Derivation of conservation laws from nonlocal symmetries of differential equations

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An identity is derived which yields a correspondence between symmetries and conservation laws for self-adjoint differential equations. This identity does not rely on use of a Lagrangian as needed to obtain conservation laws by Noether's theorem. Moreover, unlike Noether's theorem, which can only generate conservation laws from local symmetries, the derived identity generates conservation laws from *nonlocal* as well as local symmetries. It is explicitly shown how Noether's theorem is extended by the identity. Conservation laws arising from nonlocal symmetries are obtained for a class of scalar wave equations with variable wave speeds. The constants of motion resulting from these nonlocal conservation laws are shown to be linearly independent of all constants of motion resulting from local conservation laws. © *1996 American Institute of Physics.* [S0022-2488(96)02405-2]

I. INTRODUCTION

Conservation laws can be found for self-adjoint systems of differential equations by Noether's theorem.[1–3] If a local symmetry admitted by a given system leaves invariant the variational principle of the system, Noether's theorem yields a corresponding conservation law of local type. Conversely, all conservation laws of local type for a given system arise from the local symmetries admitted by the system. A limitation of Noether's theorem, however, is that it can only directly deal with local symmetries and hence conservation laws of local type. This poses a significant incompleteness in the study of differential equations since conservation laws of nonlocal type are equally as useful as those of local type. In particular, as will be shown in this article, conservation laws of nonlocal type yield additional constants of motion and thus expand the utility of methods of analysis which depend on conservation laws.

In this article we introduce an expression that yields conservation laws from nonlocal symmetries as well as local symmetries admitted by an arbitrary self-adjoint system of differential equations. Significantly, in contrast to the formulation of Noether's theorem, the expression is derived from a bilinear identity that makes no use of a Lagrangian. As preliminaries to the derivation and main results, we now give definitions of local and nonlocal symmetries and conservation laws of local and nonlocal type for self-adjoint systems of differential equations.

Consider a system of differential equations (DEs) given by

$$G_\sigma(x,u,\underset{1}{u},\ldots,\underset{K}{u})=0, \quad \sigma=1,\ldots,M \tag{1.1}$$

for $M\geqslant 1$ dependent variables $u=(u^1,\ldots,u^M)$ which are functions of $N\geqslant 1$ independent variables $x=(x^1,\ldots,x^N)$, with $\underset{J}{u}$ denoting all Jth order derivatives of u with respect to x. For the sequel, we let D_i denote total differentiation with respect to x^i, where $i=1,\ldots,N$, and we use the index notation $u^\gamma_{i_1\cdots i_J} = D_{i_1}\cdots D_{i_J}u^\gamma$ for differentiations of u, where $\gamma=1,\ldots,M$, $i_J=1,\ldots,N$, and

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$J=1,2,\ldots$. Hereafter, unless otherwise stated, we use the index conventions that all Greek indices range from 1 to M, all Latin indices (lower case) range from 1 to N, while summation is assumed over any repeated indices in all expressions.

Definition 1.1: The Fréchet derivative associated with system (1.1) is the matrix linear operator

$$\mathscr{F}_{\sigma\rho}=\frac{\partial G_\sigma}{\partial u^\rho}+\frac{\partial G_\sigma}{\partial u_i^\rho}D_i+\cdots+\frac{\partial G_\sigma}{\partial u^\rho_{i_1\cdots i_K}}D_{i_1}\cdots D_{i_K}. \tag{1.2}$$

Definition 1.2: A symmetry admitted by system (1.1) is characterized by an infinitesimal generator

$$\mathbf{X}=\eta^\mu\partial/\partial u^\mu, \tag{1.3}$$

where η^μ satisfies

$$\mathscr{F}_{\sigma\rho}\eta^\rho=0 \tag{1.4}$$

for every solution $u(x)$ of system (1.1).

Definition 1.3: A local symmetry admitted by system (1.1) is a symmetry with an infinitesimal generator of the form

$$\mathbf{X}=\eta^\mu(x,u,\underset{1}{u},\ldots,\underset{P}{u})\partial/\partial u^\mu \tag{1.5}$$

such that, for all values of x, η^μ depends on $u,\underset{1}{u},\ldots,\underset{P}{u}$ only through $u(x),\underset{1}{u}(x),\ldots,\underset{P}{u}(x)$ evaluated at x.

Definition 1.4: A nonlocal symmetry admitted by system (1.1) is a symmetry with an infinitesimal generator $\mathbf{X}=\eta^\mu\partial/\partial u^\mu$ not of the form (1.5), such that η^μ has other than just a local dependence on $u(x)$ and derivatives of $u(x)$ to some finite order.

All local symmetries of system (1.1) can be determined by Lie's algorithm.[2,3] No corresponding procedure exists to find *all* nonlocal symmetries of system (1.1).

There is an algorithm[3–6] to determine special nonlocal symmetries, called *potential symmetries*, if one DE of system (1.1) is a divergence expression. These potential symmetries arise as local symmetries admitted by auxiliary systems associated to system (1.1). In the case of two independent variables ($N=2$), suppose system (1.1) has a DE of the form

$$G_\sigma(x,u,\underset{1}{u},\ldots,\underset{K}{u})=D_1f^1(x,u,\underset{1}{u},\ldots,\underset{K-1}{u})+D_2f^2(x,u,\underset{1}{u},\ldots,\underset{K-1}{u})=0,\quad \sigma=M. \tag{1.6}$$

Through Eq. (1.6) one can introduce an auxiliary potential variable v and form a potential system given by

$$G_\sigma=0,\quad \sigma=1,\ldots,M-1, \tag{1.7}$$

$$D_2v=f^1,\quad D_1v=-f^2. \tag{1.8}$$

If $(u(x),v(x))$ satisfies system (1.7)–(1.8), then $u(x)$ satisfies system (1.1); if $u(x)$ satisfies system (1.1), then there exists some $v(x)$ (unique up to the addition of an arbitrary constant) such that $(u(x),v(x))$ satisfies system (1.7)–(1.8). Since $v(x)$ is determined in terms of integrals of $u(x)$, a local symmetry of system (1.7)–(1.8) may yield a nonlocal symmetry of system (1.1). In particular, such a nonlocal symmetry arises if and only if an infinitesimal generator of a local symmetry of system (1.7)–(1.8) does not project onto an infinitesimal generator of a local symmetry admitted by system (1.1). Similar considerations hold for the case of more than two independent variables.

Definition 1.5: A conservation law of system (1.1) is a divergence free expression $D_i\Psi^i=0$ *which holds for every solution of system (1.1) and its differential consequences. The conservation law is local (conservation law of local type) if and only if it has the form* $D_i\Psi^i(x,u,\underset{1}{u},\ldots,\underset{L}{u})=0$ *where, for all values of* x, Ψ^i *depends on* $u,\underset{1}{u},\ldots,\underset{L}{u}$ *only through* $u(x),\underset{1}{u}(x),\ldots,\underset{L}{u}(x)$ *evaluated at* x*. Otherwise the conservation law is nonlocal (conservation law of nonlocal type).*

Definition 1.6: The adjoint of the Fréchet derivative (1.2) is the matrix linear operator $\mathscr{F}^*_{\sigma\rho}$ *satisfying*

$$V^\sigma\mathscr{F}_{\sigma\rho}W^\rho-W^\sigma\mathscr{F}^*_{\sigma\rho}V^\rho=D_iP^i \tag{1.9}$$

for all K times differential functions $V^\gamma(x)$ *and* $W^\gamma(x)$*, for some* P^i *which depends on* x^i, u^γ, V^γ, W^γ *and the derivatives of* u^γ, V^γ, W^γ *to some finite order.*

Definition 1.7: The system (1.1) is self-adjoint if and only if

$$\mathscr{F}_{\sigma\rho}=\mathscr{F}^*_{\sigma\rho}\ . \tag{1.10}$$

In Sec. II, we derive the bilinear identity giving a correspondence between symmetries and conservation laws for self-adjoint systems of DEs (1.1). From this identity we obtain an expression that yields a conservation law for each pair of symmetries, *local* or *nonlocal*, admitted by any such system (linear or nonlinear). Furthermore, as each such linear system admits a trivial scaling symmetry, we obtain a conservation law for all nontrivial symmetries of any self-adjoint linear system of DEs (1.1). In particular, each nonlocal symmetry admitted by such a linear system thereby leads to a corresponding nonlocal conservation law.

From the known connection between local conservation laws and local symmetries[2,7] for self-adjoint systems of DEs it follows that all local conservation laws obtained through the bilinear identity derived in Sec. II are also obtainable from Noether's theorem. In the case when such a system is linear, we show in Sec. III that each local symmetry leaving invariant a corresponding variational principle yields the same conservation law through our bilinear identity as through Noether's theorem.

In Sec. IV, as an example of a self-adjoint linear DE, we consider the two-dimensional scalar wave equation with a variable wave speed. For a large class of wave speeds this equation admits nonlocal symmetries realized as potential symmetries.[3–5] The nonlocal character of these symmetries means that we cannot obtain corresponding conservation laws by applying Noether's theorem to the variational principle of the scalar wave equation. Moreover, we show that the potential system for this equation does not have a variational principle, and hence Noether's theorem cannot be applied to the potential system to obtain any conservation laws. By using our conservation law expression derived in Sec. II, we obtain nonlocal conservation laws for the admitted nonlocal symmetries. In Sec. V, we obtain corresponding constants of motion for the scalar wave equation. We show that these constants of motion are linearly independent of each other as well as linearly independent of all constants of motion arising from local symmetries of the scalar wave equation.

In Sec. VI, we expand on some of the ideas and results presented in earlier sections.

II. DERIVATION OF THE CONSERVATION LAW EXPRESSION

We consider a system (1.1) that is self-adjoint. Then the DEs in system (1.1) must satisfy the following Helmholtz identities:[7]

$$\frac{\partial G_\sigma}{\partial u^\rho}=\frac{\partial G_\rho}{\partial u^\sigma}-D_i\left(\frac{\partial G_\rho}{\partial u^\sigma_i}\right)+\cdots+(-1)^K D_{i_1}\cdots D_{i_K}\left(\frac{\partial G_\rho}{\partial u^\sigma_{i_1\cdots i_K}}\right), \tag{2.1}$$

$$\frac{\partial G_\sigma}{\partial u^\rho_{i_1\cdots i_j}} = (-1)^J \frac{\partial G_\rho}{\partial u^\sigma_{i_1\cdots i_J}} + (-1)^{J+1} C_J^{J+1} D_{i_{J+1}}\left(\frac{\partial G_\rho}{\partial u^\sigma_{i_1\cdots i_{J+1}}}\right) + \cdots + (-1)^K C_J^K D_{i_{J+1}}\cdots D_{i_K}\left(\frac{\partial G_\rho}{\partial u^\sigma_{i_1\cdots i_K}}\right), \quad J=1,\ldots,K-1, \tag{2.2}$$

$$\frac{\partial G_\sigma}{\partial u^\rho_{i_1\cdots i_K}} = (-1)^K \frac{\partial G_\rho}{\partial u^\sigma_{i_1\cdots i_K}}, \tag{2.3}$$

where $C_J^L = L!/J!(L-J)!$ for positive integers $L \geqslant J$. As a consequence of these identities, one can verify by direct calculation that the Fréchet derivative (1.2) leads to the identity

$$\mathscr{F}_{\sigma\rho}\omega^\rho = \omega^\rho \frac{\partial G_\rho}{\partial u^\sigma} - D_i\left(\omega^\rho \frac{\partial G_\rho}{\partial u^\sigma_i}\right) + \cdots + (-1)^K D_{i_1}\cdots D_{i_K}\left(\omega^\rho \frac{\partial G_\rho}{\partial u^\sigma_{i_1\cdots i_K}}\right) \tag{2.4}$$

for arbitrary functions ω^ρ.

Using Eq. (2.4) and the Leibnitz rule for differentiation, one finds that the following bilinear skew-symmetric identity holds for arbitrary functions ω^ρ and ν^ρ:

$$\nu^\sigma \mathscr{F}_{\sigma\rho}\omega^\rho - \omega^\sigma \mathscr{F}_{\sigma\rho}\nu^\rho = D_i \Phi^i[\nu,\omega], \tag{2.5}$$

where

$$\begin{aligned}\Phi^i[\nu,\omega] = &-\frac{1}{2}\left\{\nu^\sigma \frac{\partial G_\rho}{\partial u^\sigma_i}\omega^\rho + (D_j\nu^\sigma - \nu^\sigma D_j)\left(\frac{\partial G_\rho}{\partial u^\sigma_{ji}}\omega^\rho\right) + \cdots + (D_{i_1}\cdots D_{i_{K-1}}\nu^\sigma \right.\\ &\left. + \cdots + (-1)^K \nu^\sigma D_{i_1}\cdots D_{i_{K-1}})\left(\frac{\partial G_\rho}{\partial u^\sigma_{i_1\cdots i_{K-1}i}}\omega^\rho\right)\right\}\\ &+\frac{1}{2}\left\{\omega^\sigma \frac{\partial G_\rho}{\partial u^\sigma_i}\nu^\rho + (D_j\omega^\sigma - \omega^\sigma D_j)\left(\frac{\partial G_\rho}{\partial u^\sigma_{ji}}\nu^\rho\right) + \cdots + (D_{i_1}\cdots D_{i_{K-1}}\omega^\sigma \right.\\ &\left. + \cdots + (-1)^K \omega^\sigma D_{i_1}\cdots D_{i_{K-1}})\left(\frac{\partial G_\rho}{\partial u^\sigma_{i_1\cdots i_{K-1}i}}\nu^\rho\right)\right\}.\end{aligned} \tag{2.6}$$

The functions ω^ρ and ν^ρ here can have arbitrary (local or nonlocal) dependence on u and derivatives of u.

This bilinear identity leads to a connection between symmetries and conservation laws:

Theorem 2.1: *Suppose* $\mathbf{X}_1 = \eta_1^\mu \partial/\partial u^\mu$ *and* $\mathbf{X}_2 = \eta_2^\mu \partial/\partial u^\mu$ *are infinitesimal generators of symmetries (local or nonlocal) of a self-adjoint system (1.1). The bilinear identity (2.5) then yields the conservation law*

$$D_i \Phi^i[\eta_1,\eta_2] = 0 \tag{2.7}$$

with $\Phi^i[\eta_1,\eta_2]$ *defined by Eq. (2.6).*

We now specialize to the case when system (1.1) is a linear homogeneous system

$$G_\sigma(x,u,\underset{1}{u},\ldots,\underset{K}{u}) = G_{\sigma\rho}(x)u^\rho + G^i_{\sigma\rho}(x)u^\rho_i + \cdots + G^{i_1\cdots i_K}_{\sigma\rho}(x)u^\rho_{i_1\cdots i_K} = 0 \tag{2.8}$$

with coefficients $G_{\sigma\rho}(x), G^{i}_{\sigma\rho}(x), \ldots, G^{i_1\cdots i_K}_{\sigma\rho}(x)$. Every such system admits the trivial scaling symmetry

$$\mathbf{X}_s = u^{\mu}\partial/\partial u^{\mu}. \tag{2.9}$$

Using this symmetry as one of the symmetries in Theorem 2.1 now leads to the following correspondence:

Theorem 2.2: *Suppose a self-adjoint linear system (2.8) admits a nontrivial symmetry (local or nonlocal) with infinitesimal generator* $\mathbf{X}=\eta^{\mu}\partial/\partial u^{\mu}$. *Then Eq. (2.7) yields the conservation law*

$$D_i \Phi^i[u,\eta] = 0, \tag{2.10}$$

where

$$\begin{aligned}\Phi^i[u,\eta] = &-\tfrac{1}{2}\Big\{ u^{\sigma}G^{i}_{\rho\sigma}(x)\,\eta^{\rho} + (u^{\sigma}_j - u^{\sigma}D_j)(G^{ji}_{\rho\sigma}(x)\,\eta^{\rho})\\ &+\cdots+(u^{\sigma}_{i_1\cdots i_{K-1}} + \cdots + (-1)^K u^{\sigma} D_{i_1}\cdots D_{i_{K-1}})(G^{i_1\cdots i_{K-1}i}_{\rho\sigma}(x)\,\eta^{\rho})\Big\}\\ &+\tfrac{1}{2}\Big\{ \eta^{\sigma}G^{i}_{\rho\sigma}(x)u^{\rho} + (D_j\eta^{\sigma} - \eta^{\sigma}D_j)(G^{ji}_{\rho\sigma}(x)u^{\rho}) + \cdots + (D_{i_1}\cdots D_{i_{K-1}}\eta^{\sigma}\\ &+\cdots+(-1)^K \eta^{\sigma}D_{i_1}\cdots D_{i_{K-1}})(G^{i_1\cdots i_{K-1}i}_{\rho\sigma}(x)u^{\rho})\Big\}.\end{aligned} \tag{2.11}$$

III. RELATIONSHIP TO NOETHER'S THEOREM

Noether's theorem only relates local symmetries to conservation laws (of local type) for self-adjoint systems. The variational principle for a (linear or nonlinear) self-adjoint system (1.1) has Lagrangian L given by[2,7]

$$L(x,u,\underset{1}{u},\ldots,\underset{K}{u}) = \int_0^1 u^{\sigma}G_{\sigma}(x,\lambda u,\lambda \underset{1}{u},\ldots,\lambda \underset{K}{u})\,d\lambda. \tag{3.1}$$

Definition 3.1: An infinitesimal generator $\mathbf{X}=\eta^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u})\partial/\partial u^{\mu}$ *is a variational symmetry of a self-adjoint system (1.1) if and only if*

$$\mathbf{X}^{(K)}L(x,u,\underset{1}{u},\ldots,\underset{K}{u}) = D_i A^i \tag{3.2}$$

for some $A^i(x,u,\underset{1}{u},\ldots,\underset{L}{u})$, *where* $\mathbf{X}^{(K)}$ *is the Kth prolongation generator given by*

$$\mathbf{X}^{(K)} = \eta^{\mu}\partial/\partial u^{\mu} + (D_i\eta^{\mu})\partial/\partial u^{\mu}_i + \cdots + (D_{i_1}\cdots D_{i_K}\eta^{\mu})\partial/\partial u^{\mu}_{i_1\cdots i_K}. \tag{3.3}$$

Noether's theorem yields a local conservation law for each variational symmetry admitted by system (1.1). Specifically, one can show that

$$\mathbf{X}^{(K)}L = G_{\sigma}\eta^{\sigma} + D_i S^i = D_i A^i, \tag{3.4}$$

where $S^i = \eta^{\sigma}\partial L/\partial u^{\sigma}_t + (D_j\eta^{\sigma} - \eta^{\sigma}D_j)(\partial L/\partial u^{\sigma}_{ji}) + \cdots + (D_{j_1}\cdots D_{j_{K-1}}\eta^{\sigma} + \cdots + (-1)^{K-1}\eta^{\sigma}D_{j_1}\cdots D_{j_{K-1}})(\partial L/\partial u^{\sigma}_{j_1\cdots j_{K-1}i})$.[2,3,7] Then Eq. (3.4) yields Noether's identity

$$D_i N^i[\eta] = -G_{\sigma}\eta^{\sigma} \tag{3.5}$$

with $N^i[\eta] = S^i - A^i$. Consequently, for any solution of system (1.1) we obtain the Noether conservation law

$$D_i N^i[\eta]=0. \tag{3.6}$$

Using the Helmholtz identities (2.1) to (2.3), along with Noether's identity (3.5) and the fact that the Euler–Lagrange operator annihilates divergences, one can show that any variational symmetry $\mathbf{X}=\eta^{\mu}\partial/\partial u^{\mu}$ satisfies the identity

$$\mathscr{F}_{\sigma\rho}\eta^{\rho}=-G_{\rho}\frac{\partial\eta^{\rho}}{\partial u^{\sigma}}+D_i\left(G_{\rho}\frac{\partial\eta^{\rho}}{\partial u_i^{\sigma}}\right)+\cdots+(-1)^{P+1}D_{i_1}\cdots D_{i_P}\left(G_{\rho}\frac{\partial\eta^{\rho}}{\partial u^{\sigma}_{i_1\cdots i_P}}\right). \tag{3.7}$$

From Eq. (3.7) it immediately follows that all variational symmetries are local symmetries of system (1.1). [The converse does not always hold, as seen from the fact that scaling symmetries (2.9) generally do not satisfy Eq. (3.7).]

For the rest of this section we restrict the self-adjoint system (1.1) to be a linear homogeneous system (2.8). Before relating conservation laws from Noether's theorem to conservation laws arising from the bilinear identity (2.5), we establish the following result:

Lemma 3.2: Suppose $\eta^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u})$ *is analytic in* u *and derivatives of* u. *Then any local symmetry generator of the form* $\mathbf{X}=\eta^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u})\partial/\partial u^{\mu}$ *admitted by a linear homogeneous system* (2.8) *can be expressed as a superposition of homogeneous local symmetry generators*:

$$\eta^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u})=\sum_{n=0}^{\infty}\overset{(n)}{\eta}{}^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u}), \tag{3.8}$$

where

$$\overset{(n)}{\eta}{}^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u})=\lambda^{-n}\overset{(n)}{\eta}{}^{\mu}(x,\lambda u,\lambda\underset{1}{u},\ldots,\lambda\underset{P}{u}) \tag{3.9}$$

for all positive constants λ.

Proof: Since system (2.8) admits the scaling symmetry (2.9), it must also admit the symmetry $\eta^{\mu}(x,\lambda u,\lambda\underset{1}{u},\ldots,\lambda\underset{P}{u})\partial/\partial u^{\mu}$ for all constants λ. Then the analyticity property of η^{μ} leads to

$$\eta^{\mu}(x,\lambda u,\lambda\underset{1}{u},\ldots,\lambda\underset{P}{u})=\sum_{n=0}^{\infty}\lambda^{n}\overset{(n)}{\eta}{}^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u}), \tag{3.10}$$

where $\overset{(n)}{\eta}{}^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u})=\partial^{n}\eta^{\mu}(x,\lambda u,\lambda\underset{1}{u},\ldots,\lambda\underset{P}{u})/\partial\lambda^{n}|_{\lambda=0}$ and $\overset{(n)}{\eta}{}^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u}$ $=\eta^{\mu}(x,\lambda,u,\lambda\underset{1}{u},\ldots,\lambda\underset{P}{u})|_{\lambda=0}$. It then follows that each $\overset{(n)}{\eta}{}^{\mu}(x,u,\underset{1}{u},\ldots,\underset{P}{u})\partial/\partial u^{\mu}$, for $n=0,1,2,\ldots$, is a local symmetry of system (2.8). Setting $\lambda=1$ in the superposition (3.10) then yields Eq. (3.8).□

As an aside we remark that every infinitesimal generator of a point symmetry $\mathbf{X}=\eta^{\mu}(x,u,\underset{1}{u})\partial/\partial u^{\mu}$ has $\overset{(n)}{\eta}{}^{\mu}(x,u,\underset{1}{u})=0$ for $n\neq 1$ when system (2.8) is a scalar PDE of order $K\geqslant 2$ (with $N\geqslant 2$).[8]

Without loss of generality we assume that each infinitesimal generator of a symmetry admitted by system (2.8) satisfies the homogeneity property (3.9). We then have the following identity

$$u^{\sigma}\frac{\partial\eta^{\rho}}{\partial u^{\sigma}}+u_i^{\sigma}\frac{\partial\eta^{\rho}}{\partial u_i^{\sigma}}+\cdots+u^{\sigma}_{i_1\cdots i_P}\frac{\partial\eta^{\rho}}{\partial u^{\sigma}_{i_1\cdots i_P}}=n\,\eta^{\rho} \tag{3.11}$$

for some integer $n\geqslant 0$.

We now establish the relationship between Noether's conservation law expression (3.6) and our conservation law expression (2.10):

Theorem 3.3: *Suppose a variational symmetry of a self-adjoint linear system (2.8) has an infinitesimal generator* $\mathbf{X}=\eta^{\mu}\partial/\partial u^{\mu}$ *satisfying Eq. (3.11). Then up to the addition of a divergence free expression, one has*

$$\Phi^i[u,\eta]=(1+n)N^i[\eta], \tag{3.12}$$

for every solution of the system (2.8).

Proof. From the bilinear identity (2.5) we have

$$D_i\Phi^i[u,\eta]=u^{\sigma}\mathscr{F}_{\sigma\rho}\eta^{\rho}-\eta^{\sigma}\mathscr{F}_{\sigma\rho}u^{\rho}. \tag{3.13}$$

Since system (2.8) is linear, it satisfies the identity

$$\mathscr{F}_{\sigma\rho}u^{\rho}=G_{\sigma}. \tag{3.14}$$

Using the Leibnitz rule for differentiation to manipulate Eq. (3.7), we get

$$u^{\sigma}\mathscr{F}_{\sigma\rho}\eta^{\rho}=-G_{\rho}\left(u^{\sigma}\frac{\partial\eta^{\rho}}{\partial u^{\sigma}}+u_i^{\sigma}\frac{\partial\eta^{\rho}}{\partial u_i^{\sigma}}+\cdots+u^{\sigma}_{i_1\cdots i_P}\frac{\partial\eta^{\rho}}{\partial u^{\sigma}_{i_1\cdots i_P}}\right)+D_iB^i, \tag{3.15}$$

where

$$B^i=u^{\sigma}G_{\rho}\frac{\partial\eta^{\rho}}{\partial u_i^{\sigma}}+(u_j^{\sigma}-u^{\sigma}D_j)\left(G_{\rho}\frac{\partial\eta^{\rho}}{\partial u_{ji}^{\sigma}}\right)+\cdots$$

$$+(u^{\sigma}_{i_1\cdots i_{P-1}}+\cdots+(-1)^{P-1}u^{\sigma}D_{i_1}\cdots D_{i_{P-1}})\left(G_{\rho}\frac{\partial\eta^{\rho}}{\partial u^{\sigma}_{i_1\cdots i_{K-1}i}}\right). \tag{3.16}$$

Consequently, after substituting Eqs. (3.14) and (3.15) into Eq. (3.13) and then using Eq. (3.11), we obtain

$$D_i\Phi^i[u,\eta]=-(1+n)\eta^{\sigma}G_{\sigma}-D_iB^i. \tag{3.17}$$

Then Noether's identity (3.5) yields

$$D_i\Phi^i[u,\eta]=(1+n)D_iN^i[\eta]-D_iB^i. \tag{3.18}$$

Now observe that $B^i=0$ when $G_p=0$, and hence $B^i=0$ for every solution of system (2.8). Thus we arrive at Eq. (3.12). □

IV. NONLOCAL CONSERVATION LAWS FOR SCALAR WAVE EQUATIONS

Throughout the sequel, we set $x^1=x$, $x^2=t$, and we use a subscript notation for total differentiation with respect to x and t.

Consider the scalar wave equation

$$u_{xx}-c^{-2}u_{tt}=0 \tag{4.1}$$

with a variable wave speed $c(x)$. From Eq. (1.8) we introduce the corresponding potential system

$$v_t=u_x,\quad v_x=c^{-2}u_t. \tag{4.2}$$

The wave equation (4.1) has nonlocal symmetries which are realized as potential symmetries resulting from local point symmetries of potential system (4.2) if and only if the wave speed satisfies the fourth order DE[3,4]

$$(cc'(c/c')'')'=0. \tag{4.3}$$

Such wave speeds are bounded away from zero for $-\infty<x<\infty$ when $c(x)$ satisfies the first order DE

$$c'=\nu^{-1}\sin(\nu\log c),\quad \nu=\text{const}, \tag{4.4}$$

up to arbitrary scalings of c and x.[9]

Classification of the point symmetries of system (4.2) yielding nonlocal (potential) symmetries of the wave equation (4.1) leads to two cases[3–5] with, respectively, one and two admitted infinitesimal generators $\mathbf{X}=\eta\partial/\partial u$ of the form

$$\eta=f(x,t)u+g(x,t)v-\xi(x,t)u_x-\tau(x,t)u_t, \tag{4.5}$$

where $g(x,t)$ is not identically zero.

Case I (one nonlocal symmetry): The wave speed $c(x)$ satisfies

$$(c/c')'=\gamma=\text{const}. \tag{4.6}$$

Here we have

$$\begin{aligned} f(x,t)&=a'(t)(1-\tfrac{1}{2}\gamma),\quad g(x,t)=-\tfrac{1}{2}a''(t)c(x)/c'(x),\\ \xi(x,t)&=a'(t)c(x)/c'(x),\quad \tau(x,t)=a(t)(\gamma-1)+a''(t)d(x), \end{aligned} \tag{4.7}$$

where $d(x)$ is a definite integral of $1/(c(x)c'(x))$, and $a(t)$ satisfies the first order ODE $(a/t^2)'=0$, which thus leads to the existence of one generator $\mathbf{X}=\eta\partial/\partial u$.

Case II (two nonlocal symmetries): The wave speed $c(x)$ satisfies

$$cc'(c/c')''=\mu=\text{const}\neq 0. \tag{4.8}$$

Here we have

$$\begin{aligned} f(x,t)&=b'(t)(2-(c(x)/c'(x))'),\\ g(x,t)&=-\mu b(t)c(x)/c'(x),\\ \xi(x,t)&=2b'(t)c(x)/c'(x),\\ \tau(x,t)&=2b(t)((c(x)/c'(x))'-1), \end{aligned} \tag{4.9}$$

where $b(t)$ satisfies the second order ODE $b''-\mu b=0$, which thus leads to the existence of two generators $\mathbf{X}=\eta\partial/\partial u$.

Conservation laws for all symmetries admitted by the wave equation (4.1) are obtainable from Theorem 2.2 since the wave equation is linear and self-adjoint. Hence, each nonlocal symmetry $\mathbf{X}=\eta\partial/\partial u$ admitted in Cases I and II gives rise to a corresponding nonlocal conservation law. From Eqs. (2.11) and (4.5), these conservation laws are given by

$$(\Phi^1[u,\eta])_x+(\Phi^2[u,\eta])_t=0 \tag{4.10}$$

with

$$\Phi^1[u,\eta]=u\eta_x-\eta u_x+(\tau uu_x+c^{-2}\xi uu_t)_t$$
$$=g_x uv-gu_x v+f_x u^2+c^{-2}guu_t-c'c^{-1}\xi uu_x+c^{-2}\xi u_t^2+\xi u_x^2+2\tau u_x u_t, \quad (4.11)$$

$$\Phi^2[u,\eta]=c^{-2}(\eta u_t-u\eta_t)-(\tau uu_x+c^{-2}\xi uu_t)_x$$
$$=c^{-2}(-g_t uv+gu_t v-f_t u^2-guu_x+c'c^{-1}\xi uu_t-\tau u_t^2-c^2\tau u_x^2-2\xi u_x u_t), \quad (4.12)$$

where f,g,ξ,τ satisfy Eq. (4.7) in Case I and Eq. (4.9) in Case II. The identically divergence free terms in Φ^1 and Φ^2 have been added to eliminate all terms involving second order derivatives u_{xx}, u_{tt}, and u_{xt}.

These nonlocal conservation laws arising from the nonlocal symmetries $\mathbf{X}=\eta\partial/\partial u$ cannot be obtained through Noether's theorem for the scalar wave equation (4.1) since Noether's theorem is applicable only to local symmetries that leave invariant a variational principle for Eq. (4.1). Moreover, even though the symmetries $\mathbf{X}=\eta\partial/\partial u$ are realized as local symmetries of the potential system (4.2), Noether's theorem still cannot be applied since, as will now be demonstrated, the potential system is not self-adjoint and hence has no variational principle. Let

$$\begin{bmatrix}u^1\\u^2\end{bmatrix}=\begin{bmatrix}u\\v\end{bmatrix}$$

define a column vector. Then the Fréchet derivative (1.2) associated to system (4.2) is given by the matrix operator

$$\mathscr{F}=\begin{bmatrix}-\partial/\partial x & \partial/\partial t\\ -c^{-2}\partial/\partial t & \partial/\partial x\end{bmatrix}. \quad (4.13)$$

By direct calculation, using Eq. (1.9), the adjoint of the Fréchet derivative is

$$\mathscr{F}^*=-\mathscr{F}, \quad (4.14)$$

and thus the potential system is not self-adjoint.

V. NEW CONSTANTS OF MOTION FOR SCALAR WAVE EQUATIONS

Given a conservation law $(\Phi^1[u,\eta])_x+(\Phi^2[u,\eta])_t=0$ arising from Theorem 2.2 for a symmetry $\mathbf{X}=\eta\partial/\partial u$ of the scalar wave equation (4.1), we let

$$C[\eta]=\int_{-\infty}^{\infty}\Phi^2[u,\eta]dx. \quad (5.1)$$

If $u(x,t)$ has appropriate asymptotic properties as $x\to\pm\infty$, then

$$\frac{dC[\eta]}{dt}=-\Phi^1[u,\eta]\Big|_{x=-\infty}^{x=\infty}=0, \quad (5.2)$$

from which it follows that $C[\eta]$ defines a constant of motion for Eq. (4.1).

Now consider compact support initial data

$$u(x,t_0)=\varphi(x), \quad u_t(x,t_0)=\psi(x), \quad (5.3)$$

for the scalar wave equation (4.1). This determines corresponding data for the potential system (4.2), with

$$v(x,t_0)=\theta(x)=\int_{-\infty}^{x} c(\tilde{x})^{-2}\psi(\tilde{x})d\tilde{x} \tag{5.4}$$

(up to the addition of an arbitrary constant). Evaluating the nonlocal conservation laws given by Eqs. (4.10) to (4.12) with this initial data we find

$$\lim_{x\to\pm\infty} \Phi^1[u,\eta]=0 \tag{5.5}$$

and hence Eq. (5.1) yields constants of motion for the scalar wave equation (4.1). In terms of the initial data (5.3) and (5.4) we obtain

$$C[\eta]=\int_{-\infty}^{\infty} c(x)^{-2}\{(g(x,t_0)\psi(x)-g_t(x,t_0)\varphi(x))\theta(x)-g(x,t_0)\varphi(x)\varphi'(x)-f_t(x,t_0)\varphi(x)^2$$
$$+\xi(x,t_0)(c(x)^{-1}c'(x)\varphi(x)-2\varphi'(x))\psi(x)-\tau(x,t_0)(\psi(x)^2+c(x)^2\varphi'(x)^2)\}dx. \tag{5.6}$$

For each wave speed $c(x)$ satisfying Eq. (4.6), the expression $C[\eta]$ yields one constant of motion, with f,g,ξ,τ satisfying Eq. (4.7); for each wave speed $c(x)$ satisfying Eq. (4.8), the expression $C[\eta]$ yields two constants of motion, with f,g,ξ,τ satisfying Eq. (4.9).

A. Linear independence of constants of motion

Let $C[\eta_1],C[\eta_2],\ldots,C[\eta_k]$ define $k>1$ constants of motion arising for the scalar wave equation (4.1) from symmetries $\mathbf{X}_1=\eta_1\partial/\partial u$, $\mathbf{X}_2=\eta_2\partial/\partial u$,..., $\mathbf{X}_k=\eta_k\partial/\partial u$, respectively.

Definition 5.1: Suppose $c_1,\ldots,c_k$ *are constants such that* $c_1C[\eta_1]+\cdots+c_kC[\eta_k]$ *vanishes for arbitrary initial data (5.3). Then* $C[\eta_1],\ldots,C[\eta_k]$ *are linearly independent constants of motion if and only if* $c_1=\cdots=c_k=0$.

The following theorem now establishes that each constant of motion (5.6) arising from the admitted nonlocal symmetries (4.5) of Eq. (4.1) in Cases I and II is linearly independent of the constants of motion arising from all admitted point symmetries of Eq. (4.1). A subsequent theorem then establishes further that the two constants of motion (5.6) in Case II are linearly independent of each other modulo all point symmetry constants of motion.

Theorem 5.2: *For the scalar wave equation (4.1), the constants of motion (5.6) obtained from the admitted nonlocal symmetries (4.5) are each linearly independent of the constants of motion obtained from all admitted point symmetries.*

Proof: Every point symmetry admitted by a scalar linear PDE is characterized by an infinitesimal generator $\eta\partial/\partial u$ either with η linear in u and first order derivatives of u (in which case the symmetry is called *nontrivial*) or with η independent of u and derivatives of u (in which case the symmetry is called *trivial*).[8] Thus, for the scalar wave equation (4.1), every nontrivial point symmetry as well as every nonlocal symmetry (4.5) has an infinitesimal generator that is linear in u and first order derivatives of u. The constants of motion obtained from these symmetries through Theorem 2.2 are thereby quadratic expressions in terms of initial data $u(x,t_0)$ and $u_t(x,t_0)$, while the constants of motion obtained from trivial symmetries are only linear expressions in terms of this data.

These properties imply that the constants of motion obtained from nontrivial point symmetries and nonlocal symmetries (4.5) are linearly independent of all constants of motion obtained from trivial point symmetries, since these constants of motions have a different scaling dimension under scalings of initial data. Consequently, to complete the proof of the theorem, we need only establish that each constant of motion obtained from the nonlocal symmetries (4.5) is linearly independent of all constants of motion obtained from nontrivial point symmetries.

Let $\tilde{\eta}\partial/\partial u$ correspond to the generator of a nonlocal symmetry (4.5) admitted by Eq. (4.1), and let $\eta_1\partial/\partial u, \ldots, \eta_k\partial/\partial u$ correspond to the generators of all distinct nontrivial point symmetries admitted by Eq. (4.1). Let $C[\tilde{\eta}]$, $C[\eta_1],\ldots,C[\eta_k]$ denote the resulting constants of motion obtained through Theorem 2.2.

Consider the one-parameter family of nonnegative initial data:

$$u(x,t_0;\lambda)=\varphi(x;\lambda)\geqslant 0, \quad u_t(x,t_0;\lambda)=\psi(x;\lambda)\geqslant 0, \tag{5.7}$$

with

$$v(x,t_0;\lambda)=\theta(x;\lambda)=\int_{-\infty}^{x} c(\tilde{x})^{-2}\psi(\tilde{x};\lambda)\,d\tilde{x}\geqslant 0, \tag{5.8}$$

such that the supports of

$$\begin{aligned} \varphi_1(x)&=\varphi(x;0), \quad \varphi_2(x)=\frac{\partial\varphi}{\partial\lambda}(x;0),\\ \psi_1(x)&=\psi(x;0), \quad \psi_2(x)=\frac{\partial\psi}{\partial\lambda}(x;0), \end{aligned} \tag{5.9}$$

are compact and mutually disjoint. Now define

$$\begin{aligned} \theta_1(x)&=\theta(x;0)=\int_{-\infty}^{x} c(\tilde{x})^{-2}\psi_1(\tilde{x})\,d\tilde{x},\\ \theta_2(x)&=\frac{\partial\theta}{\partial\lambda}(x;0)=\int_{-\infty}^{x} c(\tilde{x})^{-2}\psi_2(\tilde{x})\,d\tilde{x}. \end{aligned} \tag{5.10}$$

If $\tilde{c},c_1,\ldots,c_k$ are constants such that

$$\tilde{c}C[\tilde{\eta}]+c_1C[\eta_1]+\cdots+c_kC[\eta_k]=0 \tag{5.11}$$

for arbitrary initial data, then

$$\tilde{c}C[\tilde{\eta};\lambda]+c_1C[\eta_1;\lambda]+\cdots c_kC[\eta_k;\lambda]=0, \tag{5.12}$$

where $C[\eta_1;\lambda],\ldots,C[\eta_k;\lambda]$ are the constants of motion evaluated for the one-parameter family of initial data (5.9) and (5.10). Hence we must have

$$\left.\left(\tilde{c}\,\frac{\partial C[\tilde{\eta};\lambda]}{\partial\lambda}+c_1\,\frac{\partial C[\eta_1;\lambda]}{\partial\lambda}+\cdots+c_k\,\frac{\partial C[\eta_k;\lambda]}{\partial\lambda}\right)\right|_{\lambda=0}=0. \tag{5.13}$$

Using the earlier remarks about the quadratic properties of $C[\eta]$ for nontrivial point symmetries, and taking account of the disjoint supports of $\varphi_1(x)$, $\varphi_2(x)$, $\psi_1(x)$, $\psi_2(x)$, we have

$$\left.\frac{\partial C[\eta;\lambda]}{\partial\lambda}\right|_{\lambda=0}=0 \tag{5.14}$$

for $\eta=\eta_1,\ldots,\eta=\eta_k$. Hence, from Eq. (5.13), we get

$$\widetilde{c}\left.\frac{\partial C[\widetilde{\eta};\lambda]}{\partial\lambda}\right|_{\lambda=0}=0, \tag{5.15}$$

where

$$\left.\frac{\partial C[\widetilde{\eta};\lambda]}{\partial\lambda}\right|_{\lambda=0}=\int_{-\infty}^{\infty}c(x)^{-2}\{g(x,t_0)(\psi_2(x)\theta_1(x)+\psi_1(x)\theta_2(x)) - g_t(x,t_0)(\varphi_2(x)\theta_1(x)+\varphi_1(x)\theta_2(x))\}dx, \tag{5.16}$$

using Eq. (5.6).

Now we further restrict the initial data so that the supports of $\varphi_1(x),\varphi_2(x),\psi_1(x),\psi_2(x)$ are to the left of each other, respectively. Then Eq. (5.16) reduces to

$$\left.\frac{\partial C[\widetilde{\eta};\lambda]}{\partial\lambda}\right|_{\lambda=0}=\int_{-\infty}^{\infty}c(x)^{-2}g(x,t_0)\psi_2(x)\theta_1(x)dx\neq 0. \tag{5.17}$$

Hence Eq. (5.15) leads to $\widetilde{c}=0$ in Eq. (5.11), which implies that the constant of motion arising from the nonlocal symmetry $\widetilde{\eta}\partial/\partial u$ is linearly independent of the constants of motion arising from the nontrivial point symmetries $\eta_1\partial/\partial u,\ldots,\eta_k\partial/\partial u$. □

Theorem 5.3: *The two constants of motion (5.6) obtained for the scalar wave equation (4.1) from the nonlocal symmetries (4.5) in Case II are linearly independent modulo all constants of motion obtained from point symmetries.*

Proof: We proceed by the same argument used in proving Theorem 5.2. Let $\widetilde{\eta}_1\partial/\partial u$ and $\widetilde{\eta}_2\partial/\partial u$ correspond to the generators of the two nonlocal symmetries (4.5) of Eq. (4.1), and let $\eta_1\partial/\partial u,\ldots,\eta_k\partial/\partial u$ correspond to the generators of all distinct nontrivial point symmetries. Let $C[\widetilde{\eta}_1],C[\widetilde{\eta}_2],C[\eta_1],\ldots,C[\eta_k]$ denote the resulting constants of motion. Consider the same one-parameter initial data used in the previous proof, with the supports of $\varphi_1(x),\varphi_2(x),\psi_1(x),\psi_2(x)$ lying to the left of each other.

If $\widetilde{c}_1,\widetilde{c}_2,c_1,\ldots,c_k$ are constants such that

$$\widetilde{c}_1C[\widetilde{\eta}_1]+\widetilde{c}_2C[\widetilde{\eta}_2]+c_1C[\eta_1]+\cdots+c_kC[\eta_k]=0 \tag{5.18}$$

for arbitrary initial data, then we have

$$\left.\left(\widetilde{c}_1\frac{\partial C[\widetilde{\eta}_1;\lambda]}{\partial\lambda}+\widetilde{c}_2\frac{\partial C[\widetilde{\eta}_2;\lambda]}{\partial\lambda}\right)\right|_{\lambda=0}=0, \tag{5.19}$$

where $C[\widetilde{\eta}_1;\lambda]$ and $C[\widetilde{\eta}_2;\lambda]$ are the constants of motion evaluated for the one-parameter family of initial data. From Eq. (5.16) we find that Eq. (5.19) simplifies to

$$\int_{-\infty}^{\infty}c(x)^{-2}(\widetilde{c}_1g_1(x,t_0)+\widetilde{c}_2g_2(x,t_0))\psi_2(x)\theta_1(x)dx=0, \tag{5.20}$$

where, by use of Eq. (4.9), we have

$$\widetilde{c}_1g_1(x,t_0)+\widetilde{c}_2g_2(x,t_0)=-\mu(\widetilde{c}_1b_1(t_0)+\widetilde{c}_2b_2(t_0))c(x)/c'(x). \tag{5.21}$$

Then Eq. (5.20) reduces to

$$-(\widetilde{c}_1b_1(t_0)+\widetilde{c}_2b_2(t_0))\mu\int_{-\infty}^{\infty}c(x)^{-1}c'(x)^{-1}\psi_2(x)\theta_1(x)dx=0 \tag{5.22}$$

with $\int_{-\infty}^{\infty} c(x)^{-1} c'(x)^{-1} \psi_2(x) \theta_1(x) dx \neq 0$. It then follows that $\tilde{c}_1 b_1(t_0) + \tilde{c}_2 b_2(t_0) = 0$, and since we can choose the value of t_0 freely, we then must have $\tilde{c}_1 b_1(t) + \tilde{c}_2 b_2(t) = 0$ for all t. However, from Eq. (4.9) we note that $b = b_1(t)$ and $b = b_2(t)$ are linearly independent functions satisfying $b'' - \mu b = 0$. Thus $\tilde{c}_1 = 0 = \tilde{c}_2$. The linear independence of the constants of motion $C[\tilde{\eta}_1]$ and $C[\tilde{\eta}_2]$ modulo the constants of motion $C[\eta_1],\ldots,C[\eta_k]$ then follows from Eq. (5.18). □

B. Analytical example of new constants of motion

Theorems 5.2 and 5.3 establish new constants of motion for the scalar wave equation (4.1) for wave speeds given by Eq. (4.6) in Case I and Eq. (4.8) in Case II. The wave speeds in Case II satisfying the ODE (4.4) have the most physical interest since they are bounded (above and below) away from zero. These wave speeds $c(x)$ are implicitly given by the integral

$$\int_{c(x_0)}^{c(x)} \frac{\nu \, dc}{\sin(\nu \log c)} = x - x_0, \tag{5.23}$$

where ν and x_0 are arbitrary constant parameters. From Eq. (5.23), $c(x)$ can be shown to increase monotonically from the asymptotic value $c \to 1$ for $x \to -\infty$ to the asymptotic value $c \to e^{\pi/\nu}$ for $x \to +\infty$. In physical terms, this describes a medium of two layers, with wave speeds $c \approx 1$ and $c \approx e^{\pi/\nu}$, separated by a smoothly varying transition layer having width $\Delta x \approx \nu(e^{\pi/\nu} - 1)$, controlled by the value of ν.[9]

The scalar wave equation (4.1) with wave speeds (5.23) has no constants of motion arising from nontrivial point symmetries other than time translation symmetries generated by $\mathbf{X} = u_t \partial/\partial u$. These symmetries give rise through Theorem 2.2 to an energy constant of motion

$$E = \int_{-\infty}^{\infty} c(x)^{-2} (\psi(x)^2 + c(x)^2 \varphi'(x)^2) dx, \tag{5.24}$$

where $\varphi(x)$ and $\psi(x)$ are initial data (5.3).

Two additional constants of motion arise from the nonlocal symmetries (4.5) admitted by the scalar wave equation (4.1) with these wave speeds. In terms of the potential v introduced through Eq. (4.2), the nonlocal symmetry generators $\mathbf{X} = \eta \partial/\partial u$ have the explicit form (4.5) with

$$\begin{aligned} f(x,t) &= \pm(1 + B(x)) e^{\pm t}, \quad g(x,t) = -A(x) e^{\pm t}, \\ \xi(x,t) &= \pm 2A(x) e^{\pm t}, \quad \tau(x,t) = -2B(x) e^{\pm t}, \end{aligned} \tag{5.25}$$

where

$$A(x) = \nu c(x) \csc(\nu \log c(x)), \quad B(x) = \nu \cot(\nu \log c(x)). \tag{5.26}$$

The corresponding constants of motion given by Eq. (5.6) are

$$\begin{aligned} C_{\pm} = \int_{-\infty}^{\infty} & c(x)^{-2} (-A(x)(\psi(x) \pm \varphi(x)) \theta(x) \pm (\varphi(x) - 2A(x)\varphi'(x)) \psi(x) \\ & - \tfrac{1}{2}(1 + B(x)) \varphi(x)^2 + 2B(x)(\psi(x)^2 + c(x)^2 \varphi'(x)^2)) dx, \end{aligned} \tag{5.27}$$

where $\varphi(x)$ and $\psi(x)$ are initial data (5.3), and $\theta(x)$ is determined nonlocally from $\psi(x)$ by Eq. (5.4). $C_{\pm}$ and E comprise a linearly independent set of constants of motion as shown by Theorems 5.2 and 5.3.

The new constants of motion $C_{\pm}$ may have utility in the mathematical analysis of wave propagation for two layered media described by wave speeds (5.23). In particular, $C_{\pm}$ may supplement the use of the energy constant of motion E in addressing certain problems, such as the time evolution analysis for dispersal of waves initially localized across the transition boundary between the layers, and the scattering theory analysis of traveling waves incident on the transition boundary.

VI. CONCLUDING REMARKS

(1) In Sec. II we presented an explicit conservation law arising from any pair of symmetries, local or nonlocal, admitted by an arbitrary self-adjoint system of (linear or nonlinear) DEs (1.1). This conservation law expression does not require use of a variational principle for the system. Specializing to self-adjoint systems of linear DEs, we obtained a conservation law from any admitted local or nonlocal symmetry, by using a scaling symmetry as a second symmetry. A similar conservation law also can be obtained for any nonlinear system which admits a scaling symmetry (e.g., the Einstein equations in General Relativity theory). For variational symmetries (which are always local symmetries) admitted in the case of self-adjoint linear systems, we showed in Sec. III that the resulting local conservation laws are the same as those obtained from Noether's theorem. (The proof can be generalized straightforwardly to the conservation laws arising in the case of nonlinear systems with a scaling symmetry.)

The following theorem shows how our conservation law for a pair of symmetries is connected to Noether's theorem.

Theorem 6.1: *Suppose* $\mathbf{X}_1 = \eta_1^\mu(x,u,\underset{1}{u},\ldots,\underset{P_1}{u})\partial/\partial u^\mu$ *and* $\mathbf{X}_2 = \eta_2^\mu(x,u,\underset{1}{u},\ldots,\underset{P_2}{u})\partial/\partial u^\mu$ *are variational symmetries of a self-adjoint (linear or nonlinear) system (1.1). Then the resulting conservation law (2.7) is the same as the conservation law obtained through Noether's theorem for the commutator symmetry*

$$[\mathbf{X}_1,\mathbf{X}_2] = \eta^\mu(x,u,\underset{1}{u},\ldots,\underset{P}{u})\partial/\partial u^\mu \tag{6.1}$$

with $P \leqslant P_1 + P_2$.

Proof: The commutator $[\mathbf{X}_1,\mathbf{X}_2] = \eta^\rho \partial/\partial u^\rho$ is given by

$$\eta^\rho = \left(\frac{\partial \eta_2^\rho}{\partial u^\sigma}\, \eta_1^\sigma + \frac{\partial \eta_2^\rho}{\partial u_i^\sigma}\, D_i \eta_1^\sigma + \cdots + \frac{\partial \eta_2^\rho}{\partial u^\sigma_{i_1 \cdots i_{P_2}}}\, D_{i_1} \cdots D_{i_{P_2}} \eta_1^\sigma \right) - \left(\frac{\partial \eta_1^\rho}{\partial u^\sigma}\, \eta_2^\sigma + \frac{\partial \eta_1^\rho}{\partial u_i^\sigma}\, D_i \eta_2^\sigma + \cdots + \frac{\partial \eta_1^\rho}{\partial u^\sigma_{i_1 \cdots i_{P_1}}}\, D_{i_1} \cdots D_{i_{P_1}} \eta_2^\sigma \right). \tag{6.2}$$

From Eq. (2.5) we see that

$$\eta_1^\sigma \mathscr{F}_{\sigma\rho} \eta_2^\rho - \eta_2^\sigma \mathscr{F}_{\sigma\rho} \eta_1^\sigma = D_i \Phi^i[\eta_1, \eta_2]. \tag{6.3}$$

Then similarly to the derivation of Eq. (3.15), the identity (3.7) now leads to

$$\eta_1^\sigma \mathscr{F}_{\sigma\rho} \eta_2^\rho = -G_\rho \left(\frac{\partial \eta_2^\rho}{\partial u^\sigma}\, \eta_1^\sigma + \frac{\partial \eta_2^\rho}{\partial u_i^\sigma}\, D_i \eta_1^\sigma + \cdots + \frac{\partial \eta_2^\rho}{\partial u^\sigma_{i_1 \cdots i_{P_2}}}\, D_{i_1} \cdots D_{i_{P_2}} \eta_1^\sigma \right) + D_i H^i[\eta_1, \eta_2] \tag{6.4}$$

for a certain $H^i[\eta_1, \eta_2]$. Hence, using Eqs. (6.2) to (6.4), we have

$$G_\rho \eta^\rho = D_i \Omega^i, \tag{6.5}$$

where $\Omega^i = -\Phi^i[\eta_1, \eta_2] + H^i[\eta_1, \eta_2] - H^i[\eta_2, \eta_1]$. As the commutator of any two variational symmetries is itself a variational symmetry, we see from Eq. (3.4) that Eq. (6.5) is a conservation law obtainable from Noether's theorem. □

The set of all variational symmetries for a given self-adjoint system (1.1) forms a Lie algebra $\mathscr{A}$. If all Lie algebra generators can be realized as commutators, in which case we say $\mathscr{A}$ is ''perfect,'' then Theorem 6.1 yields all local conservation laws for the system. We remark that all semisimple Lie algebras, as well as the Poincaré algebra (which is not semisimple), are perfect.

(2) The questions of how to find and how to characterize useful potential systems in order to find nonlocal symmetries admitted by a system of DEs is considered in Ref. 6.

Potential systems of a given system (1.1) rely on the existence of at least one divergence free equation in the system. However, if an appropriate divergence free equation cannot be found, one may still be able to embed system (1.1) as a subsystem of a related potential system.[10] This may allow one to find nonlocal symmetries which are generalizations of potential symmetries.

(3) The conservation laws derived in Sec. II for a system of DEs (1.1) require that the system is self-adjoint. If a given system (1.1) is not self-adjoint, one may still be able to find a related potential system that is self-adjoint. Through the embedding into the potential system, any symmetry (local or nonlocal) admitted by the given system will induce a symmetry of the potential system. (An induced symmetry will be a nonlocal symmetry unless its generator has strictly local dependence on the dependent variables in the potential system.) As a result, conservation laws for the given system can then be obtained as conservation laws arising from the induced symmetries (local and nonlocal) of each self-adjoint potential system. If a system (1.1) is itself self-adjoint, conservation laws from *any* admitted symmetry will correspondingly arise through each self-adjoint potential system found for system (1.1) as well as through system (1.1) itself.

For the wave equation (4.1), the first order potential system (4.2) considered in Sec. IV is not self-adjoint. There are several different ways, nevertheless, to introduce potential variables for system (4.2) leading to potential systems that are self-adjoint. As we will discuss in a forthcoming article, the conservation law expressions arising through each such potential system are different from the conservation law expressions obtained through the wave equation (4.1) itself. In particular, the nonlocal symmetries admitted by Eq. (4.1) as point symmetries of system (4.2) induce nonlocal symmetries of these potential systems, leading to corresponding nonlocal conservation laws different than the ones derived in Sec. IV. These additional conservation laws for the wave equation (4.1) are not obtainable by Noether's theorem applied to any of the self-adjoint potential systems, since Noether's theorem only deals with local symmetries.

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