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Framework for potential systems and nonlocal symmetries: Algorithmic approach

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An algorithmic framework is presented to find an extended tree of nonlocally related systems for a given system of differential equations (DEs). Each system in an extended tree is equivalent in the sense that the solution set for any system in a tree can be found from the solution set for any other system in the tree. Useful conservation laws play an essential role in the construction of an extended tree. A useful conservation law yields potential variables and equivalent nonlocally related potential systems and subsystems for any given system. Nonlocal symmetries for a given system of DEs can arise from any system in its extended tree. We construct extended trees for the systems of planar gas dynamics and nonlinear telegraph equations, and in both cases obtain new nonlocal symmetries. More importantly, due to the equivalence of solution sets, any coordinate-independent method of analysis (qualitative, numerical, perturbation, etc.) can be applied to any system within the tree, and may yield simpler computations and/or results that cannot be obtained when the method is directly applied to the given system. © *2005 American Institute of Physics.* [DOI: 10.1063/1.2142834]

I. INTRODUCTION

The potential symmetry approach[1–5] is an algorithmic procedure for seeking nonlocal symmetries of systems of differential equations (DEs). To be directly applicable, this approach requires the existence of a conservation law of a given system. Each conservation law allows the introduction of one or more auxiliary potential variables which are nonlocally defined with respect to the original dependent variables.[6–8] The resulting (extended) potential system yields nonlocal symmetries of the given system of DEs if it admits local symmetry generators that do not project onto local symmetry generators of the given system.

A symmetry of a system of DEs is any transformation of its solution manifold into itself (i.e., a symmetry transforms any solution to another solution of the same system). Hence, in general, symmetry transformations are defined topologically and are not restricted to point or contact (more generally local) transformations acting on the given system's dependent and independent variables. However, to perform calculations, a nonlocal symmetry transformation should be a local transformation acting on the space of variables of an auxiliary system equivalent to the given system. As has been shown in many examples, local symmetries obtained directly by Lie's algorithm do not include all calculable symmetries of a given system. However, the application of Lie's algorithm to related auxiliary systems systematically yields a search for nonlocal symmetries of the given system.

Further extensions arise. Starting from a potential system of DEs, one may continue to obtain a grander potential system resulting from any conservation law of the potential system. Furthermore, starting from a potential system, one may obtain nonlocally related subsystems (in addition

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to the original given one) by "excluding" dependent variables from the potential system. For example, the $(1+1)$-dimensional system of planar gas dynamics equations $\mathbf{E}\{x,t,v,p,\rho\}=0$ in Eulerian coordinates with independent variables x=position, t=time, and dependent variables v=velocity, p=pressure, ρ=density, gives rise to potential systems $\mathbf{G}\{x,t,v,p,\rho,r\}=0$, $\mathbf{W}\{x,t,v,p,\rho,r,w\}=0$, and $\mathbf{Z}\{x,t,v,p,\rho,r,w,z\}=0$ with auxiliary potential variables r,w,z and to various subsystems with fewer dependent variables (Sec. III). This allows one to introduce the notion of a "tree of nonlocally related potential systems and subsystems" originating from a given system of DEs. It is important to note that a given system, its related potential systems and subsystems contain all solutions of each other, i.e., any solution of a related potential system or subsystem yields a solution of the given system and, mutatis mutandi, any solution of the given system yields a solution of any related potential system or subsystem. But the solution relationship is nonlocal since it is never one-to-one.

In general, the admitted point symmetries of a given system, its related potential systems and subsystems can be very different (e.g., Ref. 9). It can happen that a point symmetry of a given system is a nonlocal symmetry of a related potential system and, conversely, a point symmetry of a related potential system is a nonlocal symmetry of the given system. Moreover, it can happen that a point symmetry of the given system which is a nonlocal symmetry of a related potential system reappears as a point symmetry of a grander potential system.[9,10] Within a tree of potential systems, a "grand" system may exist that incorporates all point symmetries of the related systems with fewer potential variables as point symmetries in the grand system. Moreover, the grand system could admit point symmetries that are nonlocal symmetries of all related systems.[9,10]

The problem of finding trees of potential systems is particularly important when a given DE system contains arbitrary (constitutive) functions where one is interested in the question of symmetry classification with respect to specific forms of such functions. For different forms of the constitutive functions, the sets of conservation laws and consequent trees of related potential systems and subsystems can be different. Here, isolating useful systems and subsystems is of great importance.

If a given system of DEs has one of its equations written as a conservation law, then the conservation law equation is a natural way of leading to a related potential system and subsequent tree. In general, for any system of DEs the algorithmic direct construction method[11,12] yields its conservation laws. In particular, this method obtains factors multiplying each DE in the system so that the resulting linear combination of equations leads to a conservation law whose conserved densities are found from an integral formula. A resulting potential system (and subsequent extended tree) is found by replacing one of the DEs in such a linear combination by the conservation law. The factor multiplying the replaced DE must have the property that the solution set will not be modified when the conservation law replaces this DE. A useful conservation law for obtaining a potential system from any given system has at least one factor with this property.

The outline of this paper is as follows. In Sec. II, we present the general framework for the algorithmic construction of trees of potential systems and subsystems for a given system of DEs. In Sec. III, as a first example we consider the $(1+1)$-dimensional system of planar gas dynamics (PGD) equations[5,14] in detail within the potential system framework. Some nonlocal symmetries for the PGD system were found by Akhatov *et al.*[14] through a heuristic approach. Here, we use the systematic conservation law/potential symmetry framework to derive an extended tree of potential systems which includes the PGD systems in Euler and Lagrange coordinates. Our work clarifies and extends the work presented in Ref. 14. In particular, we obtain new nonlocal symmetries for PGD systems by a direct study of systems of the extended tree. In Sec. IV, as a second example we consider the nonlinear telegraph (NLT) equation.[9,13] Here, we show how to extend the results in Ref. 9 by introducing further potential variables. Using conservation laws for particular forms of the constitutive functions,[13] we construct extended trees for each such form. Further remarks are presented in Sec. V.

The analysis of systems of DEs through consideration of trees of potential systems and subsystems has evident practical value. First, it allows one to calculate systematically nonlocal symmetries using Lie's algorithm, which in turn are useful for obtaining new exact solutions from known ones, for constructing invariant and nonclassical solutions, for discovering and constructing linearizations, etc. Second, and perhaps more importantly, any general method of analysis (qualitative, numerical, perturbation, conservation laws, etc.) that is being considered for a given DE system may be tried again on nonlocally related potential systems or subsystems, since all such related systems contain all solutions of the given system. In particular, since the systems are related in a nonlocal manner, new results may be obtained for any method of analysis that is not coordinate dependent.

II. ALGORITHMIC CONSTRUCTION OF TREES OF POTENTIAL SYSTEMS AND SUBSYSTEMS TO OBTAIN NONLOCAL SYMMETRIES

Consider a PDE system of m equations $\mathbf{R}\{x,t,\mathbf{u}\}=\{R_1\{x,t,\mathbf{u}\},\ldots,R_m\{x,t,\mathbf{u}\}\}=0$, with two independent variables (x,t), and n dependent variables $\mathbf{u}=(u^1,\ldots,u^n)$.

Suppose that the first equation $R_1\{x,t,\mathbf{u}\}=0$ of the system is written as a conservation law,

$$D_tT\{x,t,\mathbf{u}\}+D_xX\{x,t,\mathbf{u}\}=0. \tag{2.1}$$

Definition 2.1: The PDE system $\mathbf{S}\{x,t,\mathbf{u},v\}=0$ given by

$$\begin{aligned} v_x &= T\{x,t,\mathbf{u}\},\\ v_t &= -X\{x,t,\mathbf{u}\},\\ R_2\{x,t,\mathbf{u}\} &= 0,\\ &\ldots,\\ R_m\{x,t,\mathbf{u}\} &= 0, \end{aligned} \tag{2.2}$$

is a *potential system* with a potential variable $v=v(x,t)$ for $\mathbf{R}\{x,t,\mathbf{u}\}=0$ related to the conservation law $R_1\{x,t,\mathbf{u}\}=0$.

The potential system $\mathbf{S}\{x,t,\mathbf{u},v\}=0$ given by (2.2) is equivalent to the given system $\mathbf{R}\{x,t,\mathbf{u}\}=0$. In particular, if $(\mathbf{u},v)=(\tilde{\mathbf{u}}(x,t),\tilde{v}(x,t))$ solves (2.2), then $\mathbf{u}=\tilde{\mathbf{u}}(x,t)$ solves $\mathbf{R}\{x,t,\mathbf{u}\}=0$. Conversely, for any solution $\mathbf{u}=\tilde{\mathbf{u}}(x,t)$ of $\mathbf{R}\{x,t,\mathbf{u}\}=0$, there exists a pair of functions $(\mathbf{u},v)=(\tilde{\mathbf{u}}(x,t),\tilde{v}(x,t))$ that satisfies (2.2), with $\tilde{v}(x,t)$ unique to within a constant.

Suppose the system $\mathbf{S}\{x,t,\mathbf{u},v\}=0$ admits a Lie group of point transformations (point symmetry),

$$\begin{aligned} x^* &= x+\epsilon\xi_{\mathbf{S}}(x,t,\mathbf{u},v)+O(\epsilon^2),\\ t^* &= t+\epsilon\tau_{\mathbf{S}}(x,t,\mathbf{u},v)+O(\epsilon^2),\\ u^{*i} &= u^i+\epsilon\eta^i_{\mathbf{S}}(x,t,\mathbf{u},v)+O(\epsilon^2),\\ v^* &= v+\epsilon\zeta_{\mathbf{S}}(x,t,\mathbf{u},v)+O(\epsilon^2). \end{aligned} \tag{2.3}$$

$$\mathrm{X}=\xi_{\mathbf{S}}(x,t,\mathbf{u},v)\frac{\partial}{\partial x}+\tau_{\mathbf{S}}(x,t,\mathbf{u},v)\frac{\partial}{\partial t}+\eta^i_{\mathbf{S}}(x,t,\mathbf{u},v)\frac{\partial}{\partial u^i}+\zeta_{\mathbf{S}}(x,t,\mathbf{u},v)\frac{\partial}{\partial v} \tag{2.4}$$

is the infinitesimal generator of the point symmetry (2.3). (Throughout this paper, we assume summation over a repeated index).

Definition 2.2: A point symmetry (2.3) is called a *potential symmetry* of the given system $\mathbf{R}\{x,t,\mathbf{u}\}=0$ related to the potential system $\mathbf{S}\{x,t,\mathbf{u},v\}=0$ if and only if $(\partial\xi_{\mathbf{S}}/\partial v)^2+(\partial\tau_{\mathbf{S}}/\partial v)^2+\Sigma_{i=1}^n(\partial\eta^i_{\mathbf{S}}/\partial v)^2>0$, i.e., the infinitesimals $\xi_{\mathbf{S}}$, $\tau_{\mathbf{S}}$, $\eta^i_{\mathbf{S}}$ essentially depend on v. Any potential symmetry is a *nonlocal* symmetry of the given system $\mathbf{R}\{x,t,\mathbf{u}\}=0$.[4]

Definition 2.3: An equivalent system $\underline{\mathbf{S}}\{x,t,u^{i_1},\ldots,u^{i_p},v\}=0$, $p\leqslant n-1$ that can be obtained by excluding one or more dependent variables u^k of the potential system $\mathbf{S}\{x,t,\mathbf{u},v\}=0$, is called a *subsystem* of the potential system $\mathbf{S}\{x,t,\mathbf{u},v\}=0$.

Definition 2.4: A *tree of potential systems and subsystems* for a given PDE system

$\mathbf{R}\{x,t,\mathbf{u}\}=0$, with some equations of $\mathbf{R}\{x,t,\mathbf{u}\}=0$ written directly as conservation laws, is a set of PDE systems composed of $\mathbf{R}\{x,t,\mathbf{u}\}=0$, all resulting potential systems, and all possible subsystems. We will refer to the number of dependent variables in a subsystem as the *level* of that subsystem.

Remark 2.1: If a potential system, as it is written, includes conservation laws with an essential dependence on potential variables, a higher potential system can be obtained, as will be illustrated by examples.

Definition 2.5: A subsystem $\underline{\mathbf{S}}\{x,t,u^{i_1},\ldots,u^{i_{p-1}}\}=0$, obtained from a system $\mathbf{S}\{x,t,u^{j_1},\ldots,u^{j_p}\}=0$ by excluding a dependent variable u^{α}, is *locally related* to $\mathbf{S}\{x,t,u^{j_1},\ldots,u^{j_p}\}=0$ if u^{α} can be directly expressed from the equations of $\mathbf{S}\{x,t,u^{j_1},\ldots,u^{j_p}\}=0$ in terms of x,t, the remaining dependent variables and their derivatives. Otherwise the subsystem $\underline{\mathbf{S}}\{x,t,u^{i_1},\ldots,u^{i_{p-1}}\}=0$ is *nonlocally related* to $\mathbf{S}\{x,t,u^{j_1},\ldots,u^{j_p}\}=0$.

For example, the system

$$\mathbf{S}\{x,t,u,v\}=0: \quad \begin{cases} v_x - u = 0, \\ v_t - (L(u))_x = 0, \end{cases} \tag{2.5}$$

has two subsystems,

$$\underline{\mathbf{S}}_1\{x,t,u\}=0:\ u_t-(L(u))_{xx}=0 \quad \text{and} \quad \underline{\mathbf{S}}_2\{x,t,v\}=0:\ v_t=(L(v_x))_x, \tag{2.6}$$

where $\underline{\mathbf{S}}_1\{x,t,u\}=0$ is nonlocally related to $\mathbf{S}\{x,t,u,v\}=0$, and $\underline{\mathbf{S}}_2\{x,t,v\}=0$ is locally related to $\mathbf{S}\{x,t,u,v\}=0$. (Throughout this paper, subindices denote corresponding partial derivatives.)

Definition 2.6: A *tree of nonlocally related potential systems and subsystems* is obtained from a tree of potential systems and subsystems by removing all locally related subsystems.

Remark 2.2: It is important to emphasize that a given system $\mathbf{R}\{x,t,\mathbf{u}\}=0$, its related potential systems and subsystems, contain *all solutions of each other*. This directly follows from the way potentials are introduced in the potential systems and the way dependent variables are excluded in the subsystems since the integrability conditions always hold. Therefore, one may successfully apply a method of analysis (qualitative, numerical, perturbation, symmetry, conservation laws, etc.) to a potential system or a nonlocally related subsystem, even if it fails to be of use when applied to the given system $\mathbf{R}\{x,t,\mathbf{u}\}=0$.

A. Example 1: A tree of potential systems and subsystems for the nonlinear diffusion equation

For the nonlinear diffusion equation, the given PDE system is the conservation law

$$\mathbf{R}\{x,t,u\}=0:\ u_t-(L(u))_{xx}=0, \tag{2.7}$$

where the constitutive function $L(u)$ is related to the diffusion function $K(u)=L'(u)$. Consequently, the related potential system is given by

$$\mathbf{S}\{x,t,u,v\}=0: \begin{cases} v_x - u = 0, \\ v_t - (L(u))_x = 0, \end{cases} \tag{2.8}$$

and the subsystem $\underline{\mathbf{S}}\{x,t,v\}=0$ is given by the equation

$$\underline{\mathbf{S}}\{x,t,v\}=0:\ v_t=(L(v_x))_x. \tag{2.9}$$

The second equation of (2.8) is a conservation law, and hence it gives rise to another potential variable w, and higher potential system $\mathbf{T}\{x,t,u,v,w\}=0$ (Remark 2.1) given by

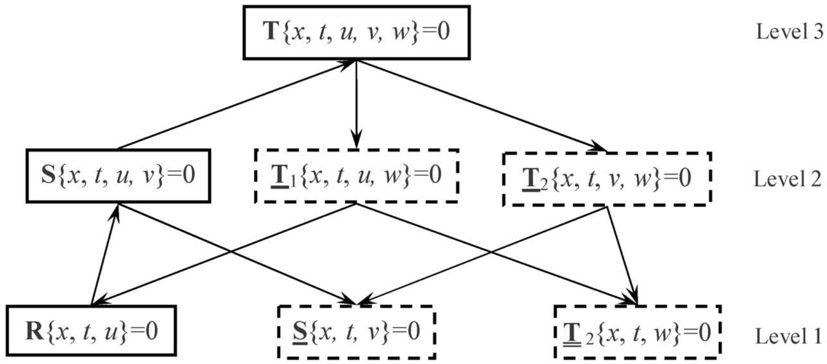

FIG. 1. The tree of potential systems and subsystems for the nonlinear diffusion equation (2.7) for an arbitrary constitutive function $L(u)$.

$$\mathbf{T}\{x,t,u,v,w\} = 0\colon \begin{cases} v_x - u = 0, \\ w_x - v = 0, \\ w_t - L(u) = 0, \end{cases} \tag{2.10}$$

and subsystems $\underline{\mathbf{T}}_1\{x,t,u,w\}=0$, $\underline{\mathbf{T}}_2\{x,t,v,w\}=0$, $\underline{\underline{\mathbf{T}}}_2\{x,t,w\}=0$ given by

$$\underline{\mathbf{T}}_1\{x,t,u,w\} = 0\colon \begin{cases} w_{xx} - u = 0, \\ w_t - L(u) = 0, \end{cases}$$

$$\underline{\mathbf{T}}_2\{x,t,v,w\} = 0\colon \begin{cases} w_x - v = 0, \\ w_t - L(v_x) = 0, \end{cases} \tag{2.11}$$

$$\underline{\underline{\mathbf{T}}}_2\{x,t,w\} = 0\colon\ w_t - L(w_{xx}) = 0.$$

Since the subsystems $\underline{\mathbf{S}}\{x,t,v\}=0$, $\underline{\mathbf{T}}_1\{x,t,u,w\}=0$, $\underline{\mathbf{T}}_2\{x,t,v,w\}=0$, and $\underline{\underline{\mathbf{T}}}_2\{x,t,w\}=0$ are locally related to the potential systems (2.8) and (2.10), they are not of any interest.

This tree of potential systems and subsystems is illustrated in Fig. 1 with arrows showing the origins of elements of the tree, with dashed lines used to denote locally related subsystems. The group classification of this tree of potential systems and subsystems is given in Ref. 5 and references therein. In particular, for certain forms of the constitutive function $L(u)$ the level two system $\mathbf{S}\{x,t,u,v\}=0$ yields nonlocal symmetries of the level one system $\mathbf{R}\{x,t,u\}=0$, and vice versa. The point symmetries of the "grand" level three system $\mathbf{T}\{x,t,u,v,w\}=0$ include all point symmetries of the lower level systems. Moreover, the "grand" system $\mathbf{T}\{x,t,u,v,w\}=0$ includes a constitutive function $L(u)$ that yields symmetries that are nonlocal for the level one and level two systems.

B. Direct construction method for finding conservation laws

A direct method for finding conservation laws using factors was presented in Refs. 11 and 12. This method follows from the fact that a function $f(\mathbf{x},\mathbf{U},\partial\mathbf{U},\ldots,\partial^p\mathbf{U})$ is a divergence expression if and only if it is annihilated by the Euler differential operators,

$$E_k = \frac{\partial}{\partial U^k} - D_i\frac{\partial}{\partial U^k_i} + D_iD_j\frac{\partial}{\partial U^k_{ij}} + \cdots + (-1)^p D_{i_1}\cdots D_{i_p}\frac{\partial}{\partial U^k_{i_1\cdots i_p}}, \tag{2.12}$$

associated with each dependent variable U^k in $f(\mathbf{x},\mathbf{U},\partial\mathbf{U},\ldots,\partial^p\mathbf{U})$, $k=1,\ldots,n$. In (2.12), D_i is the total derivative operator for the independent variable x_i and $U^k_{i_1\cdots i_p}=\partial^p U^k/\partial x_{i_1}\cdots\partial x_{i_p}$.

For a given PDE system $\mathbf{R}\{x,t,\mathbf{u}\}=0$, when *additional* conservation laws are found within its tree of potential systems and subsystems, the tree can be extended. Here $x_1=t,x_2=x$.

The procedure to find additional conservation laws is as follows:

(A) Take a linear combination of the functions $R_k\{x,t,\mathbf{U}\}$ associated with the system $\mathbf{R}\{x,t,\mathbf{u}\}=0$, with unknown factors $\Lambda_k(x,t,\mathbf{U},\partial\mathbf{U},\ldots,\partial^l\mathbf{U})$ (for some fixed l),

$$M=\Lambda_k(x,t,\mathbf{U},\partial\mathbf{U},\ldots,\partial^l\mathbf{U})R_k\{x,t,\mathbf{U}\}. \tag{2.13}$$

It is essential to note that in (2.13), $\mathbf{U}$ is an *arbitrary function*. [$\mathbf{U}=\mathbf{u}(x,t)$ is a solution of $\mathbf{R}\{x,t,\mathbf{u}\}=0$.]

(B) A set of factors $\{\Lambda_k(x,t,\mathbf{U},\partial\mathbf{U},\ldots,\partial^l\mathbf{U})\}$ yields a conservation law (2.1) of $\mathbf{R}\{x,t,\mathbf{u}\}=0$ if and only if it satisfies the linear system of determining equations,

$$\begin{aligned} E_1M&=0,\\ &\ldots,\\ E_nM&=0, \end{aligned} \tag{2.14}$$

holding for arbitrary values of x,t and the components of $\mathbf{U},\partial\mathbf{U},\partial^2\mathbf{U},\ldots$.

(C) For each set of factors $\{\Lambda_k(x,t,\partial\mathbf{U},\ldots,\partial^l\mathbf{U})\}$ satisfying (2.14), there is an integral formula to find the density T and the flux X of the corresponding conservation law (2.1).[11,12]

Remark 2.3: The procedure outlined above can be applied to find conservation laws for any potential system or subsystem in a tree.

C. Construction of extended trees of nonlocally related potential systems and subsystems using additional conservation laws

Definition 2.7: If for some additional conservation law, a factor Λ_k does not vanish or vanishes only on solutions $\mathbf{U}=\mathbf{u}(x,t)$ of the given system $\mathbf{R}\{x,t,\mathbf{u}\}=0$, then the resulting conservation law (2.1) is a *useful conservation law* and can replace the kth equation $R_k\{x,t,\mathbf{u}\}=0$ of the system $\mathbf{R}\{x,t,\mathbf{u}\}=0$.

Consequently, the resulting system

$$\widetilde{\mathbf{R}}\{x,t,\mathbf{u}\}=0:\ \begin{cases} R_1\{x,t,\mathbf{u}\}=0,\\ \cdots,\\ R_{k-1}\{x,t,\mathbf{u}\}=0,\\ D_tT\{x,t,\mathbf{u}\}+D_xX\{x,t,\mathbf{u}\}=0,\\ R_{k+1}\{x,t,\mathbf{u}\}=0,\\ \cdots,\\ R_m\{x,t,\mathbf{u}\}=0 \end{cases} \tag{2.15}$$

has the same solution set as the original DE system $\mathbf{R}\{x,t,\mathbf{u}\}=0$. In particular, the system (2.15) explicitly contains a conservation law that leads to a related higher potential system. In determining conservation laws by the direct construction method for the related higher level potential system $\mathbf{S}\{x,t,\mathbf{u},v\}=0$ arising from (2.15), for completeness it is essential to consider the potential system $\mathbf{S}\{x,t,\mathbf{u},v\}=0$ together with the replaced equation $R_k\{x,t,\mathbf{u}\}=0$, i.e., the system

$$\widetilde{\mathbf{S}}\{x,t,\mathbf{u},v\}=0:\begin{cases}R_1\{x,t,\mathbf{u}\}=0,\\ \dots\ ,\\ R_{k-1}\{x,t,\mathbf{u}\}=0\\ R_k\{x,t,\mathbf{u}\}=0,\\ v_x-T\{x,t,\mathbf{u}\}=0,\\ v_t+X\{x,t,\mathbf{u}\}=0,\\ R_{k+1}\{x,t,\mathbf{u}\}=0,\\ \dots\ ,\\ R_m\{x,t,\mathbf{u}\}=0.\end{cases} \tag{2.16}$$

If a conservation law is not useful, then one would be too restricted in considering subsystems (with the same solution sets as $\mathbf{R}\{x,t,\mathbf{u}\}=0$) that result from elimination of one or more dependent variables.

By incorporating the direct method for finding conservation laws, we are now able to outline the algorithm for constructing the *extended* tree of nonlocally related potential systems and subsystems for a given DE system $\mathbf{R}\{x,t,\mathbf{u}\}=0$. Since $\mathbf{u}=(u^1,\dots,u^n)$, the level (number of dependent variables) of the given system in the tree is n.

(1) *Construction of potential systems*: Suppose $\mathbf{R}\{x,t,\mathbf{u}\}=0$ includes explicit conservation laws as written. For each of these conservation laws (2.1), introduce a potential and construct a potential system of level $n+1$. For each of the potential systems of level $n+1$, repeat this step to obtain all potential systems of level $n+2$, etc., until higher potential systems include no more explicit conservation laws. Let $\mathcal{T}_1$ denote the resulting tree. (If $\mathbf{R}\{x,t,\mathbf{u}\}=0$ does not include explicit conservation laws as written, then $\mathcal{T}_1=\{\mathbf{R}\{x,t,\mathbf{u}\}=0\}$.)

(2) *Construction of subsystems*: For all systems of the tree $\mathcal{T}_1$, exclude where possible, one by one, dependent variables, to generate all subsystems of the systems in the tree $\mathcal{T}_1$. Eliminate subsystems that are locally related to it. This yields a possibly larger tree $\mathcal{T}_2$.

(3) *Additional conservation laws*: *Tree extension*: For each system in $\mathcal{T}_2$, find multipliers that yield useful conservation laws via the direct construction method. Use these additional useful conservation laws to obtain higher potential systems and corresponding subsystems. Eliminate locally related subsystems. Continue until no further useful conservation laws are found for any nonlocally related potential system or subsystem. This yields *an extended tree of nonlocally related potential systems and subsystems.*

D. Construction of nonlocal symmetries from an extended tree of potential systems and subsystems

The extended tree obtained by the above procedure can be used for different methods of analysis. In particular, it is useful in the search for nonlocal symmetries of the given DE system $\mathbf{R}\{x,t,\mathbf{u}\}=0$. Since each potential system and subsystem within the tree is nonlocally related to the given system, as well as other potential systems and subsystems in the tree, it follows that point symmetries of potential systems and subsystems may yield nonlocal (potential) symmetries of the given system, and/or other systems in the extended tree. Now we outline *the algorithm to construct nonlocal symmetries*.

(1) *Construction of extended trees of potential systems and nonlocally related subsystems*: For a given DE system $\mathbf{R}\{x,t,\mathbf{u}\}=0$, construct the extended tree of nonlocally related potential systems and subsystems. (If the given system contains constitutive functions, different extended trees may be obtained for particular forms of constitutive functions.)

(2) *Point symmetry analysis*: For each system in the extended tree, use Lie's algorithm to obtain its point symmetries.

(3) *Isolation of nonlocal symmetries*: From the set of point symmetries of each system in the

extended tree, isolate nonlocal symmetries of the given system $\mathbf{R}\{x,t,\mathbf{u}\}=0$.

An example of the use of this algorithm follows.

E. Example 2: Nonlocal symmetries and linearization of a nonlinear reaction-diffusion equation

We apply the above-described algorithm of construction of nonlocal symmetries to the reaction-diffusion equation

$$\mathbf{R}\{x,t,u\}=0\colon\ u_t-u^2u_{xx}-2bu^2=0. \tag{2.17}$$

First, we construct an extended tree of potential systems and subsystems. The equation (2.17) is not written as a conservation law. We look for multipliers of the form $\Lambda_1=\Lambda_1(U)$ that yield conservation laws of (2.17). Here the determining equation (2.14) becomes

$$E_1[\Lambda_1(U)(U_t-U^2U_{xx}-2bU^2)]=0,$$

and has solution $\Lambda_1(U)=-1/U^2$, with corresponding conservation law

$$\left(\frac{1}{u}\right)_t+(u+bx^2)_{xx}=0. \tag{2.18}$$

The multiplier $\Lambda_1(U)=-1/U^2$ does not vanish. Hence the conservation law (2.18) is useful and equivalent to the PDE (2.17). We let $u_1=1/u$ and denote the resulting PDE by

$$\tilde{\mathbf{R}}\{x,t,u_1\}=0\colon\ u_{1t}+\left(\frac{1}{u_1}+bx^2\right)_{xx}=0. \tag{2.19}$$

We introduce potential variables v and w and corresponding potential systems,

$$\mathbf{S}\{x,t,u_1,v\}=0\colon\ \begin{cases} v_x-u_1=0,\\ v_t+\left(\dfrac{1}{u_1}+bx^2\right)_x=0,\end{cases}$$
$$\mathbf{T}\{x,t,u_1,v,w\}=0\colon\ \begin{cases} v_x-u_1=0,\\ w_x-v=0,\\ w_t+\left(\dfrac{1}{u_1}+bx^2\right)=0.\end{cases} \tag{2.20}$$

The subsystems are

$$\underline{\mathbf{S}}\{x,t,v\}=0\colon\ v_t+\left(\frac{1}{v_x}+bx^2\right)_x=0,$$

$$\underline{\mathbf{T}}_1\{x,t,u,w\}=0\colon\ \begin{cases} w_{xx}-u_1=0,\\ w_t+\left(\dfrac{1}{u_1}+bx^2\right)=0,\end{cases}$$
$$\underline{\mathbf{T}}_2\{x,t,v,w\}=0\colon\ \begin{cases} w_x-v=0,\\ w_t+\left(\dfrac{1}{v_x}+bx^2\right)=0,\end{cases} \tag{2.21}$$

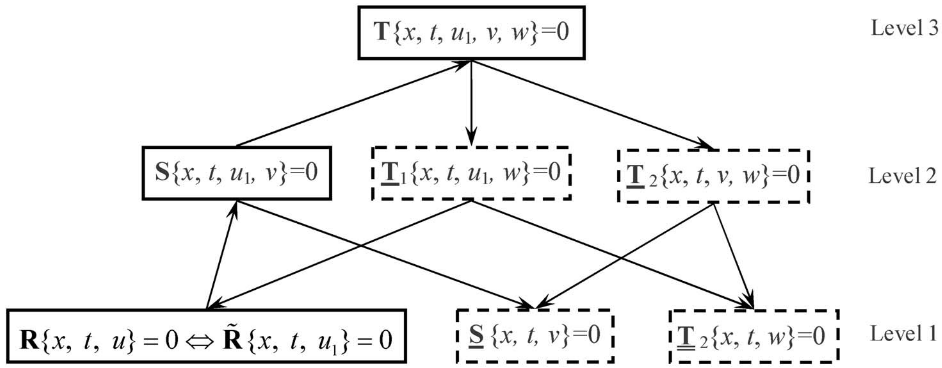

FIG. 2. The tree of potential systems and subsystems for the reaction-diffusion equation (2.17).

$$\underline{\underline{\mathbf{T}}}_2\{x,t,w\}=0\colon\ w_t+\left(\frac{1}{w_{xx}}+bx^2\right)=0.$$

The tree of potential systems and subsystems is illustrated in Fig. 2 with arrows showing the origins of elements of the tree. Locally related subsystems are outlined with dashed lines. For the analysis of nonlocal symmetries of the given system $\mathbf{R}\{x,t,u\}=0$ (2.17), only systems $\mathbf{S}\{x,t,u_1,v\}=0$ and $\mathbf{T}\{x,t,u_1,v,w\}=0$ need to be used, since all other subsystems are locally related to them.

One can show[5] that the level three potential system $\mathbf{T}\{x,t,u_1,v,w\}=0$ admits an infinitesimal generator,

$$\mathrm{X}_{\infty}^{\mathbf{T}}=e^{b(w-xv)}\left((F^1-bxF^3)\frac{\partial}{\partial x}+(2bxu_1^2F^1-u_1^2F^2+bu_1(1-bx^2u_1)F^3)\frac{\partial}{\partial u_1}\right.$$
$$\left.+(vF^1-(1+bxv)F^3)\frac{\partial}{\partial w}\right), \tag{2.22}$$

where

$$\frac{\partial F^1(v,t)}{\partial v}=F^2(v,t),\quad \frac{\partial F^3(v,t)}{\partial v}=F^1(v,t),\quad \frac{\partial F^3(v,t)}{\partial t}=F^2(v,t). \tag{2.23}$$

The point symmetry generator (2.22) is infinite dimensional; it projects to a point symmetry of $\underline{\mathbf{T}}_2\{x,t,v,w\}=0$, induces a contact symmetry of $\underline{\underline{\mathbf{T}}}_2\{x,t,w\}=0$, a Lie-Bäcklund symmetry of $\underline{\mathbf{T}}_1\{x,t,u,w\}=0$, and a nonlocal symmetry of $\mathbf{R}\{x,t,u\}=0$, $\tilde{\mathbf{R}}\{x,t,u_1\}=0$, $\mathbf{S}\{x,t,u_1,v\}=0$, and $\underline{\mathbf{S}}\{x,t,v\}=0$. Consequently, $\mathbf{T}\{x,t,u_1,v,w\}=0$, $\underline{\mathbf{T}}_2\{x,t,v,w\}=0$, and $\underline{\underline{\mathbf{T}}}_2\{x,t,w\}=0$ are linearizable by invertible mappings, and the other systems in the tree are linearizable by noninvertible mappings.[4,5]

III. TREES AND NONLOCAL SYMMETRIES FOR PLANAR GAS DYNAMICS EQUATIONS

We now use the algorithmic approach described in Sec. II to construct trees of potential systems and subsystems for the $(1+1)$-dimensional system of planar gas dynamics (PGD) equations. The point symmetries of some of these systems have been extensively considered in Ref. 14.

The two fundamental systems of differential equations that describe nonstationary $(1+1)$-dimensional gas motions are Euler and Lagrange systems. In the Eulerian description, x is a Cartesian coordinate in a fixed coordinate frame. The *Euler system* is given by

$$\mathbf{E}\{x,t,v,p,\rho\}=0: \begin{cases} \rho_t+(\rho v)_x=0, \\ \rho(v_t+vv_x)+p_x=0, \\ \rho(p_t+vp_x)+B(p,1/\rho)v_x=0. \end{cases} \tag{3.1}$$

Here v is the gas velocity, ρ is the gas density, and p is the gas pressure. In terms of the entropy density $S(p,\rho)$, the constitutive function $B(p,1/\rho)$ is given by

$$B(p,1/\rho)=-\rho^2 S_\rho/S_p.$$

In many applications, however, it is more convenient to use Lagrange mass coordinates $s=t$, $y=\int_{x_0}^{x}\rho(\xi)\mathrm{d}\xi$. In these variables, the system (3.1) takes on the equivalent form

$$\mathbf{L}\{y,s,v,p,q\}=0: \begin{cases} q_s-v_y=0, \\ v_s+p_y=0, \\ p_s+B(p,q)v_y=0, \end{cases} \tag{3.2}$$

and is called the *Lagrange system*. Here the coordinate y essentially enumerates the fluid particles; its domain does not change with time. The partial time derivative $\partial/\partial s=\partial/\partial t+v\,\partial/\partial x$ is the material derivative. The use of Lagrange mass coordinates often significantly facilitates the formulation of boundary conditions.[15–17]

We show that the potential system framework provides a direct connection between the Euler system (3.1) and the Lagrange system (3.2). Moreover, further extensions arise, and in particular, one can obtain other nonlocally related equivalent systems of equations.

We now construct a tree of nonlocally related potential systems and subsystems, with $\mathbf{E}\{x,t,v,p,\rho\}=0$ given by (3.1) serving as the given system, through the algorithm described in Sec. II.

Since the first equation of (3.1) is a conservation law, a potential variable r is naturally introduced, and the resulting level four potential system has the form

$$\mathbf{G}\{x,t,v,p,\rho,r\}=0: \begin{cases} r_x-\rho=0, \\ r_t+\rho v=0, \\ r_x(v_t+vv_x)+p_x=0, \\ r_x(p_t+vp_x)+B(p,1/r_x)v_x=0. \end{cases} \tag{3.3}$$

An obvious subsystem $\mathbf{I}\{x,t,v,p,r\}\equiv\underline{\mathbf{G}}_1\{x,t,v,p,r\}=0$ is obtained by excluding the density ρ from (3.3),

$$\mathbf{I}\{x,t,v,p,r\}=0: \begin{cases} r_t+vr_x=0, \\ r_x(v_t+vv_x)+p_x=0, \\ r_x(p_t+vp_x)+B(p,1/r_x)v_x=0. \end{cases} \tag{3.4}$$

In Ref. 14, (3.4) is referred to as the *intermediate* system. However, this system is locally related to $\mathbf{G}\{x,t,v,p,\rho,r\}=0$.

Another subsystem is $\underline{\mathbf{G}}_2\{x,t,p,\rho,r\}=0$, obtained by excluding the velocity v. This subsystem is also not of interest since it is locally related to $\mathbf{G}\{x,t,v,p,\rho,r\}=0$.

Consider a local coordinate transformation of the system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ with $r=y$, $t=s$ treated as independent variables, and x,v,p,ρ as dependent variables. Without loss of generality, $\rho\neq 0$. We let $q=1/\rho$, and obtain the system $\mathbf{G}_0\{y,s,x,v,p,\rho\}=0$ given by

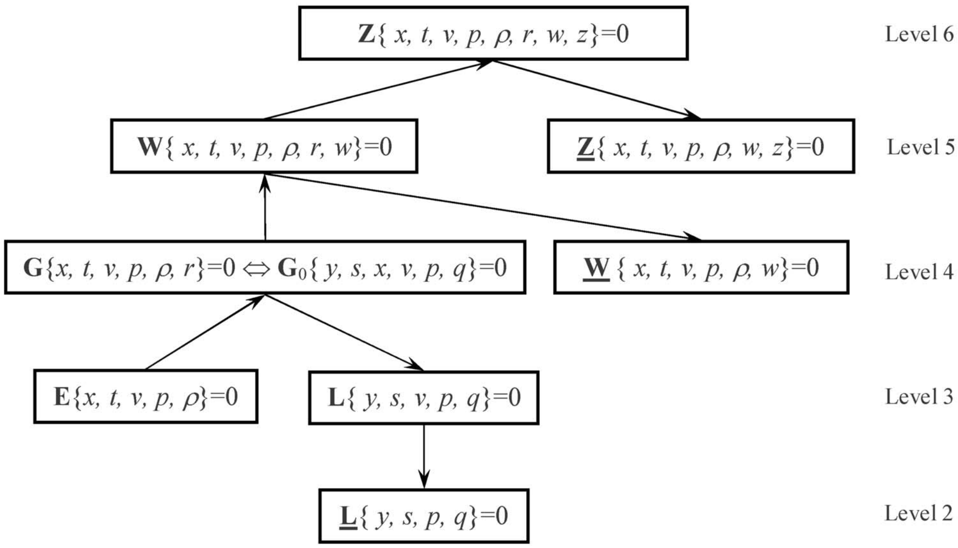

FIG. 3. The tree $\mathcal{T}_{\text{PGD}}$ of nonlocally related PGD potential systems and subsystems, for an arbitrary constitutive function $B(p,1/\rho)$; $\mathbf{E}\{x,t,v,p,\rho\}=0$ is the Euler system (3.1), $\mathbf{L}\{y,s,v,p,q\}=0$ is the Lagrange system (3.2).

$$\mathbf{G}_0\{y,s,x,v,p,\rho\}=0:\ \begin{cases} q-x_y=0, \\ v-x_s=0, \\ v_s+p_y=0, \\ p_s+B(p,q)v_y=0, \end{cases} \tag{3.5}$$

equivalent to the potential system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ and locally related to it.

A subsystem of $\mathbf{G}_0\{y,s,x,v,p,\rho\}=0$ obtained by excluding x through $x_{sy}=x_{ys}$ is the Lagrange system (3.2), $\mathbf{L}\{y,s,v,p,q\}\equiv\underline{\mathbf{G}}_0\{y,s,v,p,q\}=0$. Thus *the Euler and Lagrange systems of PGD equations are connected through a common potential system* (see Fig. 3).

We continue the construction of the tree of potential systems and subsystems for the PGD equations for a general constitutive function $B(p,1/\rho)$. We first find possible higher potential systems arising for the potential system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ given by (3.3).

The Euler system given by (3.1) with multipliers $\Lambda_1=V$, $\Lambda_2=1$, $\Lambda_3=0$ yields a *useful* conservation law,

$$(\rho v)_t+(p+\rho v^2)_x=0, \tag{3.6}$$

which also holds for the system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$. Hence we use the conservation law (3.6) to replace the equation $r_x(v_t+vv_x)+p_x=0$, and introduce a potential variable w to obtain the level five potential system $\mathbf{W}\{x,t,v,p,\rho,r,w\}=0$ given by

$$\mathbf{W}\{x,t,v,p,\rho,r,w\}=0:\ \begin{cases} r_x-\rho=0, \\ r_t+\rho v=0, \\ w_x+r_t=0, \\ w_t+p+vw_x=0, \\ r_x(p_t+vp_x)+B(p,1/r_x)v_x=0. \end{cases} \tag{3.7}$$

The third equation of (3.7) is written as a conservation law, and accordingly we introduce a potential variable z to obtain a level six potential system,

$$\mathbf{Z}\{x,t,v,p,\rho,r,w,z\} = 0: \begin{cases} r_x - \rho = 0, \\ r_t + \rho v = 0, \\ z_t - w = 0, \\ z_x + r = 0, \\ w_t + p + v w_x = 0, \\ r_x(p_t + v p_x) + B(p, 1/r_x) v_x = 0. \end{cases} \tag{3.8}$$

The only nonlocally related subsystems of (3.7) and (3.8) arise from excluding r (see Fig. 3). The Lagrange system (3.2) has a nonlocally related subsystem obtained by excluding v,

$$\underline{\mathbf{L}}\{y,s,p,q\} = 0: \begin{cases} q_{ss} + p_{yy} = 0, \\ p_s + B(p,q) q_s = 0. \end{cases} \tag{3.9}$$

The tree $\mathcal{T}_{\mathrm{PGD}}$ of useful (i.e., nonlocally related) potential systems and subsystems [for an arbitrary constitutive function $B(p,1/\rho)$] is illustrated by Fig. 3. Note that either the Euler system $\mathbf{E}\{x,t,v,p,\rho\}=0$ or the Lagrange system $\mathbf{L}\{y,s,v,p,q\}=0$ can be taken as the given system. Each of these systems gives rise to the same tree of potential systems and subsystems.

All systems in the tree $\mathcal{T}_{\mathrm{PGD}}$ are nonlocally related and equivalent (i.e., contain all solutions of each other). Therefore any general method of analysis (qualitative, numerical, perturbation, symmetry, conservation laws, etc.) may yield new results for any of these nonlocally related PGD systems. In particular, this is the case for the symmetry analysis given below.

In Ref. 14, point symmetries of three systems were studied in detail—the Euler system (3.1), the Lagrange system (3.2), and the "intermediate" system (3.4). The authors gave a classification with respect to the constitutive function $B(p,1/\rho)$ and isolated the cases in which some of the point symmetries of $\mathbf{E}\{x,t,v,p,\rho\}=0$, $\mathbf{L}\{y,s,v,p,q\}=0$ or $\mathbf{I}\{x,t,v,p,r\}=0$ were nonlocal for the other two systems. However, their approach was heuristic—the connections between their systems did not involve a general constructive framework.

Using the algorithmic approach presented in this paper, one directly arrives at the tree $\mathcal{T}_{\mathrm{PGD}}$ of nonlocally related PGD potential systems and subsystems. To find nonlocal symmetries of systems $\mathbf{E}\{x,t,v,p,\rho\}=0$ and $\mathbf{L}\{y,s,v,p,q\}=0$, one should classify the point symmetries of *all eight* systems in the tree, with respect to the constitutive function $B(p,1/\rho)$. (The system $\mathbf{I}\{x,t,v,p,r\}=0$ discussed in Ref. 14 is of no interest since it is locally related to the system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ in the tree.) For example, the subsystem $\underline{\mathbf{L}}\{y,s,p,q\}=0$ given by (3.9), in the case of a Chaplygin gas $[B(p,q)=-p/q]$, admits a point symmetry with infinitesimal generator,

$$\mathrm{X} = -y^2\frac{\partial}{\partial y} - py\frac{\partial}{\partial p} + 3yq\frac{\partial}{\partial q}, \tag{3.10}$$

which yields a nonlocal symmetry for both $\mathbf{E}\{x,t,v,p,\rho\}=0$ and $\mathbf{L}\{y,s,v,p,q\}=0$. The symmetry (3.10) was not obtained in Ref. 14, since the system (3.9) did not arise.

Furthermore, for some systems in the tree $\mathcal{T}_{\mathrm{PGD}}$, particular forms of the constitutive function $B(p,1/\rho)$ may yield useful conservation laws, which in turn would yield extended trees (cf. Sec. II). We now consider two examples.

Example A: For $B(p,1/\rho)=\rho(1+e^p)$, the system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ has a family of useful conservation laws,

$$D_t\left(\frac{e^p f(r)}{1+e^p}\right) + D_x\left(\frac{v e^p f(r)}{1+e^p}\right) = 0, \tag{3.11}$$

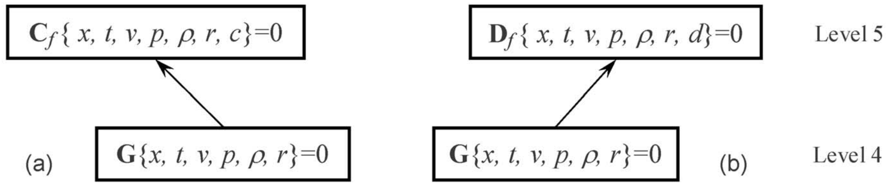

FIG. 4. PGD tree extensions for two particular forms of the constitutive function $B(p,1/\rho)$. (a) $B(p,1/\rho)=\rho(1+e^p)$. (b) Chaplygin gas $B(p,1/\rho)=-p\rho$.

for arbitrary $f(r)$. A conservation law of the form (3.11) can be used to replace the fourth equation of $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ to introduce a potential c and consequent family of potential systems $\mathbf{C}_f\{x,t,v,p,\rho,r,c\}=0$ in terms of an arbitrary function $f(r)$,

$$\mathbf{C}_f\{x,t,v,p,\rho,r,c\} = 0: \begin{cases} r_x - \rho = 0, \\ r_t + \rho v = 0, \\ r_x(v_t + v v_x) + p_x = 0, \\ c_x + e^p f(r)/(1 + e^p) = 0, \\ c_t - v e^p f(r)/(1 + e^p) = 0. \end{cases} \tag{3.12}$$

The corresponding tree extension is exhibited in Fig. 4(a).

Example B: For the Chaplygin gas $B(p,1/\rho)=-p\rho$, the family of useful conservation laws

$$D_t\left(\frac{f(r)}{p}\right) + D_x\left(\frac{vf(r)}{p}\right) = 0 \tag{3.13}$$

for arbitrary $f(r)$ yields a family of potential systems

$$\mathbf{D}_f\{x,t,v,p,\rho,r,d\} = 0: \begin{cases} r_x - \rho = 0, \\ r_t + \rho v = 0, \\ r_x(v_t + v v_x) + p_x = 0, \\ d_x + f(r)/p = 0, \\ d_t - v f(r)/p = 0, \end{cases} \tag{3.14}$$

nonlocally related to the other systems in the tree $\mathcal{T}_{\mathrm{PGD}}$. The corresponding tree extension is exhibited in Fig. 4(b). One can show that nonlocal symmetries of the Euler system $\mathbf{E}\{x,t,v,p,\rho\}=0$ only arise in the cases where $f(r)=r$, $f(r)=\text{const}$. For $f(r)=r$, the system (3.14) admits

$$\mathrm{X}_{\mathbf{D}1} = \left(-\frac{t^3}{6} + dt\right)\frac{\partial}{\partial x} + \left(d - \frac{t^2}{2}\right)\frac{\partial}{\partial v} + rt\frac{\partial}{\partial p} - \frac{rt\rho}{p}\frac{\partial}{\partial \rho}, \tag{3.15}$$

$$\mathrm{X}_{\mathbf{D}2} = \left(-\frac{t^2}{2} + d\right)\frac{\partial}{\partial x} - t\frac{\partial}{\partial v} + r\frac{\partial}{\partial p} - \frac{r\rho}{p}\frac{\partial}{\partial \rho}. \tag{3.16}$$

Symmetry (3.15) is nonlocal for both the Euler system $\mathbf{E}\{x,t,v,p,\rho\}=0$ and the Lagrange system $\mathbf{L}\{y,s,v,p,q\}=0$; symmetry (3.16) is nonlocal for the Euler system $\mathbf{E}\{x,t,v,p,\rho\}=0$ but local for the Lagrange system $\mathbf{L}\{y,s,v,p,q\}=0$. In Ref. 14, symmetry $\mathrm{X}_{\mathbf{D}1}$ was not obtained; symmetry $\mathrm{X}_{\mathbf{D}2}$ was obtained by an *ad hoc* procedure.

Through the algorithmic framework given in this paper, the symmetry results in Ref. 14 can be recovered systematically and substantially extended.

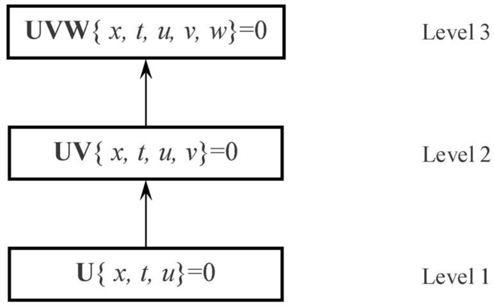

FIG. 5. NLT tree for arbitrary constitutive functions $F(u),G(u)$.

IV. EXTENDED TREES OF NONLOCALLY RELATED SYSTEMS FOR NONLINEAR TELEGRAPH EQUATIONS

We now construct trees of nonlocally related potential systems and subsystems for the nonlinear telegraph (NLT) equation, as well as further tree extensions for particular forms of constitutive functions. This allows us to extend recent results that appeared in Refs. 9, 13, and 18.

A. The tree for arbitrary constitutive functions

As a given system, we take the NLT equation,

$$\mathbf{U}\{x,t,u\} = 0\colon\ u_{tt} - (F(u)u_x)_x - (G(u))_x = 0. \tag{4.1}$$

Equation (4.1) is an explicit conservation law and hence is equivalent to the level two potential system,

$$\mathbf{UV}\{x,t,u,v\} = 0\colon\ \begin{cases} u_t - v_x = 0, \\ v_t - F(u)u_x - G(u) = 0. \end{cases} \tag{4.2}$$

NLT systems of the form (4.2) represent the equations of telegraphy of a two-conductor transmission line and equations of motion of a hyperelastic homogeneous rod whose cross-sectional area varies exponentially along the rod. For further details, see Refs. 9 and 13 and references therein.

Since the first equation of (4.2) is written as a conservation law, a level three potential system is obtained,

$$\mathbf{UVW}\{x,t,u,v,w\} = 0\colon\ \begin{cases} w_t - v = 0, \\ w_x - u = 0, \\ v_t - F(u)u_x - G(u) = 0. \end{cases} \tag{4.3}$$

For *arbitrary* constitutive functions $F(u),G(u)$, there are no further potential systems.

The complete point symmetry classifications of the scalar equation (4.1) and system (4.2) appear, respectively, in Refs. 18 and 9. The point symmetries of (4.2) yield nonlocal symmetries of (4.1) for a large class of constitutive functions.

The given equation (4.1) is the only subsystem of system (4.2). The subsystems of potential system (4.3) are obtained by excluding u and/or v, $\mathbf{UW}\{x,t,u,w\}=0$, $\mathbf{VW}\{x,t,v,w\}=0$, and $\mathbf{W}\{x,t,w\}=0$. However these subsystems are not interesting since they are locally related to the potential system $\mathbf{UVW}\{x,t,u,v,w\}=0$ given by (4.3).

The tree $\mathcal{T}_{\mathrm{NLT}}$ of useful (i.e., nonlocally related) potential systems and subsystems, for arbitrary constitutive functions $F(u),G(u)$, is exhibited in Fig. 5.

TABLE I. Conservation laws of the system (4.2) using the data presented in Ref. 13.

Case	Multipliers	Conservation law
$F(u)=G'(u)$	$\Lambda_1=0,\ \Lambda_2=e^x$	(I) $D_t(ve^x)-D_x(e^xG(u))=0$
	$\Lambda_1=e^x,\ \Lambda_2=te^x$	(II) $D_t(e^x(u+tv))-D_x(e^x(tG(u)+v))=0$
$F(u)=G'(u)+1$	$\Lambda_1=\Lambda_2=e^{x+t}$	(III) $D_t(e^{x+t}(u+v))-D_x(e^{x+t}(G(u)+u+v))=0$
	$\Lambda_1=-\Lambda_2=e^{x-t}$	(IV) $D_t(e^{x-t}(u-v))+D_x(e^{x-t}(G(u)+u-v))=0$
$F(u)=G'(u)-1$	$\Lambda_1=-i\Lambda_2=e^{x+it}$	(V,VI) Real and imaginary parts of $D_t(e^{x+it}(iu+v))-D_x(e^{x+it}(G(u)-u+iv))=0$
$F(u)$ arbitrary, $G(u)=u$	$\Lambda_1=x-t^2/2,\ \Lambda_2=t$	(VII) $D_t((x-t^2/2)u+tv)$ $+D_x((t^2/2-x)v-t\int F(u)\mathrm{d}u)=0$
	$\Lambda_1=-t,\ \Lambda_2=1$	(VIII) $D_t(v-tu)+D_x(tv-\int F(u)\mathrm{d}u)=0$

B. Tree extensions for particular constitutive functions

The complete conservation law classification of the potential system $\mathbf{UV}\{x,t,u,v\}=0$ given by (4.2) was found in Ref. 13 for multipliers of the form $\Lambda_i=\Lambda_i(x,t,U,V),\ i=1,2$. The problem of finding further potential systems from useful conservation laws of the system (4.2) was not considered in Ref. 13.

Using the data presented in Ref. 13, one sees that the following useful conservation laws (Table I) arise for (4.2).

In Table I, we exclude the cases $[G(u)=u$ with $F(u)=\text{const}$, $G(u)=0$ with $F(u)$ arbitrary$]$ for which system (4.2) is linear or linearizable by a point transformation.

The eight conservation laws in Table I are now used to obtain additional potential systems and nonlocally related subsystems, and thus to extend the tree $\mathcal{T}_{\mathrm{NLT}}$. In particular, each conservation law in Table I [except (I)] can be used to replace either equation of system $\mathbf{UV}\{x,t,u,v\}=0$ given by (4.2), since in each of these seven cases both multipliers Λ_1,Λ_2 are nonzero and have no dependence on dependent variables.

Case 1: $F(u)=G'(u)$. For conservation law (I) in Table I, one has $\Lambda_1=0$, and hence only the second equation of the system $\mathbf{UV}\{x,t,u,v\}=0$ given by (4.2) can be replaced with this conservation law. Accordingly, we introduce a potential variable $\tilde{a}$ and let $a=e^{-x}\tilde{a}$. The corresponding potential system is given by

$$\mathbf{UVA}\{x,t,u,v,a\}=0:\ \begin{cases} v_x-u_t=0,\\ a+a_x-v=0,\\ a_t-G(u)=0. \end{cases} \tag{4.4}$$

Since the first equation of (4.4) is written as a conservation law, a level four potential system is obtained,

$$\mathbf{UVWA}\{x,t,u,v,w,a\}=0:\ \begin{cases} w_t-v=0,\\ w_x-u=0,\\ a+a_x-v=0,\\ a_t-G(u)=0. \end{cases} \tag{4.5}$$

Note that the system $\mathbf{UVW}\{x,t,u,v,w\}=0$ given by (4.3) is a subsystem of the system (4.5) through excluding the variable a. The subsystems

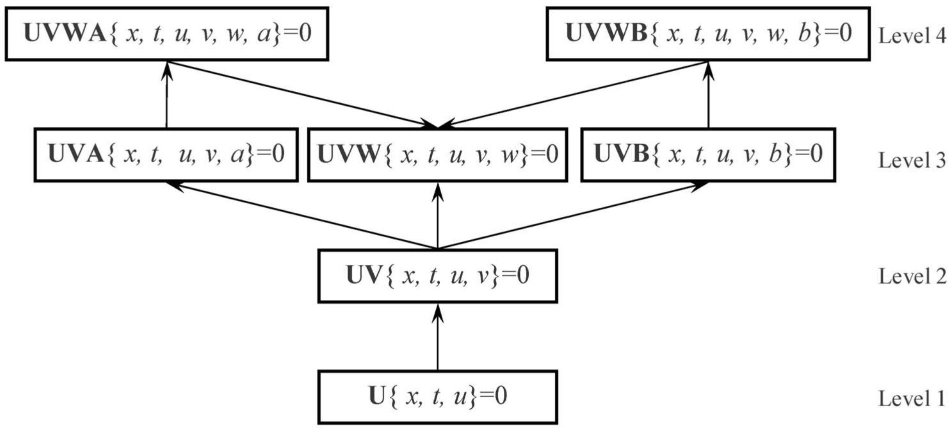

FIG. 6. The form of the extended tree of nonlocally related potential systems and subsystems of the NLT equations for Case 1 $[F(u)=G'(u),\ G(u)$ arbitrary], Case 2 $[F(u)=G'(u)+1,\ G(u)$ arbitrary], Case 3 $[F(u)=G'(u)-1,\ G(u)$ arbitrary], and Case 4 $[F(u)$ arbitrary with $G(u)=u]$.

$$\mathbf{VA}\{x,t,v,a\}=0:\ \begin{cases} a+a_x-v=0, \\ v_x-H'(a_t)a_{tt}=0 \quad (H=G^{-1}), \end{cases}$$

$$\mathbf{A}\{x,t,a\}=0:\ a_x+a_{xx}-H'(a_t)a_{tt}=0 \tag{4.6}$$

are locally related to (4.4) and therefore not interesting.

The useful conservation law (II) is equivalent to a pair of equations $b_t=tG(u)+v$, $b_x+b=u+tv$, and hence yields the level three potential system,

$$\mathbf{UVB}\{x,t,u,v,b\}=0:\ \begin{cases} v_x-u_t=0, \\ v_t-G'(u)u_x-G(u)=0, \\ b_t-tG(u)-v=0, \\ b_x+b-u-tv=0, \end{cases} \tag{4.7}$$

as well as the level four potential system,

$$\mathbf{UVWB}\{x,t,u,v,w,b\}=0:\ \begin{cases} w_t-v=0, \\ w_x-u=0, \\ b_t-tG(u)-v=0, \\ b_x+b-u-tv=0. \end{cases} \tag{4.8}$$

In Fig. 6 we exhibit the extended tree $\mathcal{T}^1_{\mathrm{NLT}}$ of nonlocally related potential systems and subsystems of the NLT equation (4.1) in the case $F(u)=G'(u)$.

In a similar manner, one can show that the three pairs of conservation laws for the other cases $[F(u)=G'(u)+1,\ F(u)=G'(u)-1,\ F(u)$ arbitrary with $G(u)=u]$ all yield extended trees of the form exhibited in Fig. 6. For each of these three cases, the nonlocally related systems are as follows.

Case 2: $F(u)=G'(u)+1$. The set of nonlocally related potential systems and subsystems is given by

$$\mathbf{UVA}\{x,t,u,v,a\} = 0: \begin{cases} v_x - u_t = 0, \\ v_t - F(u)u_x - G(u) = 0, \\ a_t + a - (G(u) + u + v) = 0, \\ a_x + a - (u + v) = 0, \end{cases}$$

$$\mathbf{UVWA}\{x,t,u,v,w,a\} = 0: \begin{cases} w_t - v = 0, \\ w_x - u = 0, \\ a_t + a - (G(u) + u + v) = 0, \\ a_x + a - (u + v) = 0, \end{cases} \tag{4.9}$$

$$\mathbf{UVB}\{x,t,u,v,b\} = 0: \begin{cases} v_x - u_t = 0, \\ v_t - F(u)u_x - G(u) = 0, \\ b_t - b - (G(u) + u - v) = 0, \\ b_x + b + (u - v) = 0, \end{cases}$$

$$\mathbf{UVWB}\{x,t,u,v,w,b\} = 0: \begin{cases} w_t - v = 0, \\ w_x - u = 0, \\ b_t - b - (G(u) + u - v) = 0, \\ b_x + b + (u - v) = 0. \end{cases} \tag{4.10}$$

Case 3: $F(u)=G'(u)-1$. The useful complex conservation law (V,VI) in Table I is equivalent to two useful real conservation laws,

$$\text{(V)} \quad D_t[e^x(v\cos t - u\sin t)] + D_x[e^x(v\sin t - (G(u)-u)\cos t)] = 0,$$

$$\text{(VI)} \quad D_t[e^x(u\cos t + v\sin t)] - D_x[e^x((G(u)-u)\sin t + v\cos t)] = 0.$$

The set of nonlocally related potential systems and subsystems is given by

$$\mathbf{UVA}\{x,t,u,v,a\} = 0: \begin{cases} v_x - u_t = 0, \\ v_t - F(u)u_x - G(u) = 0, \\ a_t + (v\sin t - (G(u)-u)\cos t) = 0, \\ a_x + a - (v\cos t - u\sin t) = 0, \end{cases} \tag{4.11}$$

$$\mathbf{UVWA}\{x,t,u,v,w,a\} = 0: \begin{cases} w_t - v = 0, \\ w_x - u = 0, \\ a_t + (v\sin t - (G(u)-u)\cos t) = 0, \\ a_x + a - (v\cos t - u\sin t) = 0, \end{cases}$$

$$\mathbf{UVB}\{x,t,u,v,b_2\} = 0: \begin{cases} v_x - u_t = 0, \\ v_t - F(u)u_x - G(u) = 0, \\ b_t + ((G(u)-u)\sin t + v\cos t) = 0, \\ b_x + b + (u\cos t + v\sin t) = 0, \end{cases} \tag{4.12}$$

$$\mathbf{UVWB}\{x,t,u,v,w,b\}=0: \begin{cases} w_t - v = 0, \\ w_x - u = 0, \\ b_t + ((G(u)-u)\sin t + v\cos t) = 0, \\ b_x + b + (u\cos t + v \sin t) = 0. \end{cases}$$

Case 4: $F(u)$ arbitrary, $G(u)=u$. In this case, the set of nonlocally related potential systems and subsystems is given by

$$\mathbf{UVA}\{x,t,u,v,a\}=0: \begin{cases} v_x - u_t = 0, \\ v_t - F(u)u_x - u = 0, \\ a_x - ((x-t^2/2)u + tv) = 0, \\ a_t + \left((t^2/2 - x)v - t\int F(u)\mathrm{d}u\right) = 0, \end{cases}$$

$$\mathbf{UVWA}\{x,t,u,v,w,a\}=0: \begin{cases} w_t - v = 0, \\ w_x - u = 0, \\ a_x - ((x-t^2/2)u + tv) = 0, \\ a_t + \left((t^2/2 - x)v - t\int F(u)\mathrm{d}u\right) = 0 \end{cases} \quad (4.13)$$

and

$$\mathbf{UVB}\{x,t,u,v,b\}=0: \begin{cases} v_x - u_t = 0, \\ v_t - F(u)u_x - u = 0, \\ b_x - (v - tu) = 0, \\ b_t + \left(tv - \int F(u)\mathrm{d}u\right) = 0, \end{cases}$$

$$\mathbf{UVWB}\{x,t,u,v,w,b\}=0: \begin{cases} w_t - v = 0, \\ w_x - u = 0, \\ b_x - (v - tu) = 0, \\ b_t + \left(tv - \int F(u)\mathrm{d}u\right) = 0. \end{cases} \quad (4.14)$$

V. FURTHER REMARKS

(1) The algorithmic framework for nonlocally related potential systems and subsystems has been demonstrated to be useful for calculating new nonlocal symmetries and new nonlocal conservation laws for a given system of PDEs. It should be important to study the applicability of other methods of analysis (qualitative, numerical, perturbation, etc.) to nonlocally related systems in extended trees, especially coordinate-independent methods.

(2) In a PDE system with $n \geqslant 3$ independent variables, a conservation law is equivalent to a set of equations involving *several potential variables*.[4] The corresponding potential system is underdetermined, and requires suitable gauge constraints (in the form of additional equations on the potential variables) to be imposed in order to find nonlocal symmetries.[19]

Although a potential system without constraints is underdetermined, its potential *subsystems* may be useful for analysis without gauge constraints.

(3) In the algorithm presented in Sec. II, the nonlocally related subsystems are obtained by

exclusion of dependent variables *as written*. Alternatively any point transformation

$$\begin{aligned} U &= U(x,t,u,v), \\ V &= V(x,t,u,v), \\ X &= X(x,t,u,v), \\ T &= T(x,t,u,v), \end{aligned} \tag{5.1}$$

could be used to exclude a dependent variable U or V to obtain *additional* nonlocally related subsystems. Indeed this is the situation within the tree of potential systems and subsystems of the PGD equations (Sec. III): the system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ as written has only a nonlocally related subsystem $\mathbf{E}\{x,t,v,p,\rho\}=0$. However after a local change of variables (to $\mathbf{G}_0\{y,s,x,v,p,\rho\}=0$), it admits the Lagrange system $\mathbf{L}\{y,s,v,p,q\}=0$ as a nonlocally related subsystem (Fig. 3).

(4) Using the algorithmic framework given in this paper, local and nonlocal symmetries for the PGD equations obtained in Ref. 14 can be recovered systematically and substantially extended (some examples of new nonlocal symmetries are given in Sec. III). The systematic classification of useful conservation laws and consequent nonlocal extensions of the PGD tree $\mathcal{T}_{\mathrm{PGD}}$ will appear in future works, as well as concomitant nonlocal symmetry analyses.

(5) An exhaustive study of nonlocal symmetries and nonlocal conservation laws of NLT equations resulting from extended trees of potential systems and subsystems is in progress. Preliminary results show that for a large class of constitutive functions, namely, $F(u)=G'(u)$, there exist point symmetries of the potential system $\mathbf{UVW}\{x,t,u,v,w\}=0$ given by (4.3) that are nonlocal for both the scalar NLT equation (4.1) and the system $\mathbf{UV}\{x,t,u,v\}=0$ given by (4.2). A particular example is a symmetry

$$\mathrm{X}_{\mathbf{UVW}} = v\frac{\partial}{\partial x} + \left(u + \frac{w}{3}\right)\frac{\partial}{\partial t} - \frac{uv}{3}\frac{\partial}{\partial u} - \frac{v^2}{3}\frac{\partial}{\partial v} + uv\frac{\partial}{\partial w} \tag{5.2}$$

for the case $F(u)=u^2, G(u)=u^3/3$.

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