $C(u)$, have appeared in problems related to plasma and solid state physics.[1,2] Interest in such processes has long occurred in other fields such as metallurgy and polymer science.[3–5]

Some exact solutions are well known for such equations.[6] These can be shown to be included in the class of all similarity solutions to such equations obtained from invariance under a Lie group of point transformations.[7,8]

Recently, it has been shown that differential equations can be invariant under continuous group transformations beyond point or contact transformation Lie groups which act on a finite dimensional space.[9] These new continuous group transformations act on an infinite dimensional space. Such infinite dimensional contact transformations have been called Noether transformations[10] or Lie–Bäcklund (LB) transformations[11] (Noether mentioned the possibility of such transformations in her celebrated paper on conservation laws[12]). Well known nonlinear partial differential equations admitting LB transformations include the Korteweg–deVries,[13,14] sine-Gordon,[10,15] cubic Schrödinger,[14] and Burgers' equations.[16] All of these known examples admit an infinite number of one-parameter LB transformations. Moreover, many of their important properties (existence of an infinite number of conservation laws,[13,14] existence of solitons,[14] and existence[17] of Bäcklund transformations[18]) are related to their invariance under LB transformations.

Any linear differential equation which admits a nontrivial one-parameter point Lie group is invariant under an infinite number of one-parameter LB transformations through superposition. Moreover, every known nonlinear p.d.e., invariant under LB transformations, can be associated with some corresponding linear p.d.e.

With the above views in mind we study the invariance properties of Eq. (1). Previously,[7,8,19] it had been shown that Eq. (1) is invariant under

a) a three-parameter point Lie group for arbitrary $C(u)$,

b) a four-parameter point Lie group if $C(u) = a\cdot(u+b)^{\nu}$,

c) a five-parameter point Lie group if $\nu = -\frac{4}{3}$.

[It is well known that a six-parameter point Lie group leaves invariant Eq. (1) in the case $C(u) = \text{const}$.[20]]

In the present work, we show that *Eq.* (1) *is invariant under LB transformations if and only if the conductivity is of the form*

$$C(u) = a\cdot(u+b)^{-2}, \qquad (2)$$

i.e., *if Eq.* (1) *is of the form*

$$\frac{\partial}{\partial x}\left[a\cdot(u+b)^{-2}\frac{\partial u}{\partial x}\right] - \frac{\partial u}{\partial t} = 0. \qquad (3)$$

Furthermore, this equation admits an infinite number of LB transformations.

In this special case, there exists a one-to-one transformation which maps Eq. (3) *into the linear diffusion equation with constant conductivity, namely, the heat equation*

$$\frac{\partial^2\bar{u}}{\partial\bar{x}^2} - \frac{\partial\bar{u}}{\partial\bar{t}} = 0. \qquad (4)$$

In the course of this paper, we find an operator connecting two infinitesimal LB transformations leaving Eq. (3) invariant. We prove that this operator is a recursion operator which generates an infinite sequence of one-parameter infinitesimal LB transformations leaving Eq. (3) invariant. Moreover, we show that no other LB transformation leaves Eq. (3) invariant.

By examining the linearization of Eq. (3), we are led to construct the transformation mapping Eq. (3) into Eq. (4). It is shown that this transformation maps the recursion operator of Eq. (3) into the spatial translation operator of Eq. (4), giving a simple interpretation of the transformation relating Eq. (3) to Eq. (4). We use this transformation to connect boundary value problems of Eq. (3) to those of Eq. (4).

We construct a new similarity solution of Eq. (3) corresponding to invariance under LB transformations.

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2. DERIVATION OF THE CLASS OF NONLINEAR DIFFUSION EQUATIONS INVARIANT UNDER LB TRANSFORMATIONS

LB transformations include Lie groups of point transformations and finite dimensional contact transformations.[11] The algorithm for calculating infinitesimal LB transformations leaving differential equations invariant is essentially the same as Lie's method[8] for calculating infinitesimal point groups.

Consider the most general one-parameter infinitesimal LB transformation that can leave invariant a time-evolution equation,[21] namely;

$$u^* = u + \epsilon U(x,t,u,u_1,\ldots,u_n) + O(\epsilon^2),$$
$$x^* = x, \tag{5}$$
$$t^* = t,$$

where $u_i = \partial^i u/\partial x^i$, $i = 1,2,\cdots$. Let $\partial u/\partial t = u_t$, $\partial u_i/\partial t = u_{it}$, $\partial U/\partial u = U_0$, $\partial U/\partial u_i = U_i$, $\partial^2 U/\partial u_i \partial u_j = U_{i,j}$, $C' = dC/du$, and $C'' = d^2C/du^2$.

In the above notation Eq. (1) becomes

$$u_t = C'(u_1)^2 + Cu_2. \tag{6}$$

Under Eqs. (5) the derivatives of u appearing in Eq. (6) transform as follows:

$$(u_t)^* = u_t + \epsilon U^t + O(\epsilon^2),$$
$$(u_1)^* = u_1 + \epsilon U^x + O(\epsilon^2),$$
$$(u_2)^* = u_2 + \epsilon U^{xx} + O(\epsilon^2),$$

where

$$U^t = D_t U = \frac{\partial U}{\partial t} + U_0 u_t + \sum_{i=1}^{n} U_i u_{it}, \tag{7}$$
$$U^x = D_x U = \frac{\partial U}{\partial x} + \sum_{i=0}^{n} U_i u_{i+1},$$
$$U^{xx} = (D_x)^2 U = \frac{\partial^2 U}{\partial x^2} + 2\sum_{i=0}^{n} \frac{\partial U_i}{\partial x} u_{i+1} + \sum_{i,j=0}^{n} U_{i,j} u_{i+1} u_{j+1} + \sum_{i=0}^{n} U_i u_{i+2}.$$

D_t and D_x are total derivative operators with respect to t and x, respectively.

The transformation (5) is said to leave Eq. (6) invariant if and only if for every solution $u = \theta(x,t)$ of Eq. (6)

$$U^t = C''U(u_1)^2 + 2C'U^x u_1 + C'Uu_2 + CU^{xx}. \tag{8}$$

The fact that U must satisfy Eq. (8) for any solution of Eq. (6) imposes severe restrictions on U. Using Eq. (6) the derivatives of u_i with respect to t, i.e., u_{it}, can be eliminated in Eq. (8). Since the invariance condition (8) must hold for every solution of Eq. (6), Eq. (8) becomes a polynomial form in u_{n+1} and u_{n+2}. As a result the coefficients of each term in this form must vanish. This leads us to the determining equations for the infinitesimal LB transformations (5).

If in Eq. (5), $n \leqslant 2$, we obtain the Lie group of point transformations leaving Eq. (6) invariant. Without loss of generality we assume $n \geqslant 3$ in Eq. (5). It turns out that for $n \geqslant 3$, U is independent of x and t.

In our polynomial form, the coefficient of u_{n+2} vanishes and the coefficients of $(u_{n+1})^2$ and u_{n+1}, respectively, lead to determining equations

$$CU_{n,n} = 0, \tag{9}$$
$$nC'U_n u_1 = 2C \sum_{i=0}^{n-1} U_{n,i} u_{i+1}. \tag{10}$$

Solving Eqs. (9) and (10) we find that

$$U = \alpha (C)^{(1/2)n} u_n + E(u,u_1,\ldots,u_{n-1}), \tag{11}$$

where E is undetermined, and $\alpha =$ arbitrary constant.

The substitution of Eq. (11) into the remaining terms of Eq. (8) leads to a polynomial form in u_n whose coefficients of $(u_n)^2$ and u_n, respectively, lead to determining equations

$$CE_{n-1,n-1} = 0, \tag{12}$$
$$2C\left[\sum_{i=0}^{n-2} E_{n-1,i} u_{i+1}\right] + (1-n)C'E_{n-1} u_1 - \frac{\alpha}{4} n(n+3)C'(C)^{(1/2)n} u_2 + \alpha\left[\tfrac{1}{4}n^2(C')^2(C)^{(1/2)n-1} - \tfrac{1}{2}n(n+2)C''(C)^{(1/2)n}\right](u_1)^2 = 0. \tag{13}$$

Solving Eqs. (12) and (13) we find that

$$U = \alpha\left[(C)^{(1/2)n} u_n + \tfrac{1}{4}n(n+3)C'(C)^{(1/2)n-1} u_1 u_{n-1}\right] + F(u)u_{n-1} + G(u,u_1,\ldots,u_{n-2}), \tag{14}$$

where F and G are undetermined and, more importantly, for $\alpha \neq 0$ it is necessary that the conductivity $C(u)$ satisfy the differential equation

$$2CC'' = 3(C')^2. \tag{15}$$

Hence, it is necessary that

$$C(u) = a\cdot(u+b)^{-2}, \tag{16}$$

where a and b are arbitrary constants for the invariance of Eq. (1) under LB transformations. Without loss of generality we can set $a = 1$, $b = 0$, i.e., from now on we consider the equivalent p.d.e.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right) \equiv B. \tag{17}$$

This particular equation has been considered as a model equation of diffusion in high-polymeric systems.[4,5]

3. CONSTRUCTION OF A RECURSION OPERATOR; AN INFINITE SEQUENCE OF INVARIANT LB TRANSFORMATIONS OF EQ. (17)

For $n = 3$ it is easy to solve the rest of the determining equations and show that the only LB transformation leaving Eq. (17) invariant is

$$U = U^{(1)} = u^{-3}u_3 - 9u^{-4}u_1u_2 + 12u^{-5}(u_1)^3. \tag{18}$$

For $n = 4$ we obtain two linearly independent LB transformations $U^{(1)}$ and

$$U^{(2)} = u^{-4}u_4 - 14u^{-5}u_1u_3 - 10u^{-5}(u_2)^2 + 95u^{-6}(u_1)^2u_2 - 90u^{-7}(u_1)^4. \tag{19}$$

The existence of $U^{(1)}$ and $U^{(2)}$, combined with the work of Olver,[16] motivates us to seek a linear recursion operator $\mathscr{D}$ leading to infinitesimal LB transformations $U^{(k)}$ defined as follows:

$$(\mathscr{D})^k B = U^{(k)}, \quad k = 1,2,\cdots. \tag{20}$$

The character of $\{B, U^{(1)}, U^{(2)}\}$ leads one to consider for $\mathscr{D}$ the form

$$\mathscr{D} = pD_x + q + r(D_x)^{-1}, \tag{21}$$

where D_x is a total derivative operator, $(D_x)\cdot(D_x)^{-1}$ is the identity operator, and $\{p,q,r\}$ are functions of $\{u,u_1,u_2\}$. Then one can show that $\mathscr{D}B = U^{(1)}$ if and only if

$$p = u^{-1}, \tag{22}$$

and

$$q[u^{-2}u_2 - 2u^{-3}(u_1)^2] + ru^{-2}u_1 = -3u^{-4}u_1u_2 + 6u^{-5}(u_1)^3. \tag{23}$$

Furthermore, $(\mathscr{D})^2 B = U^{(2)}$ if and only if

$$q = -2u^{-2}u_1, \tag{24}$$

and

$$r = -u^{-2}u_2 + 2u^{-3}(u_1)^2. \tag{25}$$

A more concise expression for the operator is

$$\mathscr{D} = (D_x)^2 \cdot (u^{-1}) \cdot (D_x)^{-1}. \tag{26}$$

We now show that the constructed operator $\mathscr{D}$ is indeed a recursion operator. Let the operator

$$\begin{aligned} A &= \sum_{i=0}^{2} B_i (D_x)^i \\ &= u^{-2}(D_x)^2 - 4u^{-3}u_1 D_x + 6u^{-4}(u_1)^2 - 2u^{-3}u_2 \\ &= (D_x)^2 \cdot u^{-2}, \end{aligned} \tag{27}$$

where $B_i = (\partial/\partial u_i)B$. Olver's work[16] shows that $\mathscr{D}$ is a recursion operator for Eq. (17) if and only if the commutator

$$[A - D_t, \mathscr{D}] = 0, \tag{28}$$

for any solution $u = \theta(x,t)$ of Eq. (17). Moreover, if $\mathscr{D}$ is a recursion operator, then the sequence $\{U^{(1)}, U^{(2)}, \cdots\}$ given by Eq. (20) is an infinite sequence of LB transformations leaving Eq. (17) invariant. It is straightforward to show that A and $\mathscr{D}$ satisfy Eq. (28).

The nature of $U^{(1)}$ and the form of a general U given by Eq. (11) show that for $n = l + 2$, there are at most $k \leqslant l$ linearly independent LB transformations leaving Eq. (17) invariant since U must depend uniquely on u_{l+2}.

The proof that $\mathscr{D}$ is a recursion operator demonstrates that $k = l$ and hence we have found all possible LB transformations leaving Eq. (17) invariant, namely, $\{U^{(k)}\}$, $k = 1,2,\cdots$.

4. A MAPPING TO THE LINEAR DIFFUSION EQUATION

As far as we know all p.d.e.'s invariant under LB transformations have a recursion operator and, moreover, can be related to linear p.d.e.'s. This suggests the possibility of seeking a transformation relating Eq. (17) to a linear equation. This leads us to consider the linearization of Eq. (17), namely,

$$(A - \partial/\partial t) f = 0, \tag{29}$$

where A is given by Eq. (27) for any solution $u = \theta(x,t)$ of Eq. (17). Introducing a new variable $\bar{u}$ by

$$f = \frac{\partial}{\partial x}(u\bar{u}), \tag{30}$$

we obtain from Eq. (29) the equation

$$\left[\left(u^{-1}\frac{\partial}{\partial x}\right)^2 + u^{-3}u_1\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right]\bar{u} = 0 \tag{31}$$

and if we set

$$\begin{aligned} \frac{\partial}{\partial \bar{x}} &= u^{-1}\frac{\partial}{\partial x}, \\ \frac{\partial}{\partial \bar{t}} &= \frac{\partial}{\partial t} - u^{-3}u_1\frac{\partial}{\partial x}, \end{aligned} \tag{32}$$

Eq. (31) becomes

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - \frac{\partial \bar{u}}{\partial \bar{t}} = 0. \tag{33}$$

Since $f = 0$ is always a solution of Eq. (29), the relation (30) suggests that we set $u\bar{u} =$ constant. This and Eqs. (32) lead us to the transformation

$$\begin{aligned} d\bar{x} &= u\,dx + u^{-2}u_1\,dt, \\ d\bar{t} &= dt, \\ \bar{u} &= u^{-1}, \end{aligned} \tag{34}$$

relating solutions $u = \theta(x,t)$ of Eq. (17) to solutions $\bar{u} = \bar{\theta}(\bar{x},\bar{t})$ of Eq. (33). Choosing a fixed point (x_0,t_0), we have the following integrated form of Eqs. (34):

$$\begin{aligned} \bar{x} &= \int_{x_0}^{x} u\,dx' - \int_{t_0}^{t}\left(\frac{\partial}{\partial x}u^{-1}\right)_{x=x_0} dt', \\ \bar{t} &= t - t_0, \\ \bar{u} &= u^{-1}. \end{aligned} \tag{35}$$

It is easy to check that Eqs. (35) indeed transform Eq. (17) to Eq. (33), and define a map relating the solutions of Eqs. (17) and (33). Moreover, if $u > 0$ $(\bar{u} > 0)$, Eqs. (35) define a one-to-one map since $\partial\bar{x}/\partial x > 0$ for each fixed t.[22]

We now show that under the transformation (34) the recursion operator $\mathscr{D}$ of Eq. (17) is transformed into the recursion operator

$$\bar{\mathscr{D}} = D_{\bar{x}}, \tag{36}$$

leading to an infinite sequence of LB transformations of the heat equation (33). The proof is as follows:

An LB transformation of the form (5) induces an LB transformation on the variables $\{\bar{x},\bar{t},\bar{u}\}$ through Eqs. (34), namely,

$$\begin{aligned} \bar{x}^* &= \bar{x} + \epsilon\bar{\xi} + O(\epsilon^2), \\ \bar{t}^* &= \bar{t}, \\ \bar{u}^* &= \bar{u} + \epsilon\bar{\eta} + O(\epsilon^2), \end{aligned} \tag{37}$$

where $\bar{\xi}$ and $\bar{\eta}$ are defined by

$$\begin{aligned} d\bar{\xi} &= \mathscr{A}\,d\bar{x} + \mathscr{B}\,d\bar{t}, \\ \mathscr{A} &= \bar{u}U, \quad \mathscr{B} = \bar{u}_{\bar{x}}U + (\bar{u})^2(U^x - 2U), \\ \bar{\eta} &= -(\bar{u})^2 U. \end{aligned} \tag{38}$$

It turns out that for any solution $\bar{u} = \bar{\theta}(\bar{x},\bar{t})$ of Eq. (33), $\mathscr{A}$ and $\mathscr{B}$ satisfy the integrability condition $D_{\bar{t}}\mathscr{A} = D_{\bar{x}}\mathscr{B}$, so that $d\bar{\xi}$ is an exact differential. The integrated form of $\bar{\xi}$ is

$$\bar{\xi} = -(D_{\bar{x}})^{-1}[\bar{\eta}\cdot(\bar{u})^{-1}] + c, \tag{39}$$

where c is an arbitrary constant. Since $U^{(i+1)} = \mathscr{D}U^{(i)}$, where $\mathscr{D}$ is given by Eq. (26), for $c = 0$ we get a corresponding infinite sequence of invariant infinitesimal LB transformations $\{\bar{U}^{(i)}\}$ for Eq. (33), namely,

$$\bar{U}^{(i)} = \bar{\eta}^i - \bar{u}_{\bar{1}}\bar{\xi}^i,$$

where

$$\bar{\eta}^i = -(\bar{u})^2 U^{(i)}, \tag{40}$$
$$\bar{\xi}^i = -(D_{\bar{x}})^{-1}[\bar{\eta}^i\cdot(\bar{u})^{-1}],$$

and $\bar{u}_{\bar{i}} = (\partial/\partial\bar{x})^i\bar{u}$. From Eqs. (40) it is simple to show that

$$\bar{U}^{(i+1)} = D_{\bar{x}}\bar{U}^{(i)}, \tag{41}$$

leading to Eq. (36). Moreover,

$$\bar{U}^{(i)} = D_{\bar{x}}(\bar{u}\bar{\xi}^i) = (D_{\bar{x}})^i\bar{u}_{\bar{2}}, \quad i = 1,2,\cdots. \tag{42}$$

$D_{\bar{x}}$ corrsponds to the obvious invariance of Eq. (33) under translations in $\bar{x}$.

It is interesting to note that the recursion operator for the invariant LB transformations of Burgers' equation is also mapped into the space translation operator under the Hopf–Cole transformation relating Burgers' equation to the heat equation. Moreover, we can obtain the Hopf–Cole transformation by examining the linearization equation (29) corresponding to Burgers' equation.

5. PROPERTIES OF SOLUTIONS OF EQ. (17) FROM THE MAPPING

We now consider the use of Eqs. (34) in constructing solutions to Eq. (17). It is easy to show that Eqs. (34) are equivalent to

$$dx = \bar{u}\,d\bar{x} + \bar{u}_{\bar{1}}\,d\bar{t},$$
$$dt = d\bar{t}, \tag{43}$$
$$u = (\bar{u})^{-1},$$

with an integrated form

$$x = \int_{\bar{x}_0}^{\bar{x}} \bar{u}\,d\bar{x}' + \int_{\bar{t}_0}^{\bar{t}} (\bar{u}_{\bar{1}})_{\bar{x}=\bar{x}_0}\,d\bar{t}',$$
$$t = \bar{t} - \bar{t}_0, \tag{44}$$
$$u = (\bar{u})^{-1},$$

for some fixed point $(\bar{x}_0,\bar{t}_0)$. In the following, we assume $u > 0$ $(\bar{u} > 0)$. Without loss of generality, we set $\bar{x}_0 = \bar{t}_0 = 0$.

A. Explicit formula connecting solutions; examples

First we consider the problem of giving a more explicit formula for relating solutions of Eq. (33) to those of Eq. (17). Let $\bar{u} = \bar{\theta}(\bar{x},\bar{t})$ be a solution of Eq. (33) on the domain $\bar{t} > 0$, $\bar{x}\in(\bar{x}_1,\bar{x}_2)$. By Eqs. (43),

$$x = X(\bar{x},\bar{t}) = \int_0^{\bar{x}} \bar{\theta}(\bar{x}',\bar{t})\,d\bar{x}' + \int_0^{\bar{t}} \left(\frac{\partial\bar{\theta}(\bar{x},\bar{t}')}{\partial\bar{x}}\right)_{\bar{x}=0} d\bar{t}'. \tag{45}$$

This uniquely determines the function X^{-1}, $\bar{x} = X^{-1}(x,t)$, where $\bar{t} = t$. Now Eqs. (44) lead to the following solution of Eq. (17):

$$u = \theta(x,t) = \frac{1}{\bar{\theta}(X^{-1}(x,t),t)},$$

on the domain $x\in(x_1(t),\ \ x_2(t))$, $t > 0$, where (46)

$$x_1(t) = X(\bar{x}_1,t), x_2(t) = X(\bar{x}_2,t).$$

In a similar manner, Eqs. (35) map a solution $u = \theta(x,t)$ of Eq. (17) to

$$\bar{u} = \bar{\theta}(\bar{x},\bar{t}) = \frac{1}{\theta(\bar{X}^{-1}(\bar{x},\bar{t}),\bar{t})}$$

on the domain $\bar{x}\in(\bar{x}_1(\bar{t}),\bar{x}_2(\bar{t}))$, $\bar{t} > 0$ where (47)

$$\bar{x}_1(\bar{t}) = \bar{X}(x_1,\bar{t}), \quad \bar{x}_2(\bar{t}) = \bar{X}(x_2,\bar{t}),$$

$$\bar{x} = \bar{X}(x,t) = \int_0^x \theta(x',t)\,dx' - \int_0^t \left[\frac{\partial}{\partial x}(\theta(x,t'))^{-1}\right]_{x=0} dt', \tag{48}$$

with the corresponding definition of the function $\bar{X}^{-1}(\bar{x},\bar{t}) = x$.

Example 1: The source solution of Eq. (33), i.e., $\bar{u} = \bar{\theta}(\bar{x},\bar{t}) = a(4\pi\bar{t})^{-1/2}e^{-(\bar{x}^2/4\bar{t})}$ on the domain $-\infty < \bar{x} < \infty, \bar{t} > 0$, is mapped by Eqs. (45) and (46) into the following separable solution of Eq. (17):

$$u = \theta(x,t) = a^{-1}(4\pi t)^{1/2}e^{v^2},$$

on the domain $-\frac{1}{2}a < x < \frac{1}{2}a$, $t > 0$, where $v(x)$ is defined by (49)

$$x = \frac{a}{\sqrt{\pi}}\int_0^v e^{-y^2}\,dy.$$

Note that $\lim_{x\to\pm\frac{1}{2}a}\theta(x,t) = +\infty$.

Example 2. The dipole solution of Eq. (33), i.e.,

$$\bar{u} = \bar{\theta}(\bar{x},\bar{t}) = -\frac{\partial}{\partial\bar{x}}[a(4\pi\bar{t})^{-1/2}e^{-(\bar{x}^2/4\bar{t})}],$$

on the domain $0 < \bar{x} < \infty$, $\bar{t} > 0$, is mapped by Eqs. (45) and (46) into the following self-similar solution of Eq. (17):

$$u = \theta(x,t) = x^{-1}(2t)^{1/2}\left[\ln\left(\frac{a^2}{4\pi t x^2}\right)\right]^{-1/2} \tag{50}$$

on the shrinking domain $0 < x < a(4\pi t)^{-1/2}$, $t > 0$.

B. Connection between initial conditions; connection between boundary conditions

The mapping formulas (34) and (43) demonstrate a one-to-one correspondence (within translation of x,t) between initial conditions for Eq. (17) and those for Eq. (33). As for the connection between boundary conditions, from the same formula it is easy to see that $x = s(t)$ is an insulating boundary of Eq. (17), i.e., $[\partial\theta(x,t)/\partial x]_{x=s(t)} = 0$, if and only if the corresponding boundary $\bar{x} = \bar{s}(\bar{t})$ is an insulating boundary of Eq. (33), i.e., the corresponding solution $\bar{u} = \bar{\theta}(\bar{x},\bar{t})$ satisfies $[\partial\bar{\theta}(\bar{x},\bar{t})/\partial\bar{x}]_{\bar{x}=\bar{s}(\bar{t})} = 0$. Moreover, $s(t) = \text{const}$ if and only if $\bar{s}(\bar{t}) = \text{const}$, i.e., there is a one-to-one correspondence between fixed insulating boundaries of Eqs. (17) and (33).

In general, a noninsulating boundary condition for Eq. (17), on a fixed boundary $x = \text{const} = c$, is mapped into a noninsulating boundary condition of Eq. (33) with a corresponding moving boundary $\bar{x} = \bar{s}(\bar{t}) \neq \text{const}$ with speed

$$\frac{d\bar{s}}{d\bar{t}} = \left[[\theta(x,t)]^{-2} \frac{\partial \theta(x,t)}{\partial x} \right]_{\substack{x=c \\ t=\bar{t}}}, \tag{51}$$

where, as previously mentioned, $u = \theta(x,t) > 0$.

6. CONCLUDING REMARKS

(a) From invariance under the LB transformations $\{U^{(i)}\}$, $i = 1,2,\cdots$, there exist similarity solutions of Eq. (17), i.e., $u = \theta(x,t;n)$, whose similarity forms satisfy

$$U^{(n)} + \sum_{k=1}^{n-1} c_k U^{(k)} = 0, \tag{52}$$

where $\{c_1, c_2, \ldots, c_{n-1}\}$ are arbitrary constants, $n = 1,2,\cdots$. For example, for $n = 1$, Eq. (52) leads to the similarity form

$$u = \theta(x,t;1) = [a(t)\cdot(x + b(t))^2 + c(t)]^{-1/2}, \tag{53}$$

where $\{a(t), b(t), c(t)\}$ are arbitrary. Substituting Eq. (53) into Eq. (17) we find that Eq. (53) solves Eq. (17) if and only if $a = \alpha$, $b = \beta$, and $c = \gamma e^{2\alpha t}$, where $\{\alpha,\beta,\gamma\}$ are arbitrary constants. This solution is not contained in the class of similarity solutions of Eq. (17) obtained from invariance under a four-parameter point Lie group.[7,8]

(b) The infinitesimal transformations (5) of the four-parameter point group of Eq. (17) are given by

$$U^a = u + xu_1, \quad U^b = xu_1 + 2tu_t,$$
$$U^c = u_1, \quad U^d = B. \tag{54}$$

Under the mapping (34), these are transformed, respectively, to corresponding infinitesimals of invariant point group transformations of Eq. (33):

$$\bar{U}^a = \bar{u}, \quad \bar{U}^b = \bar{x}\bar{u}_{\bar{1}} + 2\bar{t}\bar{u}_{\bar{t}},$$
$$\bar{U}^c = 0, \quad \bar{U}^d = \bar{B} = \bar{u}_{\bar{2}}. \tag{55}$$

Conversely, the mapping (34) transforms the six-parameter point Lie group of Eq. (33) as follows: The three-parameter subgroup of infinitesimals given by Eq. (55) transforms to $\{U^a, U^b, U^d\}$ given by Eqs. (54) and $\bar{U} = \bar{u}_{\bar{1}}$ transforms to $U = 0$; the remaining infinitesimal point group transformations $\bar{U}^e = \bar{x}\bar{u} + 2\bar{t}\bar{u}_{\bar{t}}$ and $\bar{U}^f = (\frac{1}{4}\bar{x}^2 + \frac{1}{2}\bar{t})\bar{u} + \bar{x}\bar{t}\bar{u}_{\bar{1}} + \bar{t}^2\bar{u}_{\bar{t}}$ are mapped, respectively, into infinitesimals which depend on $\{x,t,u,u_1\}$ and integrals of u.

(c) Generally speaking, the action of a recursion operator $\mathscr{D}$ on any infinitesimal invariance transformation U of the form (5) (whether of point group or LB type) yields a new infinitesimal transformation $U' = \mathscr{D}U$ if $\mathscr{D}U \neq 0$. For Eq. (17), we can show that $\mathscr{D}U^a = \mathscr{D}U^b = \mathscr{D}U^c = 0$.

(d) The heat equation is a special limiting case of Eq. (3) obtained by setting $a = b^2$ and then observing $\lim_{b\to\infty} b^2(u+b)^{-2} = 1$. As one might expect if $a = b^2$, for the corresponding recursion operator $\mathscr{D}$, $\lim_{b\to\infty}\mathscr{D} = \partial/\partial x$, and the mapping formulas reduce to identity mappings.

(e) Since Eq. (1) admits an infinite sequence of LB transformations if and only if $C(u)$ satisfies Eq. (15) with associated mapping (34) whereas Eq. (4) admits an infinite sequence of LB transformations, there is no point transformation of the form

$$\bar{x} = K(x,t,u),$$
$$\bar{t} = L(x,t,u),$$
$$\bar{u} = M(x,t,u),$$

relating solutions of Eq. (1) and those of Eq. (4).

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[21]An infinitesimal LB transformation

$$x^* = x + \epsilon\xi + O(\epsilon^2),$$
$$t^* = t + \epsilon\tau + O(\epsilon^2),$$
$$u^* = u + \epsilon\mu + O(\epsilon^2),$$

acts on a surface $F(x,t,u) = 0$ in the same manner as

$$x^* = x,$$
$$t^* = t,$$
$$u^* = u + \epsilon U + O(\epsilon^2),$$

where $U = \mu - \xi u_x - \tau u_t$.

[22]G. Rosen, Phys. Rev. B **19**, 2398 (1979). After submitting this paper we discovered the above reference through Nonlinear Science Abstracts. Here Rosen discovered transformation (35) and worked out some examples. We are also grateful to the referee for bringing the above paper to our attention.