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INVARIANT SOLUTIONS FOR ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract. An invariant solution of a differential equation is a solution of the differential equation which is also an invariant curve (surface) of a group admitted by the differential equation. For an ordinary differential equation (ODE) such solutions can be found without determining its general solution.

A theorem is proved which shows that for an ODE invariant solutions can be found by solving an algebraic equation derived from the given ODE and the infinitesimals of an admitted Lie group of transformations. For first-order ODEs it is shown that separatrices and envelope solutions are invariant solutions for *any* admitted Lie group. Several examples are given.

Key words. invariant solution, Lie group, separatrix, envelope, singular solution, phase plane, Clairaut's equation, limit cycle

AMS(MOS) subject classifications. primary 34A05, 34A08, 58F35; secondary 34C05, 58F12

1. Introduction. If an ordinary differential equation admits a one-parameter Lie group of transformations, then special solutions called invariant solutions can be constructed without knowledge of the general solution of the ordinary differential equation (ODE). Such solutions are invariant curves of the group. Usually at most a finite number of invariant curves of the group yield invariant solutions of the ODE.

A theorem is proved which shows that it is unnecessary to find explicitly all invariant curves of a group admitted by an ODE in order to determine which invariant curves yield invariant solutions. This theorem shows that invariant solutions of an ODE are essentially found by solving algebraic equations derived from the given ODE and infinitesimals of admitted one-parameter Lie groups of transformations.

Invariant solutions are especially interesting for first-order ODEs. It turns out that separatrices and/or envelope solutions, if they exist, are invariant solutions for *every* admitted nontrivial one-parameter Lie group of transformations.

Wulfman (1979) considered group aspects of separatrices which are limit cycles. Dresner (1983) discussed the construction of separatrices when a first-order ODE admits scalings. Page (1896) (see also Page (1897), Cohen (1911)) considered the construction of envelope solutions as invariant solutions. In this paper we present general results on the algebraic construction and geometrical significance of invariant solutions of ODEs. These results have appeared in Bluman and Kumei (1989) for ODEs written in solved form for the highest derivative. Here we make no such restriction.

We briefly review some background material on symmetries and differential equations relevant to the rest of the paper. For details see Bluman and Cole (1974), Olver (1986), Ovsiannikov (1982), or Bluman and Kumei (1989).

Consider an nth-order ODE

$$F(x, y, y_1, \cdots, y_n) = 0, \tag{1}$$

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where

$$y_k = y^{(k)} = \frac{d^k y}{dx^k}, \qquad k = 1, 2, \cdots, n,$$

and a one-parameter Lie group of transformations

$$\begin{aligned} x^* &= X(x, y; \varepsilon) = x + \varepsilon\xi(x, y) + O(\varepsilon^2), \\ y^* &= Y(x, y; \varepsilon) = y + \varepsilon\eta(x, y) + O(\varepsilon^2), \end{aligned} \tag{2}$$

with infinitesimal generator

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{3}$$

DEFINITION 1. A curve $\phi(x, y) = 0$ is an *invariant curve* of the group (2) if and only if $\phi(x^*, y^*) = 0$ when $\phi(x, y) = 0$.

We can show that $\phi(x, y) = 0$ is an invariant curve of (2) if and only if $\mathbf{X}\phi(x, y) = 0$ when $\phi(x, y) = 0$. This is equivalent to saying that $\phi(x, y) = 0$ is an invariant curve of (2) if and only if $\phi(x, y) = 0$ is a solution of the first-order ODE $y' = \eta(x, y)/\xi(x, y)$.

The kth extended infinitesimal generator of (3) is given by

$$\mathbf{X}^{(k)} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta^{(1)}(x, y, y_1)\frac{\partial}{\partial y_1} + \cdots + \eta^{(k)}(x, y, y_1, \cdots, y_k)\frac{\partial}{\partial y_k},$$

where, in terms of the total derivative operator

$$\begin{aligned} \frac{D}{Dx} &= \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + \cdots + y_n\frac{\partial}{\partial y_{n-1}}, \\ \eta^{(k)}(x, y, y_1, \cdots, y_k) &= \frac{D\eta^{(k-1)}}{Dx} - y_k\frac{D\xi}{Dx}, \qquad k = 1, 2, \cdots, n \end{aligned} \tag{4}$$

with

$$\eta^{(0)} = \eta(x, y).$$

Then we can show that the one-parameter Lie group of transformations (2) is admitted by ODE (1) (ODE (1) is invariant under (2)) if and only if the invariance condition

$$\mathbf{X}^{(n)}F(x, y, y_1, \cdots, y_n) = 0 \quad \text{when } F(x, y, y_1, \cdots, y_n) = 0 \tag{5}$$

holds.

DEFINITION 2. $\phi(x, y) = 0$ is an *invariant solution* of ODE (1) related to its invariance under the one-parameter Lie group of transformations (2) (infinitesimal generator (3)) if and only if ODE (1) admits group (2) and

(i) $\phi(x, y) = 0$ is an invariant curve of (2);

(ii) $\phi(x, y) = 0$ solves ODE (1).

(Note that if $\phi(x, y) = 0$ then $D^k\phi/Dx^k = 0$, $k = 1, 2, \cdots, n$.)

Two types of one-parameter Lie groups of transformations are admitted by any first-order ODE $F(x, y, y_1) = 0$:

(i) *Trivial one-parameter Lie groups* for *any* infinitesimals $(\xi(x, y), \eta(x, y))$ which satisfy the algebraic identity

$$F\left(x, y, \frac{\eta(x, y)}{\xi(x, y)}\right) \equiv 0 \tag{6}$$

in some domain $D \subset \mathbf{R}^2$. In this case each solution curve of $F(x, y, y_1) = 0$ is an invariant curve of the group (2) and, conversely, each invariant curve of (2) is a solution curve of $F(x, y, y_1) = 0$.

(ii) *Nontrivial one-parameter Lie groups* whose infinitesimals $(\xi(x, y), \eta(x, y))$ do not satisfy (6) in any domain in $\mathbf{R}^2$ but do satisfy the invariance condition

$$\mathbf{X}^{(1)} F(x, y, y_1) = 0 \quad \text{when } F(x, y, y_1) = 0.$$

In this case the family of solution curves of $F(x, y, y_1) = 0$ is invariant under the group (2) but an arbitrary solution curve of $F(x, y, y_1) = 0$ is not an invariant curve of (2). Here only special solution curves of the ODE $F(x, y, y_1) = 0$ are invariant curves of (2).

It is immediately evident from (6) that every first-order ODE admits a trivial infinite-parameter Lie group of transformations. It is easy to show that every first-order ODE also admits a nontrivial infinite-parameter Lie group of transformations.

2. Theorem on invariant solutions. An obvious way to find invariant solutions of ODE (1) related to its invariance under group (2) is to first obtain the general solution

$$g(x, y; C) = 0 \tag{7}$$

of the ODE $y' = \eta(x, y)/\xi(x, y)$ (the one-parameter family of invariant curves of (2)) and then substitute (7) into ODE (1) to find which values of $C = C^*$ lead to (7) solving ODE (1). Any such value of $C = C^*$ yields an invariant solution

$$\phi(x, y) = g(x, y; C^*) = 0$$

of (1) related to its invariance under (2). It is now shown that usually it is unnecessary to solve ODE $y' = \eta(x, y)/\xi(x, y)$ or any other ODE in order to find the invariant solutions of (1) related to the invariance of ODE (1) under group (2).

THEOREM 1. *Suppose* ODE (1) *admits the one-parameter Lie group of transformations* (2) *in domain* $D \subset \mathbf{R}^2$. *Without loss of generality assume that* $\xi(x, y) \neq 0$ *in D. Let*

$$\psi(x, y) = \frac{\eta(x, y)}{\xi(x, y)}, \qquad \mathbf{Y} = \frac{\partial}{\partial x} + \psi(x, y) \frac{\partial}{\partial y} = \frac{1}{\xi(x, y)} \mathbf{X},$$

and

$$y_k = \mathbf{Y}^{k-1} \psi, \qquad k = 1, 2, \cdots, n.$$

For ODE (1) *consider the corresponding algebraic function* $Q(x, y)$ *defined by*

$$Q(x, y) = F(x, y, \psi, \mathbf{Y}\psi, \cdots, \mathbf{Y}^{n-1}\psi) \tag{8}$$

in domain D. Three cases arise for the algebraic equation $Q(x, y) = 0$:

(I) $Q(x, y) = 0$ *defines no curves in D*;

(II) $Q(x, y) \equiv 0$ *in D*;

(III) $Q(x, y) \not\equiv 0$ *in D but* $Q(x, y) = 0$ *defines curves in D.*

In Case (I) ODE (1) *has no invariant solutions related to its invariance under* (2).

In Case (II) *each solution of the* ODE $y' = \eta(x, y)/\xi(x, y)$, *i.e., each invariant curve of* (2), *is an invariant solution of* ODE (1) *related to its invariance under* (2).

In Case (III) *an invariant solution of* ODE (1) *related to its invariance under* (2) *must satisfy* $Q(x, y) = 0$ *and, conversely, any curve satisfying* $Q(x, y) = 0$ *is an invariant solution of* ODE (1) *related to its invariance under* (2).

Proof. If $y_1 = y' = \eta/\xi = \psi$, then $y_k = y^{(k)} = \mathbf{Y}^{k-1}\psi$, $k = 1, 2, \cdots, n$. Hence an invariant solution of ODE (1) related to its invariance under (2) must satisfy the algebraic equation

$$Q(x, y) = 0.$$

From this it immediately follows that (I) if $Q(x, y) = 0$ defines no curves in D then ODE (1) has no invariant solutions related to its invariance under (2), and (II) if $Q(x, y) \equiv 0$ in D then *any* solution of $y' = \eta/\xi$ is an invariant solution of ODE (1). (Clearly $\mathbf{X}Q = 0$ when $Q \equiv 0$.)

In Case (III) consider any curve satisfying $Q(x, y) = 0$. By construction such a curve is a solution curve of ODE (1). We show that such a curve is an invariant solution of ODE (1) related to its invariance under (2) by showing that $\mathbf{X}Q = 0$ when $Q = 0$ as follows.

Since ODE (1) admits (2), it follows that the invariance condition (5) must hold. If $y_k = \mathbf{Y}^{k-1}\psi$, $k = 1, 2, \cdots, n$, then ODE (1) becomes $Q(x, y) = 0$ and the total derivative operator becomes $D/Dx \equiv \mathbf{Y}$ since for any function $G(x, y, y_1, \cdots, y_j)$, $j < n$, we have

$$\begin{aligned}\left.\frac{DG}{Dx}\right|_{y_k = \mathbf{Y}^{k-1}\psi, \quad k=1,2,\cdots,j+1} &= \left(\frac{\partial G}{\partial x} + y_1\frac{\partial G}{\partial y} + y_2\frac{\partial G}{\partial y_1} + \cdots + y_{j+1}\frac{\partial G}{\partial y_j}\right)\Bigg|_{y_k = \mathbf{Y}^{k-1}\psi, \quad k=1,2,\cdots,j+1} \\ &= \mathbf{Y}G(x, y, \psi, \cdots, \mathbf{Y}^{j-1}\psi),\end{aligned}$$

i.e.,

$$\frac{DG}{Dx} \equiv \mathbf{Y}G$$

when $y_k = \mathbf{Y}^{k-1}\psi$, $k = 1, 2, \cdots, j+1$. Hence if $y_k = \mathbf{Y}^{k-1}\psi$, $k = 1, 2, \cdots$, then

$$\begin{aligned}\eta^{(1)} &= \mathbf{Y}\eta - \psi\mathbf{Y}\xi \\ &= \mathbf{Y}(\xi\psi) - \psi\mathbf{Y}\xi \\ &= \xi\mathbf{Y}\psi.\end{aligned}$$

If $\eta^{(k)} = \xi\mathbf{Y}^k\psi$, then

$$\begin{aligned}\eta^{(k+1)} &= \mathbf{Y}\eta^{(k)} - (\mathbf{Y}^k\psi)(\mathbf{Y}\xi) \\ &= \mathbf{Y}(\xi\mathbf{Y}^k\psi) - (\mathbf{Y}^k\psi)(\mathbf{Y}\xi) \\ &= \xi\mathbf{Y}^{k+1}\psi.\end{aligned}$$

Hence it follows inductively that if $y_1 = \psi$, then $\eta^{(k)} = \xi\mathbf{Y}^k\psi$, $k = 1, 2, \cdots, n$.

Consequently, if ODE (1) admits (2), then from the invariance condition (5) it follows that for any curve $Q(x, y) = 0$ we have

$$\xi\left[\frac{\partial F}{\partial x} + \psi\frac{\partial F}{\partial y} + (\mathbf{Y}\psi)\frac{\partial F}{\partial y_1} + \cdots + (\mathbf{Y}^n\psi)\frac{\partial F}{\partial y_n}\right] = 0 \tag{9}$$

when $y_k = \mathbf{Y}^{k-1}\psi$, $k = 1, 2, \cdots, n$. But from (8) we get

$$\begin{aligned}\mathbf{X}Q &= \xi\frac{\partial Q}{\partial x} + \eta\frac{\partial Q}{\partial y} = \xi\mathbf{Y}Q \\ &= \xi\left[\frac{\partial F}{\partial x} + \psi\frac{\partial F}{\partial y} + (\mathbf{Y}\psi)\frac{\partial F}{\partial y_1} + \cdots + (\mathbf{Y}^n\psi)\frac{\partial F}{\partial y_n}\right]\end{aligned}$$

when $y_k = \mathbf{Y}^{k-1}\psi$, $k = 1, 2, \cdots, n$. Hence from (9) it follows that $\mathbf{X}Q = 0$ when $Q = 0$. □

As a first example consider the nth-order linear homogeneous ODE with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0. \tag{10}$$

We find all invariant solutions of (10) related to its invariance under translations in x and scalings in y generated by the infinitesimal generator $\mathbf{X} = \alpha(\partial/\partial x) + \beta y(\partial/\partial y)$ with arbitrary constants α, β. Let $\lambda = \beta/\alpha$. Then $\psi = \eta/\xi = \lambda y$, $\mathbf{Y} = \partial/\partial x + \lambda y \partial/\partial y$. Consequently,

$$y^{(k)} = \mathbf{Y}^{k-1}\psi = \lambda^k y, \qquad k = 1, 2, \cdots, n. \tag{11}$$

Substituting (11) into (10) we get

$$Q(x, y) = p(\lambda) y = 0,$$

where

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

Thus we obtain the familiar characteristic polynomial equation $p(\lambda) = 0$ which λ must satisfy if $y \neq 0$:

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0. \tag{12}$$

Any solution $\lambda = r$ of (12) leads to an invariant solution of (10). In terms of Theorem 1, Case (II) corresponds to λ being a root of $p(\lambda) = 0$; Case (III) corresponds to λ not being a root of $p(\lambda) = 0$ and in this case $y = 0$ is the (trivial) invariant solution. In Case (II) the invariant solution is any solution of $y' = ry$, i.e., $y = Ce^{rx}$, $p(r) = 0$.

Moreover, if $y = e^{rx}$ solves (10), then ODE (10) admits

$$\mathbf{X} = \alpha \frac{\partial}{\partial x} + (\beta y + \gamma e^{rx}) \frac{\partial}{\partial y}$$

for arbitrary constants α, β, γ. Let $\lambda = \beta/\alpha$, $\upsilon = \gamma/\alpha \neq 0$. Then $\psi = \eta/\xi = \lambda y + \upsilon e^{rx}$, $\mathbf{Y} = \partial/\partial x + (\lambda y + \upsilon e^{rx})\, \partial/\partial y$. Thus

$$y^{(k)} = \mathbf{Y}^{k-1}\psi = \upsilon(r^{k-1} + \lambda r^{k-2} + \cdots + \lambda^{k-2} r + \lambda^{k-1})\, e^{rx} + \lambda^k y, \qquad k = 1, 2, \cdots, n.$$

Correspondingly,

$$Q(x, y) = p(\lambda)\left[\frac{\upsilon}{\lambda - r} e^{rx} + y\right].$$

Four cases arise:

(i) If $p(\lambda) \neq 0$, $\mu = 0$, then $Q(x, y) = 0$ yields the trivial solution $y = 0$ of (10).

(ii) If $p(\lambda) \neq 0$, $\mu \neq 0$, then $Q(x, y) = 0$ yields the known solution $y = Ce^{rx}$ of (10) for arbitrary constant C.

(iii) If r is a simple root of $p(\lambda) = 0$ and $\lambda = r$, then $Q(x, y) \neq 0$ and hence no solution of (10) is obtained (Case I).

(iv) If $p(\lambda) = 0$ and r is not a simple root of $p(\lambda) = 0$, then $Q(x, y) \equiv 0$ and *any* solution of $y' = \lambda y + \upsilon e^{rx}$ yields a solution of (10). If $\lambda \neq r$, then this invariant solution is $y = C_1 e^{rx} + C_2 e^{\lambda x}$ for arbitrary constants C_1, C_2; if $\lambda = r$, then the corresponding invariant solution is $y = C_1 e^{rx} + C_2 x e^{rx}$ for arbitrary constants C_1, C_2.

As a second example consider the Blasius equation

$$y''' + \tfrac{1}{2} y y'' = 0, \tag{13}$$

which admits

$$\mathbf{X} = (\alpha x + \beta)\frac{\partial}{\partial x} - \alpha y \frac{\partial}{\partial y} \tag{14}$$

for arbitrary constants α, β. For $\alpha = 0$, the resulting invariant solution is $y = \text{const.} = C$ for any constant C. For $\alpha \neq 0$, let $\lambda = \beta/\alpha$. First consider the obvious approach. Then an invariant solution $\phi(x, y) = 0$ satisfies

$$y' = \frac{n}{\xi} = -\frac{y}{x+\lambda}$$

whose general solution is

$$y = \frac{C}{x+\lambda} \tag{15}$$

where C and λ are arbitrary constants. Substituting (15) into (13), we find that $C = 0$ or 6, which leads to invariant solutions

$$y = 0, \qquad y = \frac{6}{x+\lambda} \tag{16}$$

of (13) related to its invariance under (14).

Alternatively, we derive the invariant solutions (16) using Theorem 1. Here

$$\psi = -\frac{y}{x+\lambda}, \qquad \mathbf{Y} = \frac{\partial}{\partial x} - \frac{y}{x+\lambda}\frac{\partial}{\partial y},$$

$$\mathbf{Y}\Psi = \frac{2y}{(x+\lambda)^2}, \quad \mathbf{Y}^2\Psi = -\frac{6y}{(x+\lambda)^3}.$$

Then

$$Q(x, y) = \frac{y^2}{(x+\lambda)^2} - \frac{6y}{(x+\lambda)^3}, \qquad Q(x, y) = 0$$

leads to invariant solutions (16).

Note that if we are unable to obtain the general solution of ODE $y' = \eta/\xi$ then we must use the algorithm developed in Theorem 1 to determine invariant solutions of ODE (1) related to its invariance under (2).

3. Invariant solutions for first-order ODEs—separatrices and envelopes. In the case of a first-order ODE

$$F(x, y, y_1) = 0 \tag{17}$$

it only makes sense to consider invariant solutions for a nontrivial infinitesimal generator (3) admitted by (1) so that $Q(x, y) = F(x, y, \eta(x, y)/\xi(x, y)) \not\equiv 0$ in D. (If $Q(x, y) \equiv 0$ in D, then the corresponding one-parameter family of invariant solutions is a general solution of (17)!) For such a nontrivial infinitesimal generator, from Theorem 1 it follows that any curve satisfying

$$Q(x, y) = F\left(x, y, \frac{\eta(x, y)}{\xi(x, y)}\right) = 0 \tag{18}$$

is an invariant solution of ODE (17).

Consider the set of all solution curves in the xy-plane (phase plane) of an ODE (17) of the form

$$y_1 = f(x, y) \tag{19}$$

($F(x, y, y_1) = y_1 - f(x, y)$). This set may include *exceptional paths* (*separatrices*) which are solution curves that behave topologically "abnormally" in comparison with neighboring solution curves (cf. Lefschetz (1963)). Consequently, a separatrix must be an invariant solution of ODE (19) for *all* Lie groups of transformations admitted by ODE (19). We can see this as follows: Two solutions of ODE (19) which are topologically different cannot be continuously deformed into each other and hence cannot be mapped into each other by any Lie group of transformations admitted by ODE (19) as the group parameters vary. Since a group of transformations admitted by ODE (19) maps any solution of ODE (19) into another solution of ODE (19), it follows that a separatrix is an invariant solution of ODE (19) for all Lie groups of transformations admitted by ODE (19).

By the same argument as that for separatrices it will follow that all singular *envelope solutions* (if any exist) for a first-order ODE (17) must be invariant solutions for any admitted Lie group of transformations.

As a first example consider the first-order ODE

$$y' = y^2, \tag{20}$$

which obviously admits

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

From $\mathbf{X}_1$ it follows that a separatrix solution of (20) must satisfy

$$Q(x, y) = \frac{\eta(x, y)}{\xi(x, y)} - f(x, y) = -y^2 = 0,$$

which leads to the only possible candidate $y = 0$.

From $\mathbf{X}_2$ it follows that a separatrix solution of (20) must satisfy

$$Q(x, y) = -\frac{y}{x} - y^2 = 0,$$

which leads to the candidates $y = 0$, $y = -1/x$.

Since $y = -1/x$ is not an invariant solution of $\mathbf{X}_1$, it cannot be a separatrix of (20).

The solution curves of (20) are illustrated in Fig. 1 from which we can see that $y = 0$ is indeed a separatrix.

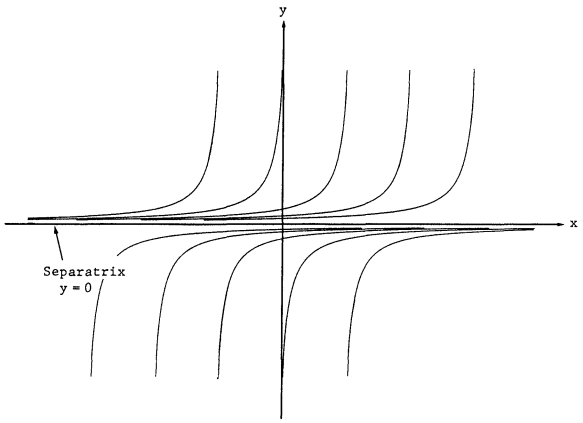

FIG. 1. *Solution curves of* $y' = y^2$.

As a second example consider the first-order ODE

$$y' = \frac{x\sqrt{x^2+y^2}+y(x^2+y^2-1)}{x(x^2+y^2-1)-y\sqrt{x^2+y^2}}, \tag{21}$$

which admits the rotation group

$$\mathbf{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Then

$$Q(x, y) = -\left(\frac{y}{x} + \frac{x\sqrt{x^2+y^2}+y(x^2+y^2-1)}{x(x^2+y^2-1)-y\sqrt{x^2+y^2}}\right)$$

and $Q(x, y) = 0$ leads to

$$\frac{(x^2+y^2)(x^2+y^2-1)}{y[x(x^2+y^2-1)-y\sqrt{x^2+y^2}]} = 0.$$

The only invariant solution and hence the only possible separatrix is the circle

$$x^2+y^2 = 1. \tag{22}$$

We can show that (22) is a limit cycle of ODE (21). Typical solution curves of (21) are illustrated in Fig. 2.

As a third example consider the Clairaut equation

$$F(x, y, y_1) = xy_1 + \frac{m}{y_1} - y = 0 \tag{23}$$

where m is a constant.

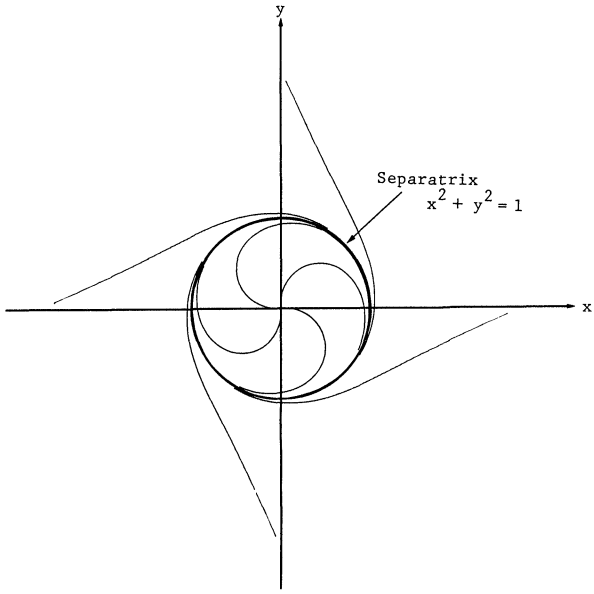

FIG. 2. *Solution curves of* ODE (21).

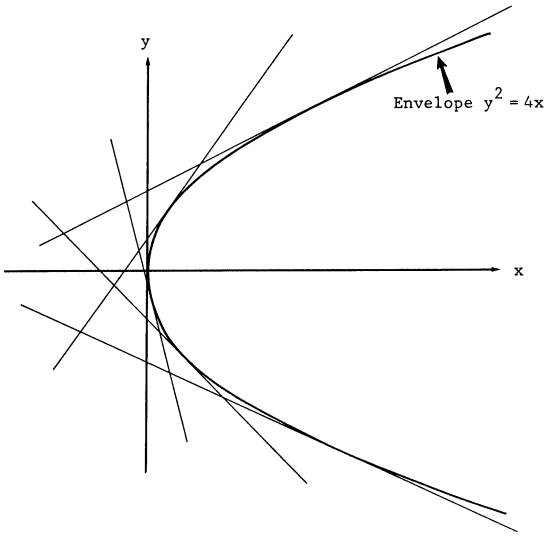

FIG. 3. *Solution curves of* ODE (23) *for* $m = 1$.

Clearly, (23) admits

$$\mathbf{X} = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{24}$$

An envelope solution of (23) must satisfy (18) for $\eta/\xi = y/2x$:

$$y = \frac{y}{2} + \frac{2mx}{y}$$

or

$$y^2 = 4mx. \tag{25}$$

Hence $y^2 = 4mx$ is the only possible envelope solution of ODE (23).

The invariance of (23) under (24) leads to its general solution

$$y = cx + \frac{m}{c} \tag{26}$$

where c is an arbitrary constant. Clearly, the parabola (25) is the envelope of the family of straight lines (26). Typical solution curves of (23) are illustrated in Fig. 3 for $m = 1$.

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REFERENCES

G. W. BLUMAN AND J. D. COLE (1974), *Similarity Methods for Differential Equations*, Applied Mathematical Sciences, No. 13, Springer-Verlag, New York.

G. W. BLUMAN AND S. KUMEI (1989), *Symmetries and Differential Equations*, Applied Mathematical Sciences, No. 81, Springer-Verlag, New York.

A. COHEN (1911), *An Introduction to the Lie Theory of One-Parameter Groups, with Applications to the Solution of Differential Equations*, D. C. Heath, New York.

L. DRESNER (1983), *Similarity Solutions of Nonlinear Partial Differential Equations*, Research Notes in Mathematics, No. 88, Pitman, Boston.

S. LEFSCHETZ (1963), *Differential Equations: Geometric Theory*, 2nd ed., Interscience, New York.

P. J. OLVER (1986), *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, No. 107, Springer-Verlag, New York.

L. V. OVSIANNIKOV (1982), *Group Analysis of Differential Equations*, Academic Press, New York.

J. M. PAGE (1896), *Note on singular solutions*, Amer. J. Math., 18, pp. 95–97.

——— (1897), *Ordinary Differential Equations with an Introduction to Lie's Theory of the Group of One Parameter*, Macmillan, London.

C. E. WULFMAN (1979), *Limit cycles as invariant functions of Lie groups*, J. Phys. A, 12, pp. L73–L75.