Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations

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(Received 12 August 1996; accepted for publication 18 February 1997)

Nonlocal symmetries are obtained for Maxwell's equations in three space–time dimensions through the use of two potential systems involving scalar and vector potentials for the electromagnetic field. Corresponding nonlocal conservation laws are derived from these symmetries. The conservation laws yield nine functionally independent constants of motion which cannot be expressed in terms of the constants of motion arising from local conservation laws for space–time symmetries. These nine constants of motion represent additional conserved quantities for the electromagnetic field in three space–time dimensions. © *1997 American Institute of Physics.* [S0022-2488(97)00706-8]

I. INTRODUCTION

Conservation laws are important in the study of evolutionary partial differential equations (PDEs) since they lead to constants of motion for the time evolution of field variables. The familiar conservation laws such as energy, momentum, and angular momentum all involve local expressions in terms of given field variables. Conservation laws given by nonlocal expressions can yield additional constants of motion not obtainable from local conservation laws. As an example,[1] we have recently derived nonlocal conservation laws arising through nonlocal symmetries for the scalar wave equation in two space–time dimensions with a spatially variable wave speed. For physically interesting wave speeds these conservation laws yield new constants of motion which cannot be linearly expressed in terms of the constants of motion yielded by local conservation laws arising through any local symmetries.

The nonlocal conservation laws for the two-dimensional wave equation were obtained through a general identity[1] which generates conservation laws from symmetries, local or nonlocal, admitted by any given self-adjoint system of PDEs. For local symmetries, the identity yields the same conservation laws as those generated through Noether's theorem, whereas for nonlocal symmetries, the identity yields additional conservation laws.

The nonlocal symmetries for the two-dimensional wave equation arise by a systematic method which uses potentials as a starting point.[2–4] The method can be extended to *any* PDEs in two or more dimensions to find nonlocal symmetries systematically in terms of local symmetries of associated potential systems. In three and higher dimensions the potentials have a natural gauge arbitrariness. To obtain nonlocal symmetries in this case, we show that the associated potential systems must be augmented by gauge constraints.

In this paper we focus on Maxwell's equations in three space–time dimensions. Through the use of two self-adjoint potential systems, we obtain gauge-dependent nonlocal symmetries of Maxwell's equations and derive corresponding nonlocal conservation laws which lead to six new constants of motion. One system is given by the wave equation for a scalar potential, and the other system involves an equivalent vector wave equation for scalar and vector potentials together with a Lorentz gauge. In terms of these potentials the conservation laws have an essential nonlocal

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0022-2488/97/38(7)/3508/25/$10.00

dependence on the electromagnetic field. The new constants of motion arising from these conservation laws are shown to be functionally independent of the seven constants of motion arising from the local conservation laws for space–time symmetries of Maxwell's equations.

In Sec. II we present the basic formulation for obtaining nonlocal symmetries for systems of PDEs by use of potential systems. Nonlocal symmetries are then found for Maxwell's equations. Corresponding nonlocal conservation laws are derived in Sec. III. The functional independence of the associated constants of motion is discussed in Sec. IV. In Sec. V we summarize the main results of the paper, with three tables presenting the symmetries, conservation laws, and constants of motion for Maxwell's equations. Some concluding remarks expanding on the results are made in Sec. VI.

II. BASIC FORMULATION

Consider a system of PDEs given by divergence expressions

$$G_\sigma(x,u,\underset{1}{u},\ldots,\underset{K}{u})=D_i H^i_\sigma(x,u,\underset{1}{u},\ldots,\underset{K-1}{u})=0, \quad \sigma=1,\ldots,M, \tag{2.1}$$

for N field variables $u=(u^1,\ldots,u^N)$ on a space of three independent variables $x=(x^0,x^1,x^2)$. Here $\underset{J}{u}$ denotes all Jth-order derivatives of u with respect to x, and D_i denotes total differentiation with respect to x^i, where $i=0,1,2$. We use the index notation $u^\tau_{i_1\cdots i_j}=D_{i_1}\cdots D_{i_j}u^\tau$ for differentiations of u^τ, where $\tau=1,\ldots,N$, $i_J=0,1,2$, and $J=1,2,\ldots$. We also use the convention that summation is assumed over any repeated index in all expressions. All lower case latin indices run over 0,1,2 unless otherwise stated.

It is important to note that any given linear system of PDEs can be transformed to the form (2.1).[4]

Definition 2.1: The Fréchet derivative associated with system (2.1) is the matrix linear operator

$$\mathscr{F}_{\sigma\rho}=\frac{\partial G_\sigma}{\partial u^\rho}+\frac{\partial G_\sigma}{\partial u^\rho_i}D_i+\cdots+\frac{\partial G_\sigma}{\partial u^\rho_{i_1\cdots i_K}}D_{i_1}\cdots D_{i_K}. \tag{2.2}$$

Definition 2.2: A symmetry admitted by system (2.1) is characterized by an infinitesimal generator

$$\mathbf{X}=\eta^\mu\partial/\partial u^\mu, \tag{2.3}$$

where η^μ satisfies

$$\mathscr{F}_{\sigma\rho}\eta^\rho=0 \tag{2.4}$$

for every solution $u(x)$ of system (2.1).

Definition 2.3: A local symmetry admitted by system (2.1) is a symmetry with an infinitesimal generator of the form

$$\mathbf{X}=\eta^\mu(x,u,\underset{1}{u},\ldots,\underset{P}{u})\partial/\partial u^\mu, \tag{2.5}$$

such that, for each value of x, η^μ depends on $u,\underset{1}{u},\ldots,\underset{P}{u}$ only through $u(x),\underset{1}{u}(x),\ldots,\underset{P}{u}(x)$ evaluated at x.

Definition 2.4: A point symmetry admitted by system (2.1) is a local symmetry with an infinitesimal generator of the form

$$\mathbf{X} = (\alpha^\mu(x,u) - \xi^i(x,u)u_i^\mu)\partial/\partial u^\mu, \tag{2.6}$$

where α^μ and ξ^i depend only on x and u.

Definition 2.5: A nonlocal symmetry admitted by system (2.1) is a symmetry with an infinitesimal generator $\mathbf{X} = \eta^\mu \partial/\partial u^\mu$ not of the form (2.5), i.e., η^μ has other than just a local dependence on $u(x)$ and derivatives of $u(x)$ to some finite order.

All local symmetries of system (2.1) can be determined by Lie's algorithm.[3,5] No corresponding algorithm exists to find *all* nonlocal symmetries of system (2.1).

We now present a general method to find special classes of nonlocal symmetries, called *potential symmetries*, of system (2.1).

Through Eq. (2.1) we introduce $3M$ auxiliary potential variables $v = (v_i^1, \ldots, v_i^M)$ and form a potential system given by PDEs

$$H_\sigma^i = \epsilon^{ijk} D_j v_k^\sigma, \quad \sigma = 1, \ldots, M, \tag{2.7}$$

where ϵ^{ijk} is the antisymmetric symbol with $\epsilon^{012} = 1$. The solution space of system (2.1) is embedded in the solution space of the potential system (2.7). In particular, if $(u(x), v(x))$ satisfies system (2.7), then $u(x)$ satisfies system (2.1); if $u(x)$ satisfies system (2.1), then there exists some nonunique $v(x)$ such that $(u(x), v(x))$ satisfies system (2.7). This nonuniqueness is represented by the invariance of system (2.7) under the transformations

$$v_k^\sigma \to v_k^\sigma + D_k \phi^\sigma \tag{2.8}$$

for arbitrary functions $\phi^\sigma(x)$, $\sigma = 1, \ldots, M$.

Definition 2.6: A potential symmetry admitted by system (2.1) through potential system (2.7) is a local symmetry of system (2.7) that does not project onto a local symmetry of system (2.1).

From this definition it immediately follows that a potential symmetry of system (2.1) is a nonlocal symmetry. In particular, suppose

$$\mathbf{X} = \eta^\mu(x,u,\underset{1}{u},\ldots,\underset{P}{u},v,\underset{1}{v},\ldots,\underset{Q}{v})\partial/\partial u^\mu + \zeta_i^\mu(x,u,\underset{1}{u},\ldots,\underset{P}{u},v,\underset{1}{v},\ldots,\underset{Q}{v})\partial/\partial v_i^\mu, \tag{2.9}$$

is a local symmetry of potential system (2.7). Through Eq. (2.7), η^μ and ζ_i^μ depend on v only through its symmetrized derivatives, since all antisymmetrized derivatives of v and their differential consequences can be eliminated in terms of u and its derivatives. Consequently, as v is determined nonlocally in terms of u from Eq. (2.7), the symmetry (2.9) defines a potential symmetry of system (2.1) if and only if at least one component of η^μ depends essentially on v or symmetrized derivatives of v.

For the sequel we now assume that the given system (2.1) is *determined* in the sense that it does not admit any symmetries that involve an arbitrary function of *all* the independent variables x. We see immediately that the potential system (2.7) is, in contrast, not determined since it admits *gauge symmetries*

$$\mathbf{X}_\phi = (D_i \phi^\sigma(x))\partial/\partial v_i^\sigma, \tag{2.10}$$

arising from the natural gauge freedom (2.8).

The following theorem now shows that unless gauge constraints are introduced, the potential system (2.7) cannot yield potential symmetries of the given system (2.1).

Theorem 2.7: *Every local symmetry admitted by the nondetermined potential system (2.7) projects onto a local symmetry of the determined system (2.1).*

Proof: Suppose system (2.7) admits a symmetry (2.9). Then the system must also admit the commutator symmetry $[\mathbf{X}_\phi, \mathbf{X}]$ which projects to the symmetry

$$\mathbf{X}' = \left(\frac{\partial \eta^\mu}{\partial v_i^\sigma} D_i \phi^\sigma(x) + \frac{\partial \eta^\mu}{\partial v_{(ik)}^\sigma} D_k D_i \phi^\sigma(x) + \cdots + \frac{\partial \eta^\mu}{\partial v_{(ik_1 \cdots k_Q)}^\sigma} D_{k_Q} \cdots D_{k_1} D_i \phi^\sigma(x) \right) \partial/\partial u^\mu \tag{2.11}$$

admitted by system (2.1), where $v^\sigma_{ik_1\cdots k_J} = D_{k_1} \cdots D_{k_J} v_i^\sigma$ denotes differentiations of v_i^σ, with $\sigma = 1,\ldots,M$, and $J = 1,2,\ldots$, and round brackets denote symmetrization of the enclosed indices.

Since system (2.1) is assumed to be determined, the symmetry $\mathbf{X}'$ cannot depend on $\phi^\sigma(x)$. Hence

$$\frac{\partial \eta^\mu}{\partial v_i^\sigma} = \frac{\partial \eta^\mu}{\partial v_{(ik)}^\sigma} = \cdots \frac{\partial \eta^\mu}{\partial v_{(ik_1 \cdots k_Q)}^\sigma} = 0 \tag{2.12}$$

and thus η^μ has no dependence on v and its symmetrized derivatives. Consequently, the symmetry (2.9) projects onto to a local symmetry (2.5) admitted by system (2.1). □

From Theorem 2.7 it follows that in order to use potential system (2.7) as a means to obtain potential symmetries of system (2.1) we must augment system (2.7) with auxiliary constraint equations relating the potentials without destroying the embedding of the solution space of system (2.1) in the solution space of the augmented system. There is considerable freedom in choosing such *gauge constraints.*

Gauge constraints to consider include:

(1) algebraic constraints, such as the *temporal gauge*

$$v_0^\sigma = 0, \quad \sigma = 1,\ldots,M \tag{2.13}$$

or *axial gauges*

$$n^i v_i^\sigma = 0, \quad \sigma = 1,\ldots,M \tag{2.14}$$

where n^1 and n^2 are components of a fixed spatial vector and $n^0 = 0$, and

(2) differential constraints, such as the *divergence gauge*

$$D_1 v_1^\sigma + D_2 v_2^\sigma = 0, \quad \sigma = 1,\ldots,M \tag{2.15}$$

or the *Lorentz gauge*

$$-D_0 v_0^\sigma + D_1 v_1^\sigma + D_2 v_2^\sigma = 0, \quad \sigma = 1,\ldots,M \tag{2.16}$$

The gauges (2.13)–(2.16) preserve the embedding property of the solution space of system (2.1) in the solution space of the potential system (2.7) augmented by any one of these gauge constraints.

For any such augmented system there are equivalent subsystems involving a subset of the (u,v) variables whose solution spaces each yield the complete solution space of system (2.1). Examples of such equivalent subsystems naturally include the given system (2.1) and augmented systems arising from algebraic constraints, and also include any systems only involving potential variables arising in algebraic and differential consequences of a given augmented system. Definition 2.6 extends as follows to such equivalent subsystems: A potential symmetry admitted by system (2.1) through an equivalent subsystem is a local symmetry of the equivalent subsystem that does not project onto a local symmetry of system (2.1).

Most importantly, equivalent subsystems are useful since:

(1) The solution space of any equivalent subsystem yields the complete solution space of system (2.1) and thereby inherits all symmetries of system (2.1);

(2) Each equivalent subsystem provides a means of determining nonlocal symmetries of system (2.1) in terms of local symmetries, which can be found by Lie's algorithm.

A. Scalar wave equation

For later use, we first consider the scalar wave equation

$$g^{ij}D_iD_ju=0 \tag{2.17}$$

in three space–time dimensions, where $g^{ij}=g_{ij}=\mathrm{diag}(-1,1,1)$ is a diagonal Lorentz metric on the space of independent variables $x=(x^0,x^1,x^2)$. Here Eq. (2.17) is already of the form (2.1) with

$$H_1^i=g^{ij}D_ju. \tag{2.18}$$

The corresponding potential system (2.7) of three PDEs involving the potentials $v=(v_0,v_1,v_2)$ is given by

$$g^{ij}D_ju=\epsilon^{ijk}D_jv_k. \tag{2.19}$$

To obtain potential symmetries of the wave equation (2.17) we consider augmented systems arising from Eq. (2.19) through specific gauge constraints. No potential symmetries are yielded by point symmetries of the augmented system arising through algebraic gauges of the form

$$n^iv_i=0 \tag{2.20}$$

for any components $n^i(x)$.[6] In contrast, the augmented system arising through the Lorentz gauge

$$g^{ij}D_iv_j=0 \tag{2.21}$$

does yield potential symmetries.[6] In particular, the augmented system consisting of Eqs. (2.19) and (2.21) admits the following six point symmetries:

$$\mathbf{X}=(\alpha(x,u,v)-\xi^j(x)D_ju)\partial/\partial u+(\beta_i(x,u,v)-\xi^j(x)D_jv_i)\partial/\partial v_i. \tag{2.22}$$

Class I (three of conformal type):

$$\xi^j=\lambda^jg_{kl}x^kx^l-2\lambda^kg_{kl}x^jx^l, \tag{2.23a}$$

$$\alpha=-\tfrac{1}{3}(D_k\xi^k)u+\tfrac{1}{4}\epsilon^{klm}(D_l\xi_m)v_k, \tag{2.23b}$$

$$\beta_i=-\tfrac{1}{2}(D_k\xi^k)v_i+\tfrac{1}{2}g^{kl}(D_l\xi_i)v_k+\tfrac{1}{4}g_{ik}\epsilon^{klm}(D_l\xi_m)u, \tag{2.23c}$$

where $\{\lambda^i\}_{i=0,1,2}$ are arbitrary constants, and $\xi_k=g_{kl}\xi^l$.

Class II (three of duality type):

$$\zeta^j=0,$$

$$\alpha=\lambda_kg^{kl}v_l, \tag{2.24}$$

$$\beta_i=\lambda_iu+\lambda_lg_{ik}\epsilon^{klm}v_m,$$

where $\{\lambda_i\}_{i=0,1,2}$ are arbitrary constants.

These two classes of point symmetries yield six potential symmetries of the wave equation (2.17) since $\alpha(x,u,v)$ depends explicitly on the potentials v. In Class I we call the symmetries *conformal type* since they project to conformal transformations in the space of independent variables $x=(x^0,x^1,x^2)$. In Class II we call the symmetries *duality type* since they represent rotations on the space of (u,v_i) variables.

All other nontrivial point symmetries admitted by the augmented system (2.19) and (2.21) project onto point symmetries of the wave equation. For later reference we now list all the nontrivial point symmetries admitted by the wave equation:

(i) three translations

$$\mathbf{X}=(-\lambda^i D_i u)\partial/\partial u \tag{2.25}$$

for arbitrary constants $\{\lambda^i\}_{i=0,1,2}$.

(ii) one rotation and two boosts

$$\mathbf{X}=(-\lambda_j g_{kl}\epsilon^{ijk}x^l D_i u)\partial/\partial u \tag{2.26}$$

for arbitrary constants $\{\lambda_j\}_{i=0,1,2}$.

(iii) one dilation

$$\mathbf{X}=(-x^i D_i u)\partial/\partial u. \tag{2.27}$$

(iv) three conformal transformations

$$\mathbf{X}=(-\tfrac{1}{6}\, u D_k \xi^k - \xi^j D_j u)\partial/\partial u, \tag{2.28}$$

where ξ^k is given by Eq. (2.23a).

B. Maxwell's equations

We now consider the source-free Maxwell's equations in three space–time dimensions

$$D_0 E^1 = D_2 B, \tag{2.29a}$$

$$D_0 E^2 = -D_1 B, \tag{2.29b}$$

$$D_0 B = D_2 E^1 - D_1 E^2, \tag{2.29c}$$

$$D_1 E^1 + D_2 E^2 = 0, \tag{2.29d}$$

where E^1 and E^2 are the components of the electric vector field and B is the magnetic scalar field. These field equations represent the components of the tensorial equations

$$\begin{aligned} D_i F^{ij} &= 0, \\ D_{[k} F_{ij]} &= 0, \end{aligned} \tag{2.30}$$

for the antisymmetric electromagnetic field tensor $F^{ij}=-F^{ji}$ and $F_{ij}=g_{ik}g_{jl}F^{kl}$, with

$$E^1 = F^{01}, \quad E^2 = F^{02}, \quad B = F^{12}, \tag{2.31}$$

where $g_{ij}=g^{ij}$ is the Lorentz metric on the space of independent variables $x=(x^0,x^1,x^2)$, and square brackets denote antisymmetrization of the enclosed indices.

Here the field equations (2.30) are of the form (2.1) with

$$H_1^i = F^{i0}, H_2^i = F^{i1}, H_3^i = F^{i2},$$
$$H_4^i = -g^{0i}F^{12} + g^{1i}F^{02} - g^{2i}F^{01} = \tfrac{1}{2}\epsilon^{ijk}F_{jk}, \tag{2.32}$$

for $u^1 = E^1$, $u^2 = E^2$, and $u^3 = B$. The corresponding potential system (2.7) of 12 PDEs is given by

$$F^{ij} = \epsilon^{ikl}D_k w_l^j,$$
$$\epsilon^{ijk}F_{jk} = \epsilon^{ijk}D_j w_k, \tag{2.33}$$

involving the 12 potentials

$$v_l^1 = w_l^0, \quad v_l^2 = w_l^1, \quad v_l^3 = w_l^2, \quad v_l^4 = w_l. \tag{2.34}$$

In terms of the potentials (2.34) the potential system (2.33) admits the gauge symmetries

$$\mathbf{X}_\phi = (D_l \phi^j(x))\partial/\partial w_l^j + (D_l \phi(x))\partial/\partial w_l \tag{2.35}$$

for arbitrary functions $\{\phi^0(x), \phi^1(x), \phi^2(x), \phi(x)\}$. Since Maxwell's equations (2.30) are a determined system, Theorem 2.7 shows that in order to obtain potential symmetries of Maxwell's equations we must augment the potential system by choosing gauge constraints.

We now impose a Lorentz gauge on w_j and an algebraic gauge on w_l^j as follows:

$$g^{ij}D_i w_j = 0, \tag{2.36}$$

$$w_l^j - \tfrac{1}{3}w_k^k \delta_l^j = 0. \tag{2.37}$$

From Eq. (2.37) it follows that the nine potentials w_l^j can be expressed in terms of a single potential w through

$$w_l^j = \tfrac{1}{2}\delta_l^j w, \tag{2.38}$$

where δ_l^j is the Kronecker symbol. As a result, from the augmented system given by Eqs. (2.33), (2.36), and (2.38), we arrive at the following equivalent system of seven PDEs:

$$F^{ij} = -\tfrac{1}{2}\epsilon^{ijk}D_k w, \tag{2.39a}$$

$$\epsilon^{ijk}F_{jk} = \epsilon^{ijk}D_j w_k, \tag{2.39b}$$

$$g^{ij}D_i w_j = 0, \tag{2.39c}$$

in terms of the electric and magnetic fields $\{E^1, E^2, B\}$ and the potentials $\{w, w_0, w_1, w_2\}$. The residual gauge freedom in this system is given by

$$w \to w + \text{const}, \tag{2.40a}$$

$$w_k \to w_k + D_k\phi, \tag{2.40b}$$

where ϕ is an arbitrary solution of the wave equation $g^{ij}D_iD_j\phi = 0$.

With the imposed gauge constraints, one can show that the system (2.39) is determined. Most importantly, the solution space of this system yields the complete solution space of Maxwell's equations (2.30), as shown by the following more transparent way of arriving at system (2.39).

Since we are in three space–time dimensions, the antisymmetry of the electromagnetic field tensor allows it to be expressed as

$$F^{ij}=\epsilon^{ijk}\widetilde{F}_k \tag{2.41}$$

in terms of a dual field vector $\widetilde{F}_k$, with

$$\widetilde{F}_0=B,\quad \widetilde{F}_1=-E^2,\quad \widetilde{F}_2=E^1 \tag{2.42}$$

Then Maxwell's equations (2.30) respectively become

$$\epsilon^{ijk}D_j\widetilde{F}_k=0, \tag{2.43a}$$

$$g^{jk}D_j\widetilde{F}_k=0, \tag{2.43b}$$

for the dual electromagnetic field $\widetilde{F}_k$. Hence from the curl form of Eq. (2.43a) it directly follows that the dual electromagnetic field is the gradient of a scalar, which leads to the three PDEs (2.39a). Similarly, the divergence form of Eq. (2.43b) means that the dual electromagnetic field is the curl of a vector, which directly gives the three PDEs (2.39b).

Rather than continue to consider system (2.39) we now algebraically eliminate the electromagnetic field tensor from Eqs. (2.39a) and (2.39b) and obtain the equivalent subsystem of four PDEs

$$g^{il}D_lw=\epsilon^{ijk}D_jw_k, \tag{2.44a}$$

$$g^{ij}D_iw_j=0, \tag{2.44b}$$

in terms of the four potentials $\{w,w_0,w_1,w_2\}$ alone. From the differential consequences of Eq. (2.44a) we note that w satisfies the wave equation

$$g^{ij}D_iD_jw=0. \tag{2.45}$$

System (2.44) is identical to the augmented potential system given by Eqs. (2.19) and (2.21) for the wave equation (2.17), under the correspondence $u\rightarrow w$ and $v_i\rightarrow w_i$. Through this correspondence we now obtain point symmetries of system (2.44) corresponding to the six point symmetries (2.22)–(2.24) admitted by system (2.19) and (2.21). We show that these point symmetries yield six potential symmetries of Maxwell's equations (2.30).

The induced symmetries of Eq. (2.30) arising from point symmetries of Eq. (2.44) have the form

$$\mathbf{X}=\widetilde{\eta}_i(x,\widetilde{F}_0,\widetilde{F}_1,\widetilde{F}_2,w,w_0,w_1,w_2)\partial/\partial\widetilde{F}_i \tag{2.46}$$

in terms of the dual electromagnetic field $\widetilde{F}_k$, where all derivatives of w and antisymmetrized derivatives of w_k are expressed in terms of $\widetilde{F}_k$ through Eqs. (2.39a), (2.39b), and (2.41). From the point symmetries (2.22)–(2.24), we obtain two corresponding classes of induced symmetries (2.46):

Class I:

$$\widetilde{\eta}_i=-\tfrac{2}{3}(D_k\xi^k)\widetilde{F}_i-\tfrac{5}{4}g^{kl}(D_{[i}\xi_{k]})\widetilde{F}_l-\xi^kD_k\widetilde{F}_i-\tfrac{1}{8}\epsilon^{klm}(D_iD_l\xi_m)w_k$$
$$+\tfrac{1}{6}(D_iD_k\xi^k)w-\tfrac{1}{8}\epsilon^{jkl}(D_k\xi_l)D_{(i}w_{j)}, \tag{2.47}$$

where ξ^k is given by Eq. (2.23a), and $\xi_k=g_{kl}\xi^l$.

Class II:

$$\tilde{\eta}_i = -\tfrac{1}{2}\lambda_k (g_{ij}\epsilon^{jkl}\tilde{F}_l + g^{jk}D_{(i}w_{j)}), \tag{2.48}$$

where $\{\lambda_j\}_{j=0,1,2}$ are arbitrary constants.

In both of these classes the components of $\tilde{\eta}_i$ have an essential dependence on the potentials w and w_i. Since these potentials are determined nonlocally in terms of the electromagnetic field from Eqs. (2.39a), (2.39b), and (2.41), we see that each of the six induced symmetries (2.46)–(2.48) is a potential symmetry of Maxwell's equations (2.30).

One can show that no other point symmetries of system (2.44) yield potential symmetries of Maxwell's equations (2.30).

III. NONLOCAL CONSERVATION LAWS

In Ref. 1 we derived an identity which generates conservation laws from symmetries admitted by any given self-adjoint system of PDEs. Given a linear homogeneous self-adjoint system of PDEs for field variables $u=(u^1,\ldots,u^N)$,

$$G_\sigma(x,u,\underset{1}{u},\ldots,\underset{K}{u}) = G_{\sigma\rho}(x)u^\rho + G^i_{\sigma\rho}(x)u^\rho_i + \cdots + G^{i_1\cdots i_K}_{\sigma\rho}(x)u^\rho_{i_1\cdots i_K} = 0, \quad \sigma=1,\ldots,N, \tag{3.1}$$

then for any nontrivial local or nonlocal symmetry (2.3) admitted by system (3.1) we have a corresponding conservation law on solutions $u(x)$,

$$D_i\Phi^i[u,\eta]=0, \tag{3.2}$$

where

$$\begin{aligned}\Phi^i[u,\eta] = -\tfrac{1}{2}\{&u^\sigma G^i_{\rho\sigma}\eta^\rho + (u^\sigma_j - u^\sigma D_j)(G^{ji}_{\rho\sigma}\eta^\rho) + \cdots + (u^\sigma_{i_1\cdots i_{K-1}} + \cdots \\ &+ (-1)^{K-1}u^\sigma D_{i_1}\cdots D_{i_{K-1}})(G^{i_1\cdots i_{K-1}i}_{\rho\sigma}\eta^\rho)\} + \tfrac{1}{2}\{\eta^\sigma G^i_{\rho\sigma}u^\rho + (D_j\eta^\sigma - \eta^\sigma D_j)(G^{ji}_{\rho\sigma}u^\rho) \\ &+ \cdots + (D_{i_1}\cdots D_{i_{K-1}}\eta^\sigma + \cdots + (-1)^{K-1}\eta^\sigma D_{i_1}\cdots D_{i_{K-1}})(G^{i_1\cdots i_{K-1}i}_{\rho\sigma}u^\rho)\}.\end{aligned} \tag{3.3}$$

For each symmetry admitted by system (3.1) we can obtain additional conservation laws through any self-adjoint equivalent system related to system (3.1), with expression (3.3) applied to the corresponding induced symmetry of the equivalent system.

In particular, for symmetries of the wave equation (2.17), we can obtain conservation laws from the wave equation itself as well as from its equivalent potential system given by Eqs. (2.19) and (2.21), since both Eq. (2.17) and Eqs. (2.19) and (2.21) are self-adjoint systems of PDEs. This leads to two conservation laws for any admitted symmetry $\mathbf{X}=\eta\partial/\partial u$ of the wave equation (2.17). We obtain

$$\Phi^i[u,\eta] = g^{ij}(uD_j\eta - \eta D_j u) \tag{3.4}$$

directly through Eq. (2.17), and using the induced symmetry $\mathbf{X}=\eta\partial/\partial u + \eta_i\partial/\partial v_i$ of Eqs. (2.19) and (2.21) we obtain

$$\Phi^i[u,\eta] = g^{ij}(u\eta_j - v_j\eta) - \epsilon^{ijk}v_j\eta_k. \tag{3.5}$$

Now suppose we are given a linear homogeneous system (3.1) that is not self-adjoint. If there exists an equivalent system that is self-adjoint, then any of its admitted symmetries yield conservation laws for the given system (3.1). Thus, since every such equivalent system inherits all symmetries of the given system, we can obtain corresponding conservation laws for any symmetry of the given system (3.1).

Maxwell's equations (2.30) and its potential system (2.33) are not self-adjoint. However, the two equivalent systems (2.44) and (2.45) are self-adjoint. Hence we can use both systems (2.44) and (2.45) to obtain conservation laws for any symmetry, local or nonlocal, admitted by Maxwell's equations.

Most importantly, since the equivalent systems (2.45) and (2.44) for Maxwell's equations correspond respectively to the wave equation (2.17) and its augmented potential system consisting of Eqs. (2.19) and (2.21), all conservation laws (3.4) and (3.5) obtained for symmetries of the wave equation yield conservation laws for the induced symmetries of Maxwell's equations through this correspondence.

Definition 3.1: A conservation law of system (3.1) is a local conservation law if and only if, on solutions of system (3.1) it has the form $D_i\Phi^i(x,u,\underset{1}{u},\dots,\underset{L}{u})=0$ *such that for each value of* x, Φ^i *depends on u only through* $u(x)$, $\underset{1}{u}(x),\dots,\underset{L}{u}(x)$ *evaluated at* x. *Otherwise a conservation law of system (3.1) is a nonlocal conservation law.*

A. Nonlocal conservation laws for the wave equation

We now obtain conservation laws for the wave equation (2.17) from the two classes of nonlocal symmetries (2.22)–(2.24). Each nonlocal symmetry yields two conservation laws (3.4) and (3.5) derived from the wave equation and its augmented potential system. These conservation laws are nonlocal as shown by their explicit dependence on the potentials v_i.

1. Conservation laws derived from the wave equation

Class I: The nonlocal symmetries of conformal type (2.23) and the conformal point symmetries (2.28) both project to the same conformal transformations on the space of independent variables x. To obtain conservation laws (3.4), we use symmetries given by subtracting the point symmetries (2.28) from the nonlocal symmetries (2.23). This leads to

$$\Phi^i[u,\eta]=-g^{ij}u(\tfrac{1}{6}(D_jD_k\xi^k)u-\tfrac{1}{4}\epsilon^{klm}((D_jD_l\xi_m)v_k+(D_l\xi_m)D_{(j}v_{k)})-\tfrac{1}{4}g^{kl}(D_{[k}\xi_{j]})D_lu)$$
$$+g^{ij}D_ju(\xi^kD_ku-\tfrac{1}{4}\epsilon^{klm}(D_l\xi_m)v_k), \tag{3.6}$$

where $\xi_j=g_{jk}\xi^k$, ξ^k is given by Eq. (2.23a), and (u,v_i) is any solution of Eqs. (2.19) and (2.21).

Class II: Here we directly use the nonlocal symmetries of duality type (2.24) to obtain conservation laws (3.4). This yields

$$\Phi^i[u,\eta]=\lambda_lg^{kl}g^{ij}(uD_{(j}v_{k)}-v_kD_ju)-\tfrac{1}{2}\lambda_j\epsilon^{ijk}uD_ku, \tag{3.7}$$

where λ_j is an arbitrary constant and (u,v_i) is any solution of Eqs. (2.19) and (2.21).

2. Conservation laws derived from the augmented potential system

Here we use the nonlocal symmetries (2.23) and (2.24) directly to obtain conservation laws (3.5).

Class I:

$$\Phi^i[u,\eta]=-(g^{ij}u+\epsilon^{ijk}v_k)(\xi^lD_{(l}v_{j)}+\tfrac{1}{2}g_{jl}\epsilon^{lmn}\xi_mD_nu-\tfrac{1}{2}g^{lm}(D_{[m}\xi_{j]})v_l-\tfrac{1}{4}g_{jl}\epsilon^{lmn}(D_m\xi_n)u)$$
$$+g^{ij}v_j(\xi^kD_ku-\tfrac{1}{4}\epsilon^{klm}(D_l\xi_m)v_k), \tag{3.8}$$

where $\xi_j=g_{jk}\xi^k$, ξ^k is given by Eq. (2.23a), and (u,v_i) is any solution of Eqs. (2.19) and (2.21).

Class II:

$$\Phi^i[u,\eta]=g^{ij}\lambda_j(u^2+g^{kl}v_kv_l)+2\epsilon^{ijk}\lambda_jv_ku-2g^{ij}g^{kl}\lambda_kv_jv_l, \tag{3.9}$$

where λ_j is an arbitrary constant and (u,v_i) is any solution of Eqs. (2.19) and (2.21).

B. Nonlocal conservation laws for Maxwell's equations

For Maxwell's equations (2.30), we now derive corresponding conservation laws from the two classes of nonlocal symmetries (2.46)–(2.48). The conservation laws are obtained directly through the conservation laws (3.6)–(3.9) for the wave equation (2.17) by using the correspondence $(u,v_i)\to(w,w_i)$ and eliminating all derivatives of w as well as antisymmetrized derivatives of w_k in terms of the dual electromagnetic field $\widetilde{F}_k$ through use of expressions (2.39a) and (2.39b). This leads to conservation laws which are nonlocal as shown by their essential dependence on w and w_i.

1. Conservation laws derived from the corresponding the wave equation for w

Class I:

$$\Phi^i[\widetilde{F},\widetilde{\eta}]=-g^{ij}w\left(\tfrac{1}{2}g^{kl}(D_{[k}\xi_{j]})\widetilde{F}_l-\tfrac{1}{4}\epsilon^{klm}(D_jD_l\xi_m)w_k+\tfrac{1}{6}(D_jD_k\xi^k)w-\tfrac{1}{4}\epsilon^{klm}(D_l\xi_m)D_{(j}w_{k)}\right)$$
$$+g^{ij}\widetilde{F}_j\left(4\xi^k\widetilde{F}_k+\tfrac{1}{2}\epsilon^{klm}(D_l\xi_m)w_k\right), \tag{3.10}$$

where $\xi_j=g_{jk}\xi^k$, ξ^k is given by Eq. (2.23a) and $(\widetilde{F}_k,w,w_j)$ is any solution of system (2.39).

Class II:

$$\Phi^i[\widetilde{F},\widetilde{\eta}]=\lambda_l g^{kl}g^{ij}(2w_k\widetilde{F}_j+wD_{(j}w_{k)})+\lambda_j\epsilon^{ijk}w\widetilde{F}_k, \tag{3.11}$$

where λ_j is an arbitrary constant and $(\widetilde{F}_k,w,w_j)$ is any solution of system (2.39).

2. Conservation laws derived from the corresponding equivalent system for (w,w_i)

Class I:

$$\Phi^i[\widetilde{F},\widetilde{\eta}]=(\epsilon^{ijk}w_k+g^{ij}w)\left(g_{jl}\epsilon^{lmn}\xi_m\widetilde{F}_n+\tfrac{1}{2}g^{lm}(D_{[m}\xi_{j]})w_l\right.$$
$$\left.+\tfrac{1}{4}g_{jl}\epsilon^{lmn}(D_m\xi_n)w-\xi^lD_{(l}w_{j)}\right)-g^{ij}w_j\left(2\xi^k\widetilde{F}_k+\tfrac{1}{4}\epsilon^{klm}(D_l\xi_m)w_k\right), \tag{3.12}$$

where $\xi_j=g_{jk}\xi^k$, ξ^k is given by Eq. (2.23a), and $(\widetilde{F}_k,w,w_j)$ is any solution of system (2.39).

Class II:

$$\Phi^i[\widetilde{F},\widetilde{\eta}]=g^{ij}\lambda_j(w^2+g^{kl}w_kw_l)+2\epsilon^{ijk}\lambda_jw_kw-2g^{ij}g^{kl}\lambda_kw_jw_l, \tag{3.13}$$

where λ_j is an arbitrary constant and $(\widetilde{F}_k,w,w_j)$ is any solution of system (2.39).

IV. NEW CONSTANTS OF MOTION

For the sequel we use the notation $x^0=t$, $x^1=x$, and $x^2=y$ to denote time and space variables, respectively.

Given a conservation law (3.2) for a linear system (3.1), we let

$$C[\eta]=\int_{\mathbf{R}^2}\Phi^0[u,\eta]dxdy \tag{4.1}$$

evaluated for solutions $u=(u^1,\ldots,u^N)$ of the system. If $u^1(x,y,t),\ldots,u^N(x,y,t)$ have appropriate asymptotic properties in terms of polar variables $r=\sqrt{x^2+y^2}$ and $\theta=\arctan y/x$ as $r\to\infty$, then

$$\frac{dC[\eta]}{dt}=-\lim_{r\to\infty}\int_0^{2\pi}(\Phi^1[u,\eta]\cos\theta+\Phi^2[u,\eta]\sin\theta)d\theta=0, \tag{4.2}$$

from which it follows that $C[\eta]$ defines a constant of motion for system (3.1).

Definition 4.1: A simple conservation law on solutions u of a linear system (3.1) is a local conservation law $D_i\Phi^i(x,y,t,u,\underset{1}{u},\ldots,\underset{L}{u})=0$ *such that* Φ^i *depends linearly on u and its derivatives.*

If a given linear system (3.1) is self-adjoint, then all of its simple conservation laws arise from expression (3.3) applied to the trivial symmetries $\mathbf{X}=\eta^\mu\partial/\partial u^\mu$ where η^μ is any solution $(u^1,\ldots,u^N)=(\eta^1,\ldots,\eta^N)$ of the system. For a linear system (3.1) that is not self-adjoint, we can obtain its simple conservation laws by finding all factors for the system as well as for differential consequences of the system, where the factors satisfy the adjoint of the system or differential consequences of the system.[4,7]

Remark 4.2: Since every linear system admits the scaling symmetry $\mathbf{X}=u^\mu\partial/\partial u^\mu$, *then without loss of generality all nonsimple conservation laws* $D_i\Phi^i(x,y,t,u,\underset{1}{u},\ldots,\underset{L}{u})=0$ *for a linear system can be assumed to have* Φ^i *given by a homogeneous expression in* $u,\underset{1}{u},\ldots,\underset{L}{u}$ *with scaling degree of at least two.*[1]

In general, for a given linear system (3.1), one is interested in finding nonsimple conservation laws yielding constants of motion whose forms do not involve explicit solutions of the system. Such constants of motion, e.g., energy, momentum, and angular momentum, are useful since they give *a priori* constraints on all solutions.

Definition 4.3: A constant of motion of a linear system (3.1) is elementary if and only if it can be expressed in terms of a finite number of constants of motion arising from simple conservation laws for the system. Otherwise a constant of motion of a linear system (3.1) is nonelementary.

Let $C[\eta_1],\ldots,C[\eta_K]$ define K constants of motion (4.1) arising for a linear system (3.1). Then any function of $C[\eta_1],\ldots,C[\eta_K]$ also defines a constant of motion of the system.

Definition 4.4: Suppose $C[\eta_1],\ldots,C[\eta_K]$ *are nonelementary constants of motion. Then* $C[\eta_1],\ldots,C[\eta_K]$ *are functionally independent if and only if each one of the K constants of motion cannot be expressed in terms of the other* $K-1$ *constants of motion together with any finite number of elementary constants of motion*.

We now obtain the constants of motion arising from the 12 nonlocal conservation laws derived in Sec. III for potential symmetries of the wave equation and Maxwell's equations, and proceed to show that six of these constants of motion represent new nonelementary functionally independent constants of motion of the wave equation and Maxwell's equations.

A. Constants of motion for the wave equation

Consider smooth compact support initial data

$$u(x,y,t_0)=\varphi(x,y),\quad D_0u(x,y,t_0)=\psi(x,y), \tag{4.3}$$

for the wave equation (2.17). This data determines corresponding initial data $v_i(x,y,t_0)$ for the augmented potential system of the wave equation as follows.

The augmented potential system consisting of PDEs (2.19) and (2.21) has a residual gauge freedom given by

$$v_i\to v_i+D_i\phi \tag{4.4}$$

for an arbitrary $\phi(x,y,t)$ satisfying the wave equation $g^{ij}D_iD_j\phi=0$. This freedom allows one to set

$$v_0(x,y,t_0)=0,\quad D_0v_0(x,y,t_0)=0, \tag{4.5}$$

by fixing appropriate initial data for ϕ. Then the PDEs (2.19) and (2.21) evaluated at $t=t_0$ lead to

$$D_1v_1+D_2v_2=D_0v_0=0, \tag{4.6a}$$

$$D_2 v_1 - D_1 v_2 = D_0 u = \psi, \tag{4.6b}$$

$$D_0 v_1 = D_1 v_0 + D_2 u = D_2 \varphi, \tag{4.6c}$$

$$D_0 v_2 = D_2 v_0 - D_1 u = -D_1 \varphi. \tag{4.6d}$$

From Eq. (4.6a) we see that

$$v_1(x,y,t_0) = D_2 \rho(x,y), \quad v_2(x,y,t_0) = -D_1 \rho(x,y), \tag{4.7}$$

for some $\rho(x,y)$. Then Eq. (4.6b) leads to

$$\psi = \Delta \rho, \tag{4.8}$$

where $\Delta = (D_1)^2 + (D_2)^2$ is the Laplace operator. Hence, from Eqs. (4.7) and (4.8), we have the initial data

$$\begin{aligned} v_1(x,y,t_0) &= D_2 \Delta^{-1} \psi(x,y), \\ v_2(x,y,t_0) &= -D_1 \Delta^{-1} \psi(x,y), \end{aligned} \tag{4.9}$$

where Δ^{-1} is the inverse Laplace operator.

From the differential consequences of PDEs (2.19) and (2.21) it follows that both $u(x,y,t)$ and $v_i(x,y,t)$ satisfy the wave equation. One can then show that the initial data (4.5) and (4.9) for v_i along with the initial data (4.3) for u can be evolved by the wave equation to obtain a solution $(u(x,y,t), v_i(x,y,t))$ of PDEs (2.19) and (2.21) given by

$$\begin{aligned} v_0(x,y,t) &= 0, \\ v_1(x,y,t) &= D_2 \Delta^{-1} D_0 u(x,y,t), \\ v_2(x,y,t) &= -D_1 \Delta^{-1} D_0 u(x,y,t). \end{aligned} \tag{4.10}$$

Expressions (4.10) determine v_i in terms of an arbitrary solution u of the wave equation (2.17) with compact support initial data. Hence we have an explicit embedding of the solution space of the wave equation (2.17) into the solution space of the augmented potential system PDEs (2.19) and (2.21). It is useful to note that the time derivatives of v_i are expressed in terms of u from Eq. (4.10) by

$$\begin{aligned} D_0 v_0(x,y,t) &= 0, \\ D_0 v_1(x,y,t) &= D_2 u(x,y,t), \\ D_0 v_2(x,y,t) &= -D_1 u(x,y,t). \end{aligned} \tag{4.11}$$

We can now evaluate, on solutions u of the wave equation (2.17), the constants of motion arising from the nonlocal conservation laws (3.6)–(3.9) derived through the wave equation (2.17) and the augmented potential system given by PDEs (2.19) and (2.21). In order to simplify the resulting expressions (4.1) for the constants of motion it is convenient to isolate divergences $D_1 S^1 + D_2 S^2$ appearing in $\Phi^0[u,\eta]$, where the expressions S^1 and S^2 involve u, $D_0 u$, $\Delta^{-1} D_0 u$, and their spatial derivatives. The contribution of such divergences to the expressions (4.1) consists of flux integrals

$$\lim_{r \to \infty} \int_0^{2\pi} (S^1 \cos\theta + S^2 \sin\theta) d\theta, \tag{4.12}$$

which can be simplified using the compact spatial support of u and $D_0 u$, and the asymptotic expansion of $\Delta^{-1}D_0 u$ as $r\rightarrow\infty$.[8] The flux integral appearing in condition (4.2) can be simplified similarly.

1. Constants of motion derived through the wave equation

We use tildes to indicate constants of motion derived through the wave equation (2.17).

Class I: From Eq. (3.6) the three conservation laws corresponding to $\{\lambda^i\}_{i=0,1,2}$ lead to

$$\tilde{C}_1^I=\int_{\mathbf{R}^2}(D_0 u(xD_1+yD_2)\Delta^{-1}D_0 u)dxdy, \tag{4.13a}$$

$$\tilde{C}_2^I=-\int_{\mathbf{R}^2}(uD_1\Delta^{-1}D_0 u)dxdy+t\int_{\mathbf{R}^2}(D_0 uD_1\Delta^{-1}D_0 u)dxdy, \tag{4.13b}$$

$$\tilde{C}_3^I=-\int_{\mathbf{R}^2}(uD_2\Delta^{-1}D_0 u)dxdy+t\int_{\mathbf{R}^2}(D_0 uD_2\Delta^{-1}D_0 u)dxdy. \tag{4.13c}$$

In Eq. (4.13a), the expression for $\Phi^0[u,\eta]$ can be manipulated into the form of a complete divergence, yielding a flux integral. Simplifying the integral then leads to

$$\tilde{C}_1^I=\frac{1}{4\pi}\left(\int_{\mathbf{R}^2}D_0 u\ dxdy\right)^2, \tag{4.14}$$

which is a constant of motion functionally depending on the well-known elementary constant of motion $\int_{\mathbf{R}^2}D_0 u\ dxdy$.

Through similar manipulations, the second terms in Eqs. (4.13b) and (4.13c) can be simplified to flux integrals which are found to vanish when evaluated using the asymptotic expansion of $\Delta^{-1}D_0 u$. Hence

$$\tilde{C}_2^I=-\int_{\mathbf{R}^2}(uD_1\Delta^{-1}D_0 u)dxdy, \tag{4.15}$$

$$\tilde{C}_3^I=-\int_{\mathbf{R}^2}(uD_2\Delta^{-1}D_0 u)dxdy. \tag{4.16}$$

The expressions for $\Phi^1[u,\eta]$ and $\Phi^2[u,\eta]$ corresponding to the simplified expressions for $\Phi^0[u,\ \eta]$ in Eqs. (4.15) and (4.16) lead to vanishing flux integrals in condition (4.2) when evaluated using the asymptotic expansion of $\Delta^{-1}D_0 u$. Consequently, $\tilde{C}_2^I$ and $\tilde{C}_3^I$ define constants of motion for the wave equation. Moreover, due to the compact spatial support of u and the smoothness of both u and Δ^{-1} in the simplified expressions for $\Phi^0[u,\eta]$, it immediately follows that both $\tilde{C}_2^I$ and $\tilde{C}_3^I$ are *finite*.

Class II: From Eq. (3.7) the conservation law corresponding to λ_0 has $\Phi^0[u,\eta]=0$, and hence yields an identically zero constant of motion. The conservation laws corresponding to λ_1 and λ_2 lead to expressions for $\Phi^0[u,\eta]$ identical to the expressions given by the second integrals in Eqs. (4.13b) and (4.13c), which each vanish. Hence we obtain two more identically zero constants of motion.

2. Constants of motion derived through the augmented potential system

We use hats to indicate constants of motion derived through PDEs (2.19) and (2.21).

Class I: From Eq. (3.8) the three conservation laws corresponding to $\{\lambda^i\}_{i=0,1,2}$ lead to

$$\hat{C}_1^I = 2\int_{\mathbf{R}^2}(u(yD_1 - xD_2)\Delta^{-1}D_0u)dxdy, \tag{4.17}$$

$$\hat{C}_2^I = \int_{\mathbf{R}^2} y(D_0u\Delta^{-1}D_0u - u^2)dxdy - 2t\int_{\mathbf{R}^2}(uD_2\Delta^{-1}D_0u)dxdy, \tag{4.18}$$

$$\hat{C}_3^I = -\int_{\mathbf{R}^2} x(D_0u\Delta^{-1}D_0u - u^2)dxdy + 2t\int_{\mathbf{R}^2}(uD_1\Delta^{-1}D_0u)dxdy. \tag{4.19}$$

To arrive at these expressions we manipulated $\Phi^0[u,\eta]$ to isolate divergence terms and used the asymptotic expansion of $\Delta^{-1}D_0u$ to find that the integrals contributed by these terms all vanish.

The expressions for $\Phi^1[u,\eta]$ and $\Phi^2[u,\eta]$ corresponding to $\Phi^0[u,\eta]$ in Eq. (4.17) lead to flux integrals satisfying the condition (4.2) similar to the ones arising from $\widetilde{C}_2^I$ and $\widetilde{C}_3^I$. For $\Phi^0[u,\eta]$ in Eqs. (4.18) and (4.19) the corresponding expressions for $\Phi^1[u,\eta]$ and $\Phi^2[u,\eta]$ have an explicit dependence on u and D_0u, leading directly to flux integrals satisfying the condition (4.2). Hence $\hat{C}_1^I$, $\hat{C}_2^I$, and $\hat{C}_3^I$ all define constants of motion for the wave equation. Moreover, from the compact spatial support of u and the smoothness of u and Δ^{-1} in these expressions for $\Phi^0[u,\eta]$, it immediately follows that $\hat{C}_1^I$, $\hat{C}_2^I$, and $\hat{C}_3^I$ are *finite*.

Class II: Here the conservation laws from Eq. (3.9) corresponding to λ_1 and λ_2 respectively yield

$$\hat{C}_1^{II} = 2\widetilde{C}_2^I, \quad \hat{C}_2^{II} = 2\widetilde{C}_3^I, \tag{4.20}$$

which are constants of motion obtained previously.

The remaining conservation law corresponding to λ_0 leads to

$$\int_{\mathbf{R}^2}(D_0u\Delta^{-1}D_0u - u^2)dxdy + \lim_{r\to\infty} \widetilde{C}_1^I \ln r \tag{4.21}$$

after some manipulations similar to the ones used to simplify $\widetilde{C}_1^I$. Since we see that the second term in Eq. (4.21) is an infinite constant, we now split it off in order to obtain a finite constant of motion. One can then show that

$$\hat{C}_3^{II} = \int_{\mathbf{R}^2}(D_0u\Delta^{-1}D_0u - u^2)dxdy \tag{4.22}$$

satisfies condition (4.2), since the expressions for $\Phi^1[u,\eta]$ and $\Phi^2[u,\eta]$ arising from Eq. (4.22) have compact spatial support through an explicit dependence on u. Hence $\hat{C}_3^{II}$ defines a constant of motion for the wave equation. Most importantly, $\hat{C}_3^{II}$ is *finite*, due to the compact spatial support of u and D_0u together with the smoothness of u and Δ^{-1} in Eq. (4.22).

B. Constants of motion for Maxwell's equations

Now consider solutions of Maxwell's equations (2.29) for $B(x,y,t)$, $E^1(x,y,t)$, and $E^2(x,y,t)$ with smooth compact spatial support at any fixed t. Corresponding solutions of the equivalent system (2.45) given by the wave equation for the potential $w(x,y,t)$ are determined as follows.

From the relations given by Eq. (2.39a) it directly follows that

$$2B = -D_0w, \quad 2E^1 = -D_2w, \quad 2E^2 = D_1w. \tag{4.23}$$

Through Maxwell's equation (2.29c), one can then solve Eq. (4.23) for w in terms of E^1 and E^2 at any fixed t, up to a constant which can be set to zero by the residual gauge freedom (2.40a) in system (2.39). This leads to

$$w(x,y,t)=2\int_\gamma (E^2(x',y',t)dx'-E^1(x',y',t)dy'), \tag{4.24}$$

where γ is any smooth curve in $\mathbf{R}^2$ from the point (x,y) to any point with $r\rightarrow\infty$. Maxwell's equation (2.29d) shows that w is independent of the choice of curve γ. As a result, one can show that w has spatial support contained in the union of the spatial supports of E^1 and E^2 at any fixed t.

Thus we have the following explicit correspondence of solutions.

Lemma 4.5: Every solution of Maxwell's equations (2.29) with compact spatial support yields a corresponding solution of the wave equation (2.45) through expression (4.24). Conversely, every solution of the wave equation (2.45) with compact spatial support yields a corresponding solution of Maxwell's equations (2.29) through expressions (4.23). This correspondence between solution spaces of Maxwell's equations and the wave equation is one-to-one.

Through Lemma 4.5, it follows that the constants of motion arising from the nonlocal conservation laws (3.10)–(3.13) on solutions (B,E^1,E^2) of Maxwell's equations can be obtained from the constants of motion (4.14)–(4.19) and (4.22) arising from the nonlocal conservation laws (3.6)–(3.9) for the wave equation with $u\rightarrow w$. This correspondence leads to one elementary constant of motion

$$\tilde{C}_1^I=\frac{1}{\pi}\left(\int_{\mathbf{R}^2}B\ dxdy\right)^2 \tag{4.25}$$

and the following six new constants of motion:

$$\begin{aligned}
\tilde{C}_2^I&=-4\int_{\mathbf{R}^2}(E^2\Delta^{-1}B)dxdy,\\
\tilde{C}_3^I&=4\int_{\mathbf{R}^2}(E^1\Delta^{-1}B)dxdy,\\
\hat{C}_1^I&=8\int_{\mathbf{R}^2}((yE^2+xE^1)\Delta^{-1}B)dxdy,\\
\hat{C}_2^I&=\int_{\mathbf{R}^2}y(4B\Delta^{-1}B-w^2)dxdy-8t\int_{\mathbf{R}^2}(E^1\Delta^{-1}B)dxdy,\\
\hat{C}_3^I&=-\int_{\mathbf{R}^2}x(4B\Delta^{-1}B-w^2)dxdy-8t\int_{\mathbf{R}^2}(E^2\Delta^{-1}B)dxdy,\\
\hat{C}_3^{II}&=\int_{\mathbf{R}^2}(4B\Delta^{-1}B-w^2)dxdy,
\end{aligned} \tag{4.26}$$

where w is given in terms of E^1 and E^2 by Eq. (4.24). In obtaining expressions (4.26) we have used relations (4.23) together with integrations by parts which use the compact spatial support of w.

Since (B, E^1, E^2) are solutions with compact spatial support and $\Delta^{-1}B$ has spatial support almost everywhere, it follows that the constants of motion (4.26) for Maxwell's equations are *finite* and generically nonzero.

C. Independence of the new constants of motion

We now establish that the six constants of motion (4.26) obtained from nonlocal conservation laws for Maxwell's equations are nonelementary, and that each one cannot be expressed in terms of the others together with any finite number of constants of motion arising from the local conservation laws for Maxwell's equations.

In view of the correspondence Lemma 4.5, we first establish corresponding results for the six constants of motion (4.15)–(4.19) and (4.22) for the wave equation.

Theorem 4.6: *For the wave equation (2.17), every constant of motion functionally depending on at least one of the six constants of motion from nonlocal conservation laws as well as on at most any finite number of constants of motion from local conservation laws is nonelementary.*

Proof: Let $\hat{c}_k$ for $k=1,\ldots,6$ denote respectively the constants of motion (4.15)–(4.19) and (4.22). Consider a function depending on at least one of the constants of motion $\{\hat{c}_k\}_{k=1,\ldots,6}$ as well as on a finite number $L+M$ of constants of motion $\{\overline{c_k}\}_{k=1,\ldots,L}$, $\{c_k\}_{k=1,\ldots,M}$ arising respectively from L nonsimple local conservation laws and M simple conservation laws of the wave equation (2.17). Suppose this function defines an elementary constant of motion, given by a function depending on a finite number J of constants of motion $\{c_k\}_{k=M+1,\ldots,M+J}$ arising from J simple conservation laws of the wave equation (2.17). Then we have

$$f(\hat{c}_1,\ldots,\hat{c}_6,\overline{c_1},\ldots,\overline{c_L},c_1,\ldots,c_M)=g(c_{M+1},\ldots,c_{M+J}) \tag{4.27}$$

for some functions f of $6+L+M$ variables and g of J variables, which we assume to be smooth, where f has an essential dependence on at least one of its first six variables.

Now consider an arbitrary one-parameter family of solutions $u(x,y,t;\lambda)$ of the wave equation (2.17) with smooth initial data (4.3) such that supports of

$$\begin{gathered}\varphi_0(x,y)=u(x,y,t_0;0)\geqslant 0, \quad \varphi_1(x,y)=\frac{\partial u}{\partial\lambda}(x,y,t_0;0)\geqslant 0,\\ \psi_0(x,y)=D_0u(x,y,t_0;0)\geqslant 0, \quad \psi_1(x,y)=\frac{\partial D_0 u}{\partial\lambda}(x,y,t_0;0)\geqslant 0,\end{gathered} \tag{4.28}$$

are compact and mutually disjoint. Evaluating Eq. (4.27) for this initial data then leads to

$$\sum_{k=1}^{6}\hat{f}_k\left.\frac{\partial\hat{c}_k}{\partial\lambda}\right|_{\lambda=0}=\sum_{k=1}^{L}\overline{f_k}\left.\frac{\partial\overline{c_k}}{\partial\lambda}\right|_{\lambda=0}+\sum_{k=1}^{M+J}f_k\left.\frac{\partial c_k}{\partial\lambda}\right|_{\lambda=0}, \tag{4.29}$$

where $\hat{f}_k=\partial f/\partial\hat{c}_k|_{\lambda=0}$ for $k=1,\ldots,6$, and $\overline{f_k}=-\partial f/\partial\overline{c_k}|_{\lambda=0}$ for $k=1,\ldots,L$, while $f_k=-\partial f/\partial c_k|_{\lambda=0}$ for $k=1,\ldots,M$, and $f_k=\partial g/\partial c_k|_{\lambda=0}$ for $k=M+1,\ldots,M+J$.

Since each c_k appearing in Eq. (4.29) arises from a simple conservation law, it can be expressed linearly in terms of the initial data for $u(x,y,t;\lambda)$, and hence we have

$$\left.\frac{\partial c_k}{\partial\lambda}\right|_{\lambda=0}=\int_{\mathbf{R}^2}(P_k(x,y,t_0)\varphi_0(x,y)+Q_k(x,y,t_0)\psi_0(x,y))dxdy \tag{4.30}$$

for some fixed functions $\{P_k(x,y,t),Q_k(x,y,t)\}_{k=1,\ldots,M+J}$. Furthermore, from Remark 4.2 it follows that, in terms of the initial data for $u(x,y,t;\lambda)$, each $\overline{c_k}$ appearing in Eq. (4.29) must be given by a homogeneous expression of scaling degree of at least two. Consequently, we have

$$\left.\frac{\partial \overline{c_k}}{\partial \lambda}\right|_{\lambda=0}=0 \tag{4.31}$$

since all the obtained terms are products of initial data and derivatives of initial data with respect to λ having disjoint supports when evaluated at $\lambda=0$ as given by Eq. (4.28).

Through Eq. (4.31) the relation (4.29) simplifies to

$$\sum_{k=1}^{6} \hat{f}_k \left.\frac{\partial \hat{c}_k}{\partial \lambda}\right|_{\lambda=0} = \sum_{k=1}^{M+J} f_k \left.\frac{\partial c_k}{\partial \lambda}\right|_{\lambda=0}. \tag{4.32}$$

Now, from Eqs. (4.15)–(4.19) and (4.22), we obtain

$$\begin{aligned}
\left.\frac{\partial \hat{c}_1}{\partial \lambda}\right|_{\lambda=0} &= \int_{\mathbf{R}^2} (-\varphi_1 D_1 \Delta^{-1}\psi_0 + \psi_1 \Delta^{-1} D_1 \varphi_0)\, dx dy,\\
\left.\frac{\partial \hat{c}_2}{\partial \lambda}\right|_{\lambda=0} &= \int_{\mathbf{R}^2} (-\varphi_1 D_2 \Delta^{-1}\psi_0 + \psi_1 \Delta^{-1} D_2 \varphi_0)\, dx dy,\\
\left.\frac{\partial \hat{c}_3}{\partial \lambda}\right|_{\lambda=0} &= 2\int_{\mathbf{R}^2} (\varphi_1 (yD_1 - xD_2)\Delta^{-1}\psi_0 - \psi_1 \Delta^{-1}(yD_1\varphi_0 - xD_2\varphi_0))\, dx dy,\\
\left.\frac{\partial \hat{c}_4}{\partial \lambda}\right|_{\lambda=0} &= \int_{\mathbf{R}^2} \psi_1 (y\Delta^{-1}\psi_0 + \Delta^{-1}(y\psi_0))\, dx dy + 2t_0 \left.\frac{\partial \hat{c}_2}{\partial \lambda}\right|_{\lambda=0},\\
\left.\frac{\partial \hat{c}_5}{\partial \lambda}\right|_{\lambda=0} &= -\int_{\mathbf{R}^2} \psi_1 (x\Delta^{-1}\psi_0 + \Delta^{-1}(x\psi_0))\, dx dy - 2t_0 \left.\frac{\partial \hat{c}_1}{\partial \lambda}\right|_{\lambda=0},\\
\left.\frac{\partial \hat{c}_6}{\partial \lambda}\right|_{\lambda=0} &= 2\int_{\mathbf{R}^2} (\psi_1 \Delta^{-1}\psi_0)\, dx dy,
\end{aligned} \tag{4.33}$$

where, in terms of the initial data (4.28), we have integrated by parts so that Δ^{-1} does not act on φ_1 and ψ_1, using the identity

$$\Omega\Delta^{-1}\Theta = \Theta\Delta^{-1}\Omega + \boldsymbol{\nabla}\cdot((\boldsymbol{\nabla}\Delta^{-1}\Omega)\Delta^{-1}\Theta - (\boldsymbol{\nabla}\Delta^{-1}\Theta)\Delta^{-1}\Omega) \tag{4.34}$$

together with the asymptotic expansion of Δ^{-1} for $r\to\infty$.[8]

Hence, from Eq. (4.33), we have

$$\begin{aligned}
\sum_{k=1}^{6} \hat{f}_k \left.\frac{\partial \hat{c}_k}{\partial \lambda}\right|_{\lambda=0} = \int_{\mathbf{R}^2} \Big(&\psi_1(\Delta^{-1}(bD_1\varphi_0 - aD_2\varphi_0 + c\psi_0) + d\Delta^{-1}\psi_0)\\
&+ \varphi_1(aD_2\Delta^{-1}\psi_0 + bD_1\Delta^{-1}\psi_0)\Big)\, dx dy,
\end{aligned} \tag{4.35}$$

where

$$\begin{aligned}
a(x) &= -\hat{f}_2 - 2\hat{f}_3 x - 2\hat{f}_4 t_0,\\
b(y) &= -\hat{f}_1 + 2\hat{f}_3 y + 2\hat{f}_5 t_0,\\
c(x,y) &= \hat{f}_4 y - \hat{f}_5 x,
\end{aligned} \tag{4.36}$$

$$d(x,y)=c(x,y)+2\,\hat{f}_6 .$$

From Eq. (4.30) we also have

$$\sum_{k=1}^{M+J} f_k \left.\frac{\partial c_k}{\partial\lambda}\right|_{\lambda=0} = \int_{\mathbf{R}^2}(\varphi_1 p+\psi_1 q)dxdy, \tag{4.37}$$

where

$$\begin{aligned} p(x,y)&=\sum_{k=1}^{M+J} f_k P_k(x,y,t_0),\\ q(x,y)&=\sum_{k=1}^{M+J} f_k Q_k(x,y,t_0), \end{aligned} \tag{4.38}$$

in terms of the fixed functions $\{P_k(x,y,t),Q_k(x,y,t)\}_{k=1,\ldots,M+J}$.

Since φ_1 and ψ_1 are independent data, it follows from Eq. (4.32) that the terms in Eqs. (4.35) and (4.37) involving these functions must be separately equal. This immediately leads to the separating equations

$$\int_{\mathbf{R}^2}(\varphi_1(aD_2\Delta^{-1}\psi_0+bD_1\Delta^{-1}\psi_0-p))dxdy=0,$$

$$\int_{\mathbf{R}^2}(\psi_1(\Delta^{-1}(bD_1\varphi_0-aD_2\varphi_0+c\psi_0)+d\Delta^{-1}\psi_0-q))dxdy=0,$$

with $\Delta^{-1}\psi_0(x,y)$ having support almost everywhere, and both $\varphi_1(x,y)$ and $\psi_1(x,y)$ having arbitrary compact support. Since we can vary each of φ_1 and ψ_1 arbitrarily as non-negative compactly supported functions, it follows that

$$aD_2\Delta^{-1}\psi_0+bD_1\Delta^{-1}\psi_0=p, \tag{4.39a}$$

$$\Delta^{-1}(bD_1\varphi_0-aD_2\varphi_0+c\psi_0)+d\Delta^{-1}\psi_0=q. \tag{4.39b}$$

The expressions a, b, c, d, p, q appearing in Eq. (4.39) have dependence on the initial data φ_0 and ψ_0 only through f_k and $\hat{f}_k$ which are functions of the finite number of constants of motion $\{\hat{c}_k\}_{k=1,\ldots,6}$, $\{\overline{c_k}\}_{k=1,\ldots,L}$, and $\{c_k\}_{k=1,\ldots,M+J}$ all evaluated for this initial data. Applying the Laplacian Δ to Eq. (4.39a) leads to the relation

$$a(x)D_2\psi_0(x,y)+b(y)D_1\psi_0(x,y)=\Delta p(x,y). \tag{4.40}$$

By fixing the values of the constants $\{\hat{c}_k\}_{k=1,\ldots,6}$, $\{\overline{c_k}\}_{k=1,\ldots,L}$, and $\{c_k\}_{k=1,\ldots,M+J}$ which comprise a finite number of integrals involving ψ_0, we can vary $\psi_0(x,y)$ as a smooth compactly supported function such that the values of $D_2\psi_0$ and $D_1\psi_0$ at any chosen point (x,y) are arbitrary while $a(x)$, $b(y)$, and $\Delta p(x,y)$ all remain fixed. Hence, from this arbitrariness, a and b in Eq. (4.40) must be identically zero. As a result it follows that

$$\hat{f}_1=\hat{f}_2=\hat{f}_3=\hat{f}_4=\hat{f}_5=0. \tag{4.41}$$

Then Eq. (4.39b) simplifies to

$$2\,\hat{f}_6\Delta^{-1}\psi_0(x,y)=q(x,y), \tag{4.42}$$

from which one can show that

$$\hat{f}_6 = 0 \tag{4.43}$$

by a similar argument.

Consequently, from Eqs. (4.41) and (4.43) we have that each $\hat{f}_k$ vanishes for the initial data φ_0 and ψ_0. Since this data is arbitrary, we thus have

$$\frac{\partial f}{\partial \hat{c}_k} = 0, \tag{4.44}$$

which shows that f must have no dependence on $\hat{c}_k$ for $k = 1,\ldots,6$. Hence the functional relation (4.27) cannot hold. □

From Theorem 4.6 it follows that the six constants of motion obtained from nonlocal conservation laws for the wave equation are nonelementary, and that each one cannot be expressed in terms of the others together with any finite number of constants of motion arising from local conservation laws for the wave equation. Hence, we have the following corollaries from Theorem 4.6.

Corollary 4.7: The six constants of motion (4.15)–(4.19) and (4.22) arising from nonlocal conservation laws for the wave equation are nonelementary and functionally independent.

Corollary 4.8: The six constants of motion (4.15)–(4.19) and (4.22) arising from nonlocal conservation laws for the wave equation are functionally independent of nonelementary constants of motion arising from any finite number of local conservation laws for the wave equation.

Corollaries 4.7 and 4.8 now lead to the following key theorem for the constants of motion (4.26) for Maxwell's equations.

Theorem 4.9: *For Maxwell's equations (2.29), the six constants of motion (4.26) obtained from nonlocal conservation laws are nonelementary and functionally independent. Furthermore, each of the six constants of motion (4.26) is functionally independent of nonelementary constants of motion arising from any finite number of local conservation laws of Maxwell's equations.*

Proof: From the correspondence Lemma 4.5 and the form of relations (4.23), it directly follows that any local conservation law for Maxwell's equations yields a local conservation law for the wave equation, and in particular any simple conservation law for Maxwell's equations yields a simple conservation law for the wave equation. Moreover, since through Lemma 4.5 the six constants of motion (4.26) arising from the nonlocal conservation laws (3.10)–(3.13) for Maxwell's equations correspond to the six constants of motion (4.15)–(4.19) and (4.22) arising from nonlocal conservation laws (3.6)–(3.9) for the wave equation, the proof of Theorem 4.9 reduces to the proof of Theorem 4.6. □

V. SUMMARY

We have obtained six potential symmetries (2.46)–(2.48) for Maxwell's equations (2.29) through the point symmetries (2.22)–(2.24) admitted by the equivalent system (2.44). All other point symmetries of this equivalent system yield only point symmetries of Maxwell's equations, in particular, translations, a rotation and boosts, and a dilation. One can show that Maxwell's equations admit no other nontrivial point symmetries in three space–time dimensions. Note that the admitted point symmetries of Maxwell's equations (2.29) do not include conformal transformations, unlike the case in four spacetime dimensions.

Since the wave equation (2.45) is also an equivalent system for Maxwell's equations (2.29), we can use its point symmetries to obtain symmetries of Maxwell's equations. From the ten nontrivial point symmetries admitted by the wave equation, one can easily show that the seven point symmetries given by translations (2.25), a rotation and boosts (2.26), and a dilation (2.27) yield the seven corresponding point symmetries admitted by Maxwell's equations, whereas the

TABLE I. Symmetries of Maxwell's equations (2.29).

Nonlocal symmetries	Remarks
$\mathbf{X}_1=(\xi_1^i B-4tB+\frac{5}{2}(xE^2-yE^1)+w-\frac{1}{2}(yD_{(1}w_{0)}$ $-xD_{(2}w_{0)}))\partial/\partial B+(\xi_1^i D_i E^1-4tE^1-\frac{5}{2}yB$ $+\frac{1}{2}(w_1+yD_{(2}w_{1)}-xD_2w_2))\partial/\partial E^1+(\xi_1^i D_i E^2-4tE^2$ $+\frac{5}{2}xB+\frac{1}{2}(w_2+xD_{(2}w_{1)}-yD_1w_1))\partial/\partial E^2$	$\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ are three conformal point symmetries of potential system (2.44). $\xi_1^0=-t^2-x^2-y^2,\ \xi_1^1=-2tx,\ \xi_1^2=-2yt$
$\mathbf{X}_2=(\xi_2^i D_i B+4xB-\frac{5}{2}tE^2+\frac{1}{2}(w_2+tD_{(2}w_{0)}$ $+yD_0w_0))\partial/\partial B+(\xi_2^i D_i E^1+4xE^1+\frac{5}{2}yE^2$ $+\frac{1}{2}(w_0+yD_{(2}w_{0)}+tD_2w_2))\partial/\partial E^1+(\xi_2^i D_i E^2+4xE^2$ $-\frac{5}{2}(tB+yE^1)+w-\frac{1}{2}(tD_{(1}w_{2)}+yD_{(1}w_{0)}))\partial/\partial E^2$	$\xi_2^0=2tx,\ \xi_2^1=t^2+x^2-y^2,\ \xi_2^2=2yx$
$\mathbf{X}_3=(\xi_3^i D_i B+4yB+\frac{5}{2}tE^1-\frac{1}{2}(w_1+tD_{(1}w_{0)}$ $+xD_0w_0))\partial/\partial B+(\xi_3^i D_i E^1+4yE^1+\frac{5}{2}(tB-xE^2)-w$ $-\frac{1}{2}(tD_{(2}w_{1)}+xD_{(2}w_{0)}))\partial/\partial E^1+(\xi_3^i D_i E^2+4yE^2$ $+\frac{5}{2}xE^1+\frac{1}{2}(w_0+xD_{(1}w_{0)}-tD_1w_1))\partial/\partial E^2$	$\xi_3^0=2yt,\ \xi_3^1=2yx,\ \xi_3^2=t^2-x^2+y^2$
$\mathbf{X}_4=(\frac{1}{2}D_0w_0)\partial/\partial B+(\frac{1}{2}E^2+\frac{1}{2}D_{(2}w_{0)})\partial/\partial E^1$ $+(-\frac{1}{2}E^1+\frac{1}{2}D_{(1}w_{0)})\partial/\partial E^2$	$\mathbf{X}_4$, $\mathbf{X}_5$, $\mathbf{X}_6$ are three duality point symmetries of potential system (2.44).
$\mathbf{X}_5=(-\frac{1}{2}E^1-\frac{1}{2}D_{(1}w_{0)})\partial/\partial B+(\frac{1}{2}B-\frac{1}{2}D_{(2}w_{1)})\partial/\partial E^1$ $+(\frac{1}{2}D_1w_1)\partial/\partial E^2$	
$\mathbf{X}_6=(\frac{1}{2}E^2-\frac{1}{2}D_{(2}w_{0)})\partial/\partial B+(-\frac{1}{2}D_2w_2)\partial/\partial E^1$ $+(\frac{1}{2}B+\frac{1}{2}D_{(2}w_{1)})\partial/\partial E^2$	
$\mathbf{X}_7=(\xi_1^i D_i B-3tB+2(xE^2-yE^1)+\frac{1}{2}w)\partial/\partial B+(\xi_1^i D_i E^1$ $-3tE^1-2yB)\partial/\partial E^1+(\xi_1^i D_i E^2-3tE^2+2xB)\partial/\partial E^2$	$\mathbf{X}_7$, $\mathbf{X}_8$, $\mathbf{X}_9$ are three conformal point symmetries of potential system (2.45). $\xi_1^0=-t^2-x^2-y^2,\ \xi_1^1=-2tx,\ \xi_1^2=-2yt$
$\mathbf{X}_8=(\xi_2^i D_i B+3xB-2tE^2)\partial/\partial B+(\xi_2^i D_i E^1+3xE^1$ $+2yE^2)\partial/\partial E^1+(\xi_2^i D_i E^2+3xE^2-2(tB+yE^1)$ $+\frac{1}{2}w)\partial/\partial E^2$	$\xi_2^0=2tx,\ \xi_2^1=t^2+x^2-y^2,\ \xi_2^2=2yx$
$\mathbf{X}_9=(\xi_3^i D_i B+3yB+2tE^1)\partial/\partial B+(\xi_3^i D_i E^1+3yE^1+2(tB$ $-xE^2)-\frac{1}{2}w)\partial/\partial E^1+(\xi_3^i D_i E^2+3yE^2+2xE^1)\partial/\partial E^2$	$\xi_3^0=2yt,\ \xi_3^1=2yx,\ \xi_3^2=t^2-x^2+y^2$

The potentials $\{w,w_0,w_1,w_2\}$ are determined nonlocally in terms of the fields $\{B,E^1,E^2\}$ from relations (2.44a) and (4.23) up to the residual gauge freedom (2.40).

Point symmetries	Remarks
$\mathbf{X}_{10}=(D_0B)\partial/\partial B+(D_0E^1)\partial/\partial E^1+(D_0E^2)\partial/\partial E^2$	$\mathbf{X}_{10}$, $\mathbf{X}_{11}$, $\mathbf{X}_{12}$ are three translations.
$\mathbf{X}_{11}=(D_1B)\partial/\partial B+(D_1E^1)\partial/\partial E^1+(D_1E^2)\partial/\partial E^2$	
$\mathbf{X}_{12}=(D_2B)\partial/\partial B+(D_2E^1)\partial/\partial E^1+(D_2E^2)\partial/\partial E^2$	
$\mathbf{X}_{13}=(yD_1B-xD_2B)\partial/\partial B+(yD_1E^1-xD_2E^1-E^2)\partial/\partial E^1$ $+(yD_1E^2-xD_2E^2+E^1)\partial/\partial E^2$	$\mathbf{X}_{13}$ is a rotation.
$\mathbf{X}_{14}=(-tD_2B-yD_0B-E^1)\partial/\partial B+(-tD_2E^1-yD_0E^1$ $-B)\partial/\partial E^1+(-tD_2E^2-yD_0E^2)\partial/\partial E^2$	$\mathbf{X}_{14}$, $\mathbf{X}_{15}$ are two boosts.
$\mathbf{X}_{15}=(tD_2B+xD_0B+E^2)\partial/\partial B+(tD_2E^1+xD_0E^1$ $+B)\partial/\partial E^1+(tD_2E^2+xD_0E^2)\partial/\partial E^2$	
$\mathbf{X}_{16}=(tD_0B+xD_1B+yD_2B)\partial/\partial B+(tD_0E^1+xD_1E^1$ $+yD_2E^1)\partial/\partial E^1+(tD_0E^2+xD_1E^2+yD_2E^2)\partial/\partial E^2$	$\mathbf{X}_{16}$ is a dilation.

TABLE II. Conserved densities for Maxwell's equations (2.29) from conservation laws derived through the potential system (2.44).

Symmetry	Conserved density	Remarks
$\mathbf{X}_1$	$8(yE^2(x,y,t)+xE^1(x,y,t))\Delta^{-1}B(x,y,t)$	6 new quantities [see (4.26)] from $\mathbf{X}_1,\ldots,\mathbf{X}_6$; γ is any smooth curve from (x,y) to $r\to\infty$ at fixed t.
$\mathbf{X}_2$	$-4yB(x,y,t)\Delta^{-1}B(x,y,t)$ $-4y\left(\int_\gamma(E^2(x',y',t)dx'-E^1(x',y',t)dy')\right)^2$ $+8tE^1(x,y,t)\Delta^{-1}B(x,y,t)$	
$\mathbf{X}_3$	$4xB(x,y,t)\Delta^{-1}B(x,y,t)$ $+4x\left(\int_\gamma(E^2(x',y',t)dx'-E^1(x',y',t)dy')\right)^2$ $-8tE^2(x,y,t)\Delta^{-1}B(x,y,t)$	
$\mathbf{X}_4$	$-4B(x,y,t)\Delta^{-1}B(x,y,t)$ $+\left(\int_\gamma(E^2(x',y',t)dx'-E^1(x',y',t)dy')\right)^2$	
$\mathbf{X}_5$	$-8E^2(x,y,t)\Delta^{-1}B(x,y,t)$	
$\mathbf{X}_6$	$8E^1(x,y,t)\Delta^{-1}B(x,y,t)$	
$\mathbf{X}_7$	trivial	
$\mathbf{X}_8$	trivial	
$\mathbf{X}_9$	trivial	
$\mathbf{X}_{10}$	trivial	
$\mathbf{X}_{11}$	trivial	
$\mathbf{X}_{12}$	$8E^2(x,y,t)\Delta^{-1}B(x,y,t)$	Duplicate of new quantity from $\mathbf{X}_5$.
$\mathbf{X}_{13}$	$-8E^1(x,y,t)\Delta^{-1}B(x,y,t)$	Duplicate of new quantity from $\mathbf{X}_6$.
$\mathbf{X}_{14}$	trivial	
$\mathbf{X}_{15}$	trivial	
$\mathbf{X}_{16}$	trivial	

three point symmetries given by conformal transformations (2.28) yield three potential symmetries which are *different* from the six potential symmetries (2.46)–(2.48) admitted by Maxwell's equations. Consequently, we obtain three additional potential symmetries for Maxwell's equations. The generators of the seven point symmetries and these nine potential symmetries for Maxwell's equations are exhibited in Table I.

Each symmetry of Maxwell's equations yields two conservation laws derived through the equivalent systems (2.44) and (2.45). In Sec. III B we have obtained 12 conservation laws arising for the six potential symmetries (2.46)–(2.48). We can likewise obtain 20 conservation laws arising for the seven point symmetries and three other potential symmetries discussed above. It is interesting to note that for each symmetry the conservation laws obtained from the two systems (2.44) and (2.45) are distinct. However, some symmetries yield trivial or duplicate conservation laws. The conserved densities arising from all 32 conservation laws are exhibited in Tables II and Tables III.

Altogether, these conserved densities yield 16 nonelementary functionally independent constants of motion for Maxwell's equations: seven constants of motion arising for the seven point symmetries, given by translations, a rotation and boosts, and a dilation, are obtained from local conserved densities through system (2.45); six constants of motion arising for the six potential symmetries (2.46)–(2.48) are obtained from nonlocal conserved densities through system (2.44); three constants of motion arising for the three additional potential symmetries above are obtained from nonlocal conserved densities through system (2.45). The functional independence of these 16

TABLE III. Conserved densities for Maxwell's equations (2.29) from conservation laws derived through the potential system (2.45).

Symmetry	Conserved density	Remarks
$\mathbf{X}_1-\mathbf{X}_7$	$4B(x,y,t)(xD_1+yD_2)\Delta^{-1}B(x,y,t)$	Not new [see (4.25)].
$\mathbf{X}_2-\mathbf{X}_8$	$-4E^2(x,y,t)\Delta^{-1}B(x,y,t)+$trivial	Duplicate of new quantity.
$\mathbf{X}_3-\mathbf{X}_9$	$4E^1(x,y,t)\Delta^{-1}B(x,y,t)+$trivial	Duplicate of new quantity.
$\mathbf{X}_4$	trivial	
$\mathbf{X}_5$	trivial	
$\mathbf{X}_6$	trivial	
$\mathbf{X}_7$	$-4(t^2+x^2+y^2)(B(x,y,t)^2+E^1(x,y,t)^2+E^2(x,y,t)^2)$ $+16tB(x,y,t)(xE^2(x,y,t)-yE^1(x,y,t))$ $+4tB(x,y,t)\left(\int_\gamma (E^2(x',y',t)dx'-E^1(x',y',t)dy')\right)$ $+\left(\int_\gamma (E^2(x',y',t)dx'-E^1(x',y',t)dy')\right)^2$	Three conformal quantities (see Sec. V) from $\mathbf{X}_7$, $\mathbf{X}_8$, $\mathbf{X}_9$; γ is any smooth curve, from (x,y) to $r\rightarrow\infty$ at fixed t.
$\mathbf{X}_8$	$8xt(B(x,y,t)^2+E^1(x,y,t)^2+E^2(x,y,t)^2)$ $-8(t^2+x^2-y^2)B(x,y,t)E^2(x,y,t)+16xyB(x,y,t)E^1(x,y,t)$ $-4xB(x,y,t)\left(\int_\gamma (E^2(x',y',t)dx'-E^1(x',y',t)dy')\right)$	
$\mathbf{X}_9$	$8yt(B(x,y,t)^2+E^1(x,y,t)^2+E^2(x,y,t)^2)$ $+8(t^2-x^2+y^2)B(x,y,t)E^1(x,y,t)-16xyB(x,y,t)E^2(x,y,t)$ $-4yB(x,y,t)\left(\int_\gamma (E^2(x',y',t)dx'-E^1(x',y',t)dy')\right)$	
$\mathbf{X}_{10}$	$4(B(x,y,t)^2+E^1(x,y,t)^2+E^2(x,y,t)^2)$	energy
$\mathbf{X}_{11}$	$-4B(x,y,t)E^2(x,y,t)$	spatial momentum
$\mathbf{X}_{12}$	$4B(x,y,t)E^1(x,y,t)$	spatial momentum
$\mathbf{X}_{13}$	$-8B(x,y,t)(xE^2(x,y,t)+yE^1(x,y,t))$	rotation angular momentum
$\mathbf{X}_{14}$	$-4y(B(x,y,t)^2+E^1(x,y,t)^2+E^2(x,y,t)^2)$ $-8tB(x,y,t)E^1(x,y,t)$	boost angular momentum
$\mathbf{X}_{15}$	$-4x(B(x,y,t)^2+E^1(x,y,t)^2+E^2(x,y,t)^2)$ $+8tB(x,y,t)E^2(x,y,t)$	boost angular momentum
$\mathbf{X}_{16}$	$4t(B(x,y,t)^2+E^1(x,y,t)^2+E^2(x,y,t)^2)$ $-8B(x,y,t)(E^2(x,y,t)-E^1(x,y,t))$	dilation quantity

constants of motion is established by a strengthing of Theorem 4.9, through the use of Lemma 4.5 and Corollaries 4.7 and 4.8 for the 16 corresponding constants of motion of the wave equation arising for its ten nontrivial point symmetries (2.25)–(2.28) and six potential symmetries (2.22)–(2.24).

For Maxwell's equations (2.29), the ten constants of motion arising for the seven point symmetries and the three additional potential symmetries represent energy, momentum, angular momentum, dilation, and conformal quantities for the electromagnetic field. The six constants of motion (4.26) arising for the potential symmetries (2.46)–(2.48) represent new additional conserved quantities for the electromagnetic field.

VI. CONCLUDING REMARKS

(1) Maxwell's equations (2.29) in three space–time dimensions arise from Maxwell's equations in four space–time dimensions when the electromagnetic field tensor **F** has no essential

dependence on one spatial dimension as follows: Fix spatial directions $\hat{x},\hat{y},\hat{z}$, and let $\vec{E}=E^1\hat{x}+E^2\hat{y}+E^3\hat{z}$ and $\vec{B}=B^1\hat{x}+B^2\hat{y}+B^3\hat{z}$ represent the electric and magnetic fields. If the $\hat{z}$ components of **F**, given by E^3,B^1,B^2, are constant, while the other components of **F**, given by E^1,E^2,B^3, have no dependence on z, then Maxwell's equations for $\vec{E}$ and $\vec{B}$ reduce to Eq. (2.29) for E^1, E^2, and $B^3=B$.

(2) Maxwell's equations in four space–time dimensions admit 15 point symmetries and corresponding local conservation laws.[9–12] Through the above dimensional reduction of Maxwell's equations, seven local conservation laws survive in three space–time dimensions. These local conservation laws correspond to the three translations, one rotation and two boosts each not involving the $\hat{z}$ direction, and one dilation, which are the point symmetries admitted by Maxwell's equations (2.29) in three space–time dimensions. Interestingly, local conservation laws corresponding to the four conformal transformations in four space–time dimensions are lost since conformal transformations are not admitted as point symmetries by Maxwell's equations in three space–time dimensions. Using a scalar potential for the electromagnetic field, we have obtained a group of nonlocal conformal transformations and three corresponding nonlocal conservation laws for Maxwell's equations (2.29). More importantly, through a system of scalar and vector potentials for the electromagnetic field, we have found a new group of nonlocal conformal transformations and three further nonlocal conservation laws for Maxwell's equations (2.29). From the same system of scalar and vector potentials, we also have found three additional nonlocal conservation laws corresponding to a group of nonlocal duality transformations arising as rotations on the potentials. Altogether these nonlocal conservation laws yield nine gauge-invariant conserved quantities for the electromagnetic field in three space–time dimensions.

(3) The results of this paper can be generalized to Maxwell's equations in three space–time dimensions with a curved Lorentz metric g_{ij}. Let g^{ij} denote the inverse metric, ϵ^{ijk} denote the totally-skew tensor normalized with respect to g_{ij}, and D_i denote the derivative operator determined by g_{ij}. Then, the nonlocal symmetries (2.46)–(2.48) obtained in flat space–time extend to curved space–time if and only if λ^i is a covariantly constant vector, $D_j\lambda^i=0$, and ξ^i is a conformal Killing vector of special type such that $R_{lkji}\xi^i=0$ where R_{lkji} is the curvature tensor and $g^{jk}D_k\xi^i+g^{ik}D_k\xi^j=\frac{2}{3}g^{ji}D_k\xi^k$. From these nonlocal symmetries, corresponding nonlocal conservation laws and associated constants of motion can be derived by the methods of Secs. III and IV.

(4) In a future paper we will apply our methods to Maxwell's equations in four space–time dimensions to seek nonlocal symmetries and corresponding nonlocal conservation laws, and new constants of motion.

(5) It is important to emphasize that the basic formulation presented in Sec. II can be applied to *any* system of PDEs with three or more independent variables.

ACKNOWLEDGMENTS

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under Grant No. OGP0007157.

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