Multidimensional partial differential equation systems: Nonlocal symmetries, nonlocal conservation laws, exact solutions

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(Received 11 June 2010; accepted 11 September 2010; published online 25 October 2010)

For systems of partial differential equations (PDEs) with $n \geq 3$ independent variables, construction of nonlocally related PDE systems is substantially more complicated than is the situation for PDE systems with two independent variables. In particular, in the multidimensional situation, nonlocally related PDE systems can arise as nonlocally related subsystems as well as potential systems that follow from divergence-type or lower-degree conservation laws. The theory and a systematic procedure for the construction of such nonlocally related PDE systems is presented in Part I [A. F. Cheviakov and G. W. Bluman, J. Math. Phys. **51**, 103521 (2010)]. This paper provides many new examples of applications of nonlocally related systems in three and more dimensions, including new nonlocal symmetries, new nonlocal conservation laws, and exact solutions for various nonlinear PDE systems of physical interest. © *2010 American Institute of Physics.* [doi:10.1063/1.3496383]

I. INTRODUCTION

In the situation of two independent variables, nonlocally related systems of partial differential equations (PDEs) have proven to be useful for many given nonlinear and linear PDE systems of physical interest. For a given PDE system, one can systematically construct nonlocally related potential systems and subsystems[2,3] having the same solution set as the given system. Due to nonlocal relations between solution sets, analysis of such nonlocally related systems can yield new results for the given system.

Examples include results for nonlinear wave and diffusion equations, gas dynamics equations, continuum mechanics, electromagnetism, plasma equilibria, as well as other nonlinear and linear PDE systems.[2–13] New results for such physical systems include systematic computations of nonlocal symmetries and nonlocal conservation laws, systematic constructions of further invariant and nonclassical solutions, and the systematic construction of noninvertible linearizations.

This paper follows from Ref. 1 and is concerned with the construction and use of nonlocally related PDE systems with three or more independent variables for specific examples. As shown in Ref. 1, the situation for obtaining and using nonlocally related PDE systems is considerably more complex than in the two-dimensional case. In particular, the usual (divergence-type) conservation laws give rise to vector potential variables subject to *gauge freedom*, i.e., defined to within arbitrary functions of the independent variables, making the corresponding potential system underdetermined.

Another important difference between two-dimensional and multidimensional PDE systems is that in higher dimensions, there can exist several types of conservation laws (divergence-type and

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lower-degree conservation laws). For example, in the case of $n=3$ independent variables, one can have a vanishing divergence or a vanishing curl; for $n>3$, $n-1$ types of conservation laws exist. [It is important to note that in many physical examples, the most commonly arising conservation laws are of divergence-type (degree $r=n-1$) conservation laws, which yield underdetermined potential systems.]

Due to such complexity and, furthermore, the difficulty of performing computations for PDE systems involving many dependent and independent variables, very few results have been obtained so far for multidimensional systems. In this paper, building on the framework presented in Ref. 1, we present new results for important examples as well as discuss and synthesize some previously known results. The symbolic software package GeM for MAPLE (Ref. 14) was used for the symbolic computations.

An important use of nonlocally related systems is the computation of *nonlocal symmetries* of a given PDE system. A nonlocal symmetry is a symmetry for which the components of its infinitesimal generator, corresponding to the variables of the given system, have an essential dependence on nonlocal variables. Only determined nonlocally related systems can yield nonlocal symmetries of a given system.[11] Consequently, one seeks nonlocal symmetries of a given PDE system (with $n \geq 3$ independent variables) through seeking local symmetries of the following types of nonlocally related PDE systems.[1]

- Nonlocally related subsystems (always determined).
- Potential systems of degree one (always determined). (In $\mathbb{R}^3$, such potential systems arise from curl-type conservation laws.)
- Potential systems of degree $r: 1 < r \leq n-1$, appended with an appropriate number of gauge constraints.

Examples of nonlocal symmetries arising from all three of the above types are given in this paper.

Another important use of nonlocally related systems is the computation of *nonlocal conservation laws* of a given PDE system. A nonlocal conservation law is a conservation law whose fluxes depend on nonlocal variables, and which is not equivalent to any local conservation law of the given system.[1,15] Unlike nonlocal symmetries, nonlocal conservation laws can arise from both determined and underdetermined potential systems, as illustrated by examples in this paper.

The sections of the paper below pertain to particular examples of nonlocally related PDE systems and their applications to construction of nonlocal symmetries, nonlocal conservation laws, and exact solutions of PDE systems in $n \geq 3$ dimensions. Examples of results for multidimensional PDE systems in this paper include the following (new results are marked by an asterisk).

- A nonlocal symmetry* arising from a nonlocally related subsystem of a nonlinear PDE system in (2+1) dimensions (Sec. II).
- Nonlocal symmetries* and nonlocal conservation laws* of a nonlinear "generalized plasma equilibrium" PDE system in three space dimensions (Sec. III). [These nonlocal symmetries and nonlocal conservation laws arise from local symmetries and local conservation laws of a potential system following from a lower-degree (curl-type) conservation law.]
- Nonlocal symmetries of the linear wave equation in (2+1) dimensions,[11] arising from local symmetries of an underdetermined potential system of degree of 2, appended with a Lorentz gauge (Sec. IV). (Nonlocal conservation laws of this equation were also obtained in Ref. 11.)
- Nonlocal symmetries* of dynamic Euler equations of incompressible fluid dynamics arising from axially and helically symmetric reductions (Sec. V).
- Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations in (2+1)-dimensional Minkowski space, arising from local symmetries and local conservation laws of a determined potential system of degree 1 and an underdetermined potential system of degree of 2, appended with a Lorentz gauge.[11] Additional nonlocal conservation laws arise from local conservation laws of a potential system appended with algebraic* and divergence* gauges (Sec. VI).
- Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations in

(3+1)-dimensional Minkowski space, arising from local symmetries and local conservation laws of an underdetermined potential system of degree 2, appended with a Lorentz gauge.[12] Additional nonlocal conservation laws arise from local conservation laws of a potential system appended with algebraic* and divergence* gauges. (Sec. VII).

- Nonlocal symmetries (following from a curl-type conservation law) and exact solutions of the nonlinear three-dimensional magnetohydrodynamics (MHD) equilibrium equations[16,17] (Sec. VIII).

Finally, in Sec. IX, some open problems are discussed.

II. A NONLOCAL SYMMETRY ARISING FROM A NONLOCALLY RELATED SUBSYSTEM IN THREE DIMENSIONS

The first example illustrates the use of nonlocally related subsystems to obtain nonlocal symmetries of PDE systems in higher dimensions.

Consider the PDE system $\mathbf{UV}\{t,x,y;u,v^1,v^2\}$ in one time and two space dimensions, given by

$$\mathbf{v}_t = \text{grad } u,$$

$$u_t = K(|\mathbf{v}|)\text{div } \mathbf{v}. \tag{2.1}$$

In (2.1), $\mathbf{v}=(v^1,v^2)$ is a vector function and $K(|\mathbf{v}|)$ is a constitutive function of the indicated scalar argument. In (2.1) and throughout this paper, subscripts are used to denote the corresponding partial derivatives.

PDE system (2.1) has the nonlocally related subsystem $\mathbf{V}\{t,x,y;v^1,v^2\}$, given by

$$\mathbf{v}_{tt} = \text{grad}[K(|\mathbf{v}|)\text{div } \mathbf{v}]. \tag{2.2}$$

Consider the one-parameter class of constitutive functions given by

$$K(|\mathbf{v}|) = |\mathbf{v}|^{2m} = ((v^1)^2 + (v^2)^2)^m. \tag{2.3}$$

It is interesting to compare the symmetry classifications of systems (2.1) and (2.2) with respect to the constitutive parameter $m \neq 0$.

For arbitrary m in (2.3), one can show that the point symmetries of given PDE system (2.1) are given by the seven infinitesimal generators,

$$\mathrm{X}_1 = \frac{\partial}{\partial t}, \quad \mathrm{X}_2 = \frac{\partial}{\partial x}, \quad \mathrm{X}_3 = \frac{\partial}{\partial y}, \quad \mathrm{X}_4 = \frac{\partial}{\partial u},$$

$$\mathrm{X}_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$\mathrm{X}_6 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v^2\frac{\partial}{\partial v^1} + v^1\frac{\partial}{\partial v^2},$$

$$\mathrm{X}_7 = m\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + (m+1)u\frac{\partial}{\partial u} + v^1\frac{\partial}{\partial v^1} + v^2\frac{\partial}{\partial v^2}. \tag{2.4}$$

In contrast, subsystem (2.2) has the point symmetries given by the six infinitesimal generators,

$$\mathrm{Y}_1 = \mathrm{X}_1, \quad \mathrm{Y}_2 = \mathrm{X}_2, \quad \mathrm{Y}_3 = \mathrm{X}_3, \quad \mathrm{Y}_4 = \mathrm{X}_5, \quad \mathrm{Y}_5 = \mathrm{X}_6,$$

$$\mathrm{Y}_6 = m\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + v^1\frac{\partial}{\partial v^1} + v^2\frac{\partial}{\partial v^2}. \tag{2.5}$$

Additional point symmetries arise for (2.1) if and only if $m=-1$ and for (2.2) if and only if $m=-1,-2$. In the case $m=-1$, one can show that both systems have an infinite number of point symmetries. In the case $m=-2$, subsystem (2.2) has an additional point symmetry,

$$\mathrm{Y}_7 = t^2\frac{\partial}{\partial t} + tv^1\frac{\partial}{\partial v^1} + tv^2\frac{\partial}{\partial v^2}, \tag{2.6}$$

whereas given PDE system (2.1) still has same point symmetries (2.4). It follows that (2.6) yields a nonlocal symmetry of given PDE system $\mathbf{UV}\{t,x,y;u,v^1,v^2\}$ (2.1).

III. NONLOCAL SYMMETRIES AND NONLOCAL CONSERVATION LAWS OF A NONLINEAR PDE SYSTEM IN THREE DIMENSIONS

As a second example, consider the time-independent "generalized plasma equilibrium" PDE system $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ in three space dimensions, given by

$$\mathrm{curl}(K(|\mathbf{h}|)(\mathrm{curl}\ \ \mathbf{h}) \times \mathbf{h}) = 0, \quad \mathrm{div}\ \ \mathbf{h} = 0. \tag{3.1}$$

In (3.1), $\mathbf{h}=(h^1,h^2,h^3)$ is a vector of dependent variables. The first equation in PDE system (3.1) is a conservation law of degree one (curl-type conservation law). The corresponding potential system $\mathbf{HW}\{x,y,z;h^1,h^2,h^3,w\}$ is given by

$$K(|\mathbf{h}|)(\mathrm{curl}\ \ \mathbf{h}) \times \mathbf{h} = \mathrm{grad}\ w, \quad \mathrm{div}\ \ \mathbf{h} = 0, \tag{3.2}$$

where $w(x,y,z)$ is a scalar potential variable. Potential system (3.2) is determined and hence needs no gauge constraints.

A. Nonlocal symmetries of PDE system (3.1)

First, a comparison is made of the classifications of point symmetries of the PDE systems $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ and $\mathbf{HW}\{x,y,z;h^1,h^2,h^3,w\}$ for the one-parameter family of constitutive functions $K(|\mathbf{h}|)$ given by

$$K(|\mathbf{h}|) = |\mathbf{h}|^{2m} \equiv ((h^1)^2 + (h^2)^2 + (h^3)^2)^m, \tag{3.3}$$

where m is a parameter.

For an arbitrary m, given system $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ (3.1) has eight point symmetries, given by

$$\mathrm{X}_1 = \frac{\partial}{\partial x}, \quad \mathrm{X}_2 = \frac{\partial}{\partial y}, \quad \mathrm{X}_3 = \frac{\partial}{\partial z}, \quad \mathrm{X}_4 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + y\frac{\partial}{\partial y},$$

$$\mathrm{X}_5 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} - h^3\frac{\partial}{\partial h^1} + h^1\frac{\partial}{\partial h^3}, \quad \mathrm{X}_6 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + h^2\frac{\partial}{\partial h^1} - h^1\frac{\partial}{\partial h^2},$$

$$\mathrm{X}_7 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + h^3\frac{\partial}{\partial h^2} - h^2\frac{\partial}{\partial h^3}, \quad \mathrm{X}_8 = h^1\frac{\partial}{\partial h^1} + h^2\frac{\partial}{\partial h^2} + h^3\frac{\partial}{\partial h^3}, \tag{3.4}$$

corresponding to invariance, respectively, under three spatial translations, one dilation, three rotations, and one scaling.

For $m \neq -1$, potential system $\mathbf{HW}\{x,y,z;h^1,h^2,h^3,w\}$ (3.2) has nine point symmetries, eight of them corresponding to symmetries (3.4), plus an extra translational symmetry in the potential variable,

$$\mathrm{Y}_i = \mathrm{X}_i, \quad i = 1, \ldots, 7; \quad \mathrm{Y}_8 = \mathrm{X}_8 + 2(m+1)w\frac{\partial}{\partial w}, \quad \mathrm{Y}_9 = \frac{\partial}{\partial w}. \tag{3.5}$$

For $m=-1$, the point symmetries of $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ remain the same, whereas the potential system $\mathbf{HW}\{x,y,z;h^1,h^2,h^3,w\}$ has an additional infinite number of point symmetries given by

$$\mathrm{Y}_\infty = F(w)\left(\frac{\partial}{\partial w} + h^1\frac{\partial}{\partial h^1} + h^2\frac{\partial}{\partial h^2} + h^3\frac{\partial}{\partial h^3}\right), \tag{3.6}$$

depending on an arbitrary smooth function $F(w)$. Symmetries (3.6) are nonlocal symmetries of given PDE system $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ (3.1).

Note that symmetries (3.6) cannot be used for the construction of invariant solutions since they do not involve spatial components. However, one can use symmetries (3.6) to map any known solution of PDE system (3.1) (with a corresponding potential variable w) to an infinite family of solutions of (3.1).

B. Nonlocal conservation laws arising from potential system (3.2)

We now seek divergence-type conservation laws of PDE system $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ (3.1), using the direct method, applied first to given system (3.1) itself, and then to potential system $\mathbf{HW}\{x,y,z;h^1,h^2,h^3,w\}$ (3.2) for the one-parameter family of constitutive functions $K(|\mathbf{h}|)$ given by (3.3) for an arbitrary m. (For the details on the direct method of construction of conservation laws, see Ref. 1.)

First, we seek local divergence-type conservation laws of PDE system $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ (3.1), using multipliers of the form $\Lambda_\sigma=\Lambda_\sigma(x,y,z,H^1,H^2,H^3)$, $\sigma=1,\ldots,4$. [Here and below, to underline the fact that multipliers are sought off of the solution space of a given PDE system, the arbitrary functions corresponding to dependent variables are denoted by capitals. Then if a linear combination of equations of the system with a set of multipliers gives a divergence expression, one obtains a conservation law on solutions of the system. (For details and notation, see Ref. 1 or Ref. 15, Chap. 1.)]

From solving the corresponding set of multiplier determining equations, one finds the nontrivial conservation law multipliers given by

$$\Lambda_1 = AH^1, \quad \Lambda_2 = AH^2, \quad \Lambda_3 = AH^3, \quad \Lambda_4 = B,$$

where A, B are arbitrary constants. (In particular, the conservation law corresponding to the constant B is simply the fourth PDE div $\mathbf{h}=0$.)

Second, we apply the direct method to potential system $\mathbf{HW}\{x,y,z;h^1,h^2,h^3,w\}$ (3.2), to seek additional conservation laws of given PDE system $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ (3.1). As shown in Theorem 6.3 of Ref. 1 (see also Ref. 15, Chap. 3), in order to obtain nonlocal divergence-type conservation laws, one must seek multipliers that essentially depend on potential variables. For four equations (3.2), we seek multipliers of the form $\hat{\Lambda}_\sigma=\hat{\Lambda}_\sigma(H^1,H^2,H^3,W)$, $\sigma=1,\ldots,4$. In terms of an arbitrary function $G(W)$, one finds an infinite family of such multipliers, given by

$$\hat{\Lambda}_i = H^i G'(W), \quad i = 1,2,3, \quad \Lambda_4 = G(W),$$

with the corresponding divergence-type conservation laws given by

$$\sum_{i=1}^{3} \frac{\partial}{\partial x^i}[(G(w) + 2(m+1)wG'(w))h^i] = 0. \tag{3.7}$$

In (3.7), $(x^1,x^2,x^3)=(x,y,z)$. Conservation laws (3.7) have an evident geometrical meaning. From the vector equation in (3.2), it follows that grad w is orthogonal to $\mathbf{h}$, i.e., the vector field $\mathbf{h}$ is tangent to level surfaces $w=$const. Expression (3.7) can be rewritten as div$(M(w)\mathbf{h})$

$\equiv M'(w)\text{grad}(w)\cdot\mathbf{h}+M(w)\text{div }\mathbf{h}=0$, where $M(w)=G(w)+2(m+1)wG'(w)$, and hence is equivalent to $\text{grad}(w)\cdot\mathbf{h}=0$, provided that $M'(w)\neq 0$.

By a similar argument, it follows that PDE system $\mathbf{H}\{x,y,z;h^1,h^2,h^3\}$ (3.1) has another family of nonlocal conservation laws given by

$$\text{div}[Q(w)\text{curl }\mathbf{h}]=0 \tag{3.8}$$

for an arbitrary $Q(w)$.

IV. NONLOCAL SYMMETRIES OF THE TWO-DIMENSIONAL LINEAR WAVE EQUATION

Consider the linear wave equation $\mathbf{U}\{t,x,y;u\}$ given by

$$u_{tt}=u_{xx}+u_{yy}. \tag{4.1}$$

Equation (4.1) is a divergence-type conservation law as it stands. Following Ref. 11, we introduce a vector potential $v=(v^0,v^1,v^2)$. The resulting potential equations are underdetermined, therefore in order to seek nonlocal symmetries, a gauge constraint is needed. A Lorentz gauge is chosen since it complies with the geometrical symmetries of given PDE (4.1).[11] The resulting determined potential system $\mathbf{UV}\{t,x,y;u,v\}$ is given by

$$\begin{aligned} u_t &= v^2_x - v^1_y,\\ -u_x &= v^0_y - v^2_t,\\ -u_y &= v^1_t - v^0_x,\\ v^0_t - v^1_x - v^2_y &= 0. \end{aligned} \tag{4.2}$$

A comparison is now made of the point symmetries of PDE systems $\mathbf{U}\{t,x,y;u\}$ (4.1) and $\mathbf{UV}\{t,x,y;u,v\}$ (4.2). Modulo the infinite number of point symmetries of any linear PDE system, linear wave equation (4.1) has ten point symmetries:

- three translations $\mathrm{X}_1,\mathrm{X}_2,\mathrm{X}_3$ given by

$$\mathrm{X}_1=\frac{\partial}{\partial t},\quad \mathrm{X}_2=\frac{\partial}{\partial x},\quad \mathrm{X}_3=\frac{\partial}{\partial y};$$

- one dilation given by

$$\mathrm{X}_4=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y};$$

- one rotation and two space-time rotations (boosts) given by

$$\mathrm{X}_5=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x},\quad \mathrm{X}_6=t\frac{\partial}{\partial x}+x\frac{\partial}{\partial t},\quad \mathrm{X}_7=t\frac{\partial}{\partial y}+y\frac{\partial}{\partial t};$$

- three additional conformal transformations given by

$$\mathrm{X}_8=(t^2+x^2+y^2)\frac{\partial}{\partial t}+2tx\frac{\partial}{\partial x}+2ty\frac{\partial}{\partial y}-tu\frac{\partial}{\partial u},$$

$$\mathrm{X}_9=2tx\frac{\partial}{\partial t}+(t^2+x^2-y^2)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}-xu\frac{\partial}{\partial u},$$

$$X_{10} = 2ty\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (t^2 - x^2 + y^2)\frac{\partial}{\partial y} - yu\frac{\partial}{\partial u}.$$

Potential system $\mathbf{UV}\{t,x,y;u,v^0,v^1,v^2\}$ (4.2) has seven point symmetries $Y_1,\ldots,Y_7$ that project onto the point symmetries $X_1,\ldots,X_7$ of wave equation (4.1). However, the three additional conformal symmetries of potential system (4.2) given by

$$Y_8 = X_8 + (yv^1 - xv^2 - tu)\frac{\partial}{\partial u} - (2tv^0 + xv^1 + yv^2)\frac{\partial}{\partial v^0} - (xv^0 + 2tv^1 - yu)\frac{\partial}{\partial v^1} - (yv^0 + 2tv^2 + xu)\frac{\partial}{\partial v^2},$$

$$Y_9 = X_9 - (yv^0 + tv^2 + xu)\frac{\partial}{\partial u} - (2xv^0 + tv^1 - yu)\frac{\partial}{\partial v^0} - (tv^0 + 2xv^1 + yv^2)\frac{\partial}{\partial v^1} + (yv^1 - 2xv^2 - tu)\frac{\partial}{\partial v^2},$$

$$Y_{10} = X_{10} + (xv^0 + tv^1 - yu)\frac{\partial}{\partial u} - (2yv^0 + tv^2 + xu)\frac{\partial}{\partial v^0} - (2yv^1 - xv^2 - tu)\frac{\partial}{\partial v^1} - (tv^0 + xv^1 + 2yv^2)\frac{\partial}{\partial v^2} \quad (4.3)$$

clearly yield nonlocal symmetries of wave equation (4.1). Moreover, potential system (4.2) has three duality-type point symmetries given by

$$Y_{11} = v^0\frac{\partial}{\partial u} - u\frac{\partial}{\partial v^0} - v^2\frac{\partial}{\partial v^1} + v^1\frac{\partial}{\partial v^2},$$

$$Y_{12} = v^1\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v^0} + u\frac{\partial}{\partial v^1} + v^0\frac{\partial}{\partial v^2},$$

$$Y_{13} = v^2\frac{\partial}{\partial u} - v^1\frac{\partial}{\partial v^0} - v^0\frac{\partial}{\partial v^1} + u\frac{\partial}{\partial v^2}, \quad (4.4)$$

that also yield nonlocal symmetries of wave equation $\mathbf{U}\{t,x,y;u\}$ (4.1). In summary, potential system $\mathbf{UV}\{t,x,y;u,v^0,v^1,v^2\}$ (4.2) with the Lorentz gauge yields six nonlocal symmetries of linear wave equation (4.1).[11]

One can show that no nonlocal symmetries of the wave equation arise from the potential system $\mathbf{UV}\{t,x,y;u,v^0,v^1,v^2\}$ if the Lorentz gauge is replaced by any one of the algebraic gauges $v^k=0$ for $k \in \{0,1,2\}$, the divergence gauge, the Poincaré gauge, or the Cronstrom gauge.[15]

In Ref. 11, potential system $\mathbf{UV}\{t,x,y;u,v^0,v^1,v^2\}$ (4.2) was used to obtain additional (nonlocal) conservation laws of wave equation $\mathbf{U}\{t,x,y;u\}$ (4.1).

V. NONLOCAL SYMMETRIES OF THE EULER EQUATIONS

Consider the Euler equations describing the motion for an incompressible inviscid fluid in $\mathbb{R}^3$, which in Cartesian coordinates are given by

$$\operatorname{div}\ \mathbf{u} = 0, \quad (5.1a)$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \text{grad } p = 0, \tag{5.1b}$$

where the fluid velocity vector $\mathbf{u}=u^1\mathbf{e}_x+u^2\mathbf{e}_y+u^3\mathbf{e}_z$ and fluid pressure p are functions of x,y,z,t.

The *fluid vorticity* is a local vector variable defined by

$$\boldsymbol{\omega} = \text{curl } \mathbf{u}. \tag{5.2}$$

Using the vector calculus identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \text{grad}\frac{|\mathbf{u}|^2}{2} + (\text{curl } \mathbf{u}) \times \mathbf{u},$$

vector momentum equation (5.1b) can be rewritten as

$$\mathbf{u}_t + \boldsymbol{\omega} \times \mathbf{u} + \text{grad}\left(p + \frac{|\mathbf{u}|^2}{2}\right) = 0. \tag{5.3}$$

One can construct a *vorticity subsystem* of PDE system (5.1) by taking the curl of Eq. (5.3),

$$\text{div } \mathbf{u} = 0, \tag{5.4a}$$

$$\boldsymbol{\omega}_t + \text{curl}(\boldsymbol{\omega} \times \mathbf{u}) = 0, \tag{5.4b}$$

$$\boldsymbol{\omega} = \text{curl } \mathbf{u}. \tag{5.4c}$$

PDE system (5.4) is nonlocally related to Euler equations (5.1). By definition, Euler system (5.1) is a potential system of PDE system (5.4) following from curl-type (degree one) conservation law (5.4b).

Below we compare point symmetries of Euler equations and the vorticity subsystem in two symmetric settings.[26]

A. Axially symmetric case

Rewriting Euler equations (5.1) in cylindrical coordinates (r,z,φ) with

$$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\varphi + w\mathbf{e}_z,$$

we restrict the dependence of each of u,v,w,p to the coordinates t,r,z only due to the invariance of the Euler equations under (azimuthal) rotations in φ. Consequently, one obtains the reduced axially symmetric Euler system $\mathbf{AE}\{t,r,z;u,v,w,p\}$ given by

$$u_r + \frac{1}{r}u + \frac{1}{r}v_\varphi + w_z = 0, \tag{5.5a}$$

$$u_t + uu_r + wu_z - \frac{1}{r}v^2 + p_r = 0, \tag{5.5b}$$

$$v_t + uv_r + wv_z + \frac{1}{r}uv = 0, \tag{5.5c}$$

$$w_t + uw_r + ww_z + \frac{1}{r}p_z = 0. \tag{5.5d}$$

In terms of cylindrical coordinates, the vorticity is represented in the form

$$\boldsymbol{\omega} = m\mathbf{e}_r + n\mathbf{e}_\varphi + q\mathbf{e}_z.$$

Using the invariance of (5.2) under the same azimuthal rotations, we again assume axial symmetry and rewrite three scalar equations (5.2) as

$$m + v_z = 0, \quad n - u_z + w_r = 0, \quad q - \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0, \tag{5.6}$$

where m,n,q are functions of t,r,z.

Combining PDE systems (5.5) and (5.6), we obtain the PDE system $\mathbf{AEW}\{t,r,z;u,v,w,p,m,n,q\}$, which is obviously locally related to the axially symmetric Euler system $\mathbf{AE}\{t,r,z;u,v,w,p\}$ (5.5), since vorticity components are local variables in terms of u,v,w.

The point symmetries of the system $\mathbf{AEW}\{t,r,z;u,v,w,p,m,n,q\}$ are given by

$$\mathrm{X}_1 = \frac{\partial}{\partial t},$$

$$\mathrm{X}_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z} - m\frac{\partial}{\partial m} - n\frac{\partial}{\partial n} - q\frac{\partial}{\partial q},$$

$$\mathrm{X}_3 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} + 2p\frac{\partial}{\partial p},$$

$$\mathrm{X}_4 = F(t)\frac{\partial}{\partial z} + F'(t)\frac{\partial}{\partial w} - zF''(t)\frac{\partial}{\partial p},$$

$$\mathrm{X}_5 = G(t)\frac{\partial}{\partial p},$$

$$\mathrm{X}_6 = \frac{1}{r^2v^2}\left(-v\frac{\partial}{\partial v} + v^2\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} + m\frac{\partial}{\partial m}\right), \tag{5.7}$$

in terms of arbitrary functions $F(t)$ and $G(t)$. Point symmetries (5.7) correspond to the invariance of reduced system $\mathbf{AE}\{t,r,z;u,v,w,p\}$ (5.5) under time translations, two scalings, Galilean invariance in z, pressure invariance, and the additional symmetry X_6 which corresponds to the an introduction of a vortex at the origin given by

$$(v')^2 = v^2 + \frac{2C}{r^2}, \quad p' = p - \frac{C}{r^2}, \quad C = \text{const}.$$

Now consider vorticity subsystem (5.4). Under the assumption of axial symmetry, it is denoted by $\mathbf{AW}\{t,r,z;u,v,w,m,n,q\}$ and given by

$$u_r + \frac{1}{r}u + \frac{1}{r}v_\varphi + w_z = 0, \tag{5.8a}$$

$$m_t + \frac{\partial}{\partial z}(wm - uq) = 0, \tag{5.8b}$$

$$n_t + \frac{\partial}{\partial r}(un - vm) + \frac{\partial}{\partial z}(nw - vq) = 0, \tag{5.8c}$$

$$q_t + \frac{1}{r}\frac{\partial}{\partial r}(r(uq - wm)) = 0, \tag{5.8d}$$

$$m + v_z = 0, \quad n - u_z + w_r = 0, \quad q - \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0. \tag{5.8e}$$

PDE system $\mathbf{AW}\{t,r,z;u,v,w,m,n,q\}$ (5.8) is a nonlocally related subsystem of the Euler reduced system with vorticity $\mathbf{AEW}\{t,r,z;u,v,w,p,m,n,q\}$ (5.5) and (5.6), and hence is nonlocally related to Euler reduced system $\mathbf{AE}\{t,r,z;u,v,w,p\}$ (5.5).

One can show that the point symmetries of system $\mathbf{AW}\{t,r,z;u,v,w,m,n,q\}$ (5.8) are given by

$$\mathrm{Y}_1 = \mathrm{X}_1, \quad \mathrm{Y}_2 = \mathrm{X}_2, \quad \mathrm{Y}_3 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} \sim \mathrm{X}_3,$$

$$\mathrm{Y}_4 = F(t)\frac{\partial}{\partial z} + F'(t)\frac{\partial}{\partial w} \sim \mathrm{X}_4, \tag{5.9}$$

in terms of an arbitrary function $F(t)$. It follows that the symmetry X_6 in (5.7), which is a point symmetry of PDE systems $\mathbf{AE}\{t,r,z;u,v,w,p\}$ and $\mathbf{AEW}\{t,r,z;u,v,w,p,m,n,q\}$, yields a nonlocal symmetry of the vorticity subsystem $\mathbf{AW}\{t,r,z;u,v,w,m,n,q\}$..

B. Helically symmetric case

Now consider helical coordinates (r,η,ξ) in $\mathbb{R}^3$,

$$\xi = az + b\varphi, \quad \eta = a\varphi - bz/r^2, \quad a,b = \text{const}, \quad a^2 + b^2 > 0.$$

In helical coordinates, r is the cylindrical radius; each helix is defined by r=const, ξ=const; η is a variable along a helix.

In a helically symmetric setting, the velocity and vorticity vectors are given by

$$\mathbf{u} = u^r\mathbf{e}_r + u^\eta\mathbf{e}_\eta + u^\xi\mathbf{e}_\xi, \quad \boldsymbol{\omega} = \omega^r\mathbf{e}_r + \omega^\eta\mathbf{e}_\eta + \omega^\xi\mathbf{e}_\xi,$$

where the vector components as well as the pressure p are functions of t,r,ξ. (Note that in the limit a=1, b=0, helical coordinates become cylindrical coordinates with $\eta=\varphi$, $\xi=z$.)

Rewriting Euler equations (5.1) in helical coordinates and imposing helical symmetry (independence of η),[18] one obtains the reduced helically symmetric PDE system $\mathbf{HE}\{t,r,\xi;u^r,u^\eta,u^\xi,p\}$, given by

$$\frac{u^r}{r} + \frac{\partial u^r}{\partial r} + \frac{1}{B(r)}\frac{\partial u^\xi}{\partial \xi} = 0, \tag{5.10a}$$

$$(u^r)_t + u^r(u^r)_r + \frac{1}{B(r)}u^\xi(u^r)_\xi - \frac{B^2(r)}{r}\left(\frac{b}{r}u^\xi + au^\eta\right)^2 + p_r = 0, \tag{5.10b}$$

$$(u^\eta)_t + u^r(u^\eta)_r + \frac{1}{B(r)}u^\xi(u^\eta)_\xi + \frac{a^2B^2(r)}{r}u^r u^\eta = 0, \tag{5.10c}$$

$$(u^{\xi})_t + u^r(u^{\xi})_r + \frac{1}{B(r)}u^{\xi}(u^{\xi})_{\xi} + \frac{2abB^2(r)}{r^2}u^r u^{\eta} + \frac{b^2B^2(r)}{r^3}u^r u^{\xi} + \frac{1}{B(r)}p_{\xi} = 0. \tag{5.10d}$$

In (5.10),

$$B(r) = \frac{r}{\sqrt{a^2r^2+b^2}}.$$

The helically symmetric version of (5.2) is given by the three scalar equations,

$$\omega^r = -\frac{(u^{\eta})_{\xi}}{B(r)}, \tag{5.11a}$$

$$\omega^{\eta} = -\frac{1}{r}\frac{\partial}{\partial r}(ru^{\xi}) - 2\frac{abB^2(r)}{r^2}u^{\eta} + \frac{a^2B^2(r)}{r}u^{\xi} + \frac{1}{B(r)}(u^r)_{\xi}, \tag{5.11b}$$

$$\omega^{\xi} = \frac{a^2B^2(r)}{r}u^{\eta} + (u^{\eta})_r. \tag{5.11c}$$

One can consider the system $\mathbf{HEW}\{t,r,\xi;u^r,u^{\eta},u^{\xi},\omega^r,\omega^{\eta},\omega^{\xi}\}$, given by the combination of PDE systems (5.10) and (5.11). This PDE system is locally related to helically symmetric Euler system $\mathbf{HE}\{t,r,\xi;u^r,u^{\eta},u^{\xi},p\}$ (5.10), since vorticity components are local functions of velocity components, their derivatives, and independent variables.

The point symmetries of system $\mathbf{HEW}\{t,r,\xi;u^r,u^{\eta},u^{\xi},\omega^r,\omega^{\eta},\omega^{\xi}\}$ (5.10) and (5.11) are given by

$$\mathrm{X}_1 = \frac{\partial}{\partial t}, \quad \mathrm{X}_2 = \frac{\partial}{\partial \xi},$$

$$\mathrm{X}_3 = t\frac{\partial}{\partial t} - u^r\frac{\partial}{\partial u^r} - u^{\eta}\frac{\partial}{\partial u^{\eta}} - u^{\xi}\frac{\partial}{\partial u^{\xi}} - 2p\frac{\partial}{\partial p} - \omega^r\frac{\partial}{\partial \omega^r} - \omega^{\eta}\frac{\partial}{\partial \omega^{\eta}} - \omega^{\xi}\frac{\partial}{\partial \omega^{\xi}},$$

$$\mathrm{X}_4 = t\frac{\partial}{\partial \xi} - \frac{bB(r)}{ar}\frac{\partial}{\partial u^{\eta}} + B(r)\frac{\partial}{\partial u^{\xi}},$$

$$\mathrm{X}_5 = F(t)\frac{\partial}{\partial p}, \tag{5.12}$$

in terms of an arbitrary function $F(t)$. Due to the local relation, point symmetries of helically symmetric Euler system $\mathbf{HE}\{t,r,\xi;u^r,u^{\eta},u^{\xi},p\}$ (5.10) are given by projections of symmetries (5.12) onto the space of variables $t,r,\xi,\ u^r,u^{\eta},u^{\xi},p$.

The corresponding helically symmetric version of vorticity subsystem (5.4), where pressure has been excluded through the application of a curl, is denoted by $\mathbf{HW}\{t,r,\xi;u^r,u^{\eta},u^{\xi},\omega^r,\omega^{\eta},\omega^{\xi}\}$ and given by

$$\frac{u^r}{r} + \frac{\partial u^r}{\partial r} + \frac{1}{B(r)}\frac{\partial u^{\xi}}{\partial \xi} = 0, \tag{5.13a}$$

$$(\omega^r)_t + \frac{1}{B(r)}\frac{\partial}{\partial \xi}(u^{\xi}\omega^r - u^r\omega^{\xi}) = 0, \tag{5.13b}$$

$$(\omega^\eta)_t + \frac{1}{r}\frac{\partial}{\partial r}(r(u^r\omega^\eta - u^\eta\omega^r)) - \frac{a^2B^2(r)}{r}(u^r\omega^\eta - u^\eta\omega^r) + \frac{1}{B(r)}(u^\xi\omega^\eta - u^\eta\omega^\xi)$$
$$+ \frac{2abB^2(r)}{r^2}(u^\xi\omega^r - u^r\omega^\xi) = 0, \tag{5.13c}$$

$$(\omega^\xi)_t + \frac{\partial}{\partial r}(u^r\omega^\xi - u^\xi\omega^r) + \frac{a^2B^2(r)}{r}(u^r\omega^\xi - u^\xi\omega^r) = 0, \tag{5.13d}$$

$$\omega^r = -\frac{(u^\eta)_\xi}{B(r)}, \tag{5.13e}$$

$$\omega^\eta = -\frac{1}{r}\frac{\partial}{\partial r}(ru^\xi) - 2\frac{abB^2(r)}{r^2}u^\eta + \frac{a^2B^2(r)}{r}u^\xi + \frac{1}{B(r)}(u^r)_\xi, \tag{5.13f}$$

$$\omega^\xi = \frac{a^2B^2(r)}{r}u^\eta + (u^\eta)_r. \tag{5.13g}$$

Its point symmetries are given by

$$\mathrm{Y}_1 = \mathrm{X}_1, \quad \mathrm{Y}_2 = \mathrm{X}_3 - \omega^r\frac{\partial}{\partial\omega^r} - \omega^\eta\frac{\partial}{\partial\omega^\eta} - \omega^\xi\frac{\partial}{\partial\omega^\xi},$$

$$\mathrm{Y}_3 = G(t)\frac{\partial}{\partial\xi} - \frac{bB(r)}{ar}G'(t)\frac{\partial}{\partial u^\eta} + B(r)G'(t)\frac{\partial}{\partial u^\xi}. \tag{5.14}$$

in terms of an arbitrary function $G(t)$. [Note that symmetries $\mathrm{X}_2, \mathrm{X}_4$ (5.12) are special cases of the infinite family of symmetries Y_3.]

Comparing symmetry classifications (5.12) and (5.14), one observes that the full Galilei group in the direction of ξ only occurs as a point symmetry of reduced vorticity subsystem $\mathbf{HW}\{t,r,\xi;u^r,u^\eta,u^\xi,\omega^r,\omega^\eta,\omega^\xi\}$ (5.13), and thus yields a nonlocal symmetry of helically symmetry reduced Euler system $\mathbf{HE}\{t,r,\xi;u^r,u^\eta,u^\xi,p\}$ (5.10).

VI. NONLOCAL SYMMETRIES AND NONLOCAL CONSERVATION LAWS OF MAXWELL'S EQUATIONS IN (2+1) DIMENSIONS

The linear system of Maxwell's equations in a vacuum in three space dimensions is given by

$$\operatorname{div}\ \mathbf{B} = 0, \quad \operatorname{div}\ \mathbf{E} = 0,$$

$$\mathbf{E}_t = \operatorname{curl}\ \mathbf{B}, \quad \mathbf{B}_t = -\operatorname{curl}\ \mathbf{E}, \tag{6.1}$$

where $\mathbf{B}=B^1\mathbf{e}_x+B^2\mathbf{e}_y+B^3\mathbf{e}_z$ is a magnetic field, $\mathbf{E}=E^1\mathbf{e}_x+E^2\mathbf{e}_y+E^3\mathbf{e}_z$ is an electric field, (x,y,z) are Cartesian coordinates, and t is time.

Following Ref. 11, we consider PDE system (6.1) in three-dimensional Minkowski space (t,x,y). It is assumed that $\mathbf{B}=B(x,y)\mathbf{e}_z$, $\mathbf{E}=E^1(x,y)\mathbf{e}_x+E^2(x,y)\mathbf{e}_y$. Then Maxwell's equations (6.1) can be written as the PDE system $\mathbf{M}\{t,x,y;B,E^1,E^2\}$ in terms of the four equations given by

$$R^1[e^1,e^2,b] = e^1_x + e^2_y = 0, \quad R^2[e^1,e^2,b] = e^1_t - b_y = 0,$$

$$R^3[e^1,e^2,b] = e^2_t + b_x = 0, \quad R^4[e^1,e^2,b] = b_t + e^2_x - e^1_y = 0. \tag{6.2}$$

We now seek nonlocal symmetries and nonlocal conservation laws of PDE system (6.2). Following the systematic procedure described in Ref. 1, we first construct potential systems for PDE system (6.2). Note that each of the four equations in (6.2) is a divergence expression as it stands. Hence for each equation in (6.2), one can introduce a three-component vector potential. This yields 12 potential variables. From Theorem 6.1 in Ref. 1, it follows that in order to obtain nonlocal symmetries of Maxwell's equations (6.2), gauge constraints are required. Since the form of gauge constraints that could yield nonlocal symmetries is not known *a priori*, a different approach is chosen. In particular, the system of Maxwell's equations (6.2) is equivalent to the union of a divergence-type conservation law and a curl-type lower-degree conservation law, with the latter requiring no gauge constraints.[1,11] In particular, considering the electromagnetic field tensors,

$$F_{ij} = \begin{pmatrix} 0 & -e^1 & -e^2 \\ e^1 & 0 & b \\ e^2 & -b & 0 \end{pmatrix}, \quad F^{ij} = \begin{pmatrix} 0 & e^1 & e^2 \\ -e^1 & 0 & b \\ -e^2 & -b & 0 \end{pmatrix}, \tag{6.3}$$

and the dual tensor of F_{ij}, given by $*F_k = \frac{1}{2}\varepsilon_{ijk}F^{ij}$, where ε_{ijk} is the Levi–Civita symbol, one can rewrite Maxwell's equations (6.2) as

$$\mathrm{d}F = 0, \quad \mathrm{d} * F = 0, \tag{6.4}$$

where the differential forms are given, respectively, by

$$F = -e^1 \mathrm{d}t \wedge \mathrm{d}x - e^2 \mathrm{d}t \wedge \mathrm{d}y + b\mathrm{d}x \wedge \mathrm{d}y, \quad * F = bdt - e^2 dx + e^1 dy.$$

If the three-dimensional Minkowski space (t,x,y) is treated as $\mathbb{R}^3$, Eqs. (6.4) can be written in the conserved form $\mathbf{M}\{t,x,y;e^1,e^2,b\}$,

$$\mathrm{div}_{(t,x,y)}[b,e^2,-e^1] = 0, \quad \mathrm{curl}_{(t,x,y)}[b,-e^2,e^1] = 0. \tag{6.5}$$

Using the curl-type conservation law in (6.5), one obtains a determined singlet potential system $\mathbf{MW}\{t,x,y;b,e^1,e^2,w\}$ given by

$$\begin{gathered} b = w_t, \quad -e^2 = w_x, \\ e^1 = w_y, \quad b_t + e^2_x - e^1_y = 0. \end{gathered} \tag{6.6}$$

Using the divergence-type conservation law in (6.5), one introduces a vector potential variable $a=(a^0,a^1,a^2)$ to obtain the underdetermined singlet potential system $\mathbf{MA}\{t,x,y;b,e^1,e^2,a\}$ given by

$$\begin{gathered} b = a^2_x - a^1_y, \quad e^2 = a^0_y - a^2_t, \\ -e^1 = a^1_t - a^0_x, \quad e^1_x + e^2_y = 0, \\ e^1_t - b_y = 0, \quad e^2_t + b_x = 0, \\ a^0_t - a^1_x - a^2_y = 0, \end{gathered} \tag{6.7}$$

appended by a Lorentz gauge for determinedness.

From singlet potential systems (6.6) and (6.7), one obtains the couplet potential system $\mathbf{MAW}\{t,x,y;a,w\}$ given by

$$w_t = a^2_x - a^1_y, \quad -w_x = a^0_y - a^2_t,$$

$$-w_y = a^1_t - a^0_x, \quad a^0_t - a^1_x - a^2_y = 0, \tag{6.8}$$

where the components of the electric and magnetic fields have been excluded through appropriate substitutions.

The corresponding tree of nonlocally related PDE systems for given PDE system $\mathbf{M}\{t,x,y;e^1,e^2,b\}$ (6.5) was presented in Fig. 1 in Ref. 1.

A. Nonlocal symmetries of Maxwell's equations (6.2)

Maxwell's equations (6.2) have eight point symmetries: three translations, one rotation, two space-time rotations (boosts), one dilation, and one scaling, given by the infinitesimal generators,

$$\mathrm{X}_1 = \frac{\partial}{\partial t}, \quad \mathrm{X}_2 = \frac{\partial}{\partial x}, \quad \mathrm{X}_3 = \frac{\partial}{\partial y}, \quad \mathrm{X}_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - e^2\frac{\partial}{\partial e^1} + e^1\frac{\partial}{\partial e^2},$$

$$\mathrm{X}_5 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + b\frac{\partial}{\partial e^2} + e^2\frac{\partial}{\partial b}, \quad \mathrm{X}_6 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} - b\frac{\partial}{\partial e^1} - e^1\frac{\partial}{\partial b},$$

$$\mathrm{X}_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad \mathrm{X}_8 = e^1\frac{\partial}{\partial e^1} + e^2\frac{\partial}{\partial e^2} + b\frac{\partial}{\partial b}. \tag{6.9}$$

We now seek nonlocal symmetries of PDE system (6.2) that arise as point symmetries of its potential systems. As discussed in Ref. 1, nonlocal symmetries can only arise from a potential system if the latter is determined. The point symmetries of determined singlet potential systems $\mathbf{MW}\{t,x,y;b,e^1,e^2,w\}$ (6.6), $\mathbf{MA}\{t,x,y;b,e^1,e^2,a\}$ (6.7), and determined couplet potential system $\mathbf{MAW}\{t,x,y;a,w\}$ (6.8) are as follows.

Potential system $\mathbf{MW}\{t,x,y;b,e^1,e^2,w\}$ (6.6) has eight point symmetries that project onto point symmetries (6.9) of PDE system (6.2), plus three additional conformal-type point symmetries given by

$$\mathrm{W}_1 = (t^2+x^2+y^2)\frac{\partial}{\partial t} + 2tx\frac{\partial}{\partial x} + 2ty\frac{\partial}{\partial y} - (3te^1+2yb)\frac{\partial}{\partial e^1}$$
$$- (3te^2-2xb)\frac{\partial}{\partial e^2} - (2ye^1-2xe^2+3tb+w)\frac{\partial}{\partial b} - tw\frac{\partial}{\partial w},$$

$$\mathrm{W}_2 = 2tx\frac{\partial}{\partial t} + (t^2+x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} - (3xe^1+2ye^2)\frac{\partial}{\partial e^1}$$
$$+ (2ye^1-3xe^2+2tb+w)\frac{\partial}{\partial e^2} + (2te^2-3xb)\frac{\partial}{\partial b} - xw\frac{\partial}{\partial w},$$

$$\mathrm{W}_3 = 2ty\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (t^2-x^2+y^2)\frac{\partial}{\partial y} - (3ye^1-2xe^2+2tb+w)\frac{\partial}{\partial e^1}$$
$$- (2xe^1+3ye^2)\frac{\partial}{\partial e^2} - (2te^1+3yb)\frac{\partial}{\partial b} - yw\frac{\partial}{\partial w}, \tag{6.10}$$

that yield nonlocal symmetries of Maxwell's equations (6.2).

Potential system $\mathbf{MA}\{t,x,y;b,e^1,e^2,a\}$ (6.7) has five point symmetries. They project onto point symmetries X_i, $i=1,2,3,7,8$ (6.9) of Maxwell's equations (6.2).

Couplet potential system $\mathbf{MAW}\{t,x,y;a,w\}$ (6.8) is potential system (4.2) for the wave equation (with $w=u$, $a^i=v^i$). Hence it has duality-type symmetries (4.4). In particular, one can write them as first-order symmetries,

$$\mathrm{Z}_1 = a_t^0\frac{\partial}{\partial b} + a_y^0\frac{\partial}{\partial e^1} - a_x^0\frac{\partial}{\partial e^2} + a^0\frac{\partial}{\partial w} - w\frac{\partial}{\partial a^0} - a^2\frac{\partial}{\partial a^1} + a^1\frac{\partial}{\partial a^2},$$

$$\mathrm{Z}_2 = a_t^1\frac{\partial}{\partial b} + a_y^1\frac{\partial}{\partial e^1} - a_x^1\frac{\partial}{\partial e^2} + a^1\frac{\partial}{\partial w} + a^2\frac{\partial}{\partial a^0} + w\frac{\partial}{\partial a^1} + a^0\frac{\partial}{\partial a^2},$$

$$\mathrm{Z}_3 = a_t^2\frac{\partial}{\partial b} + a_y^2\frac{\partial}{\partial e^1} - a_x^2\frac{\partial}{\partial e^2} + a^2\frac{\partial}{\partial w} - a^1\frac{\partial}{\partial a^0} - a^0\frac{\partial}{\partial a^1} + w\frac{\partial}{\partial a^2}. \tag{6.11}$$

Symmetries (6.11) yield three additional nonlocal symmetries of Maxwell's equations (6.2).[11]

B. Nonlocal conservation laws of Maxwell's equations (6.2)

(A) The potential system $\mathbf{MAW}\{t,x,y;a,w\}$ *with the Lorentz gauge*. Potential system $\mathbf{MAW}\{t,x,y;a,w\}$ (6.8) with the Lorentz gauge was used in Ref. 11 to obtain additional conservation laws with explicit dependence of the multipliers on potential variables. As an example, consider a linear combination of the equations of (6.8) with multipliers depending only on potential variables and their derivatives: $\Lambda_\sigma(A,W,\partial A,\partial W)$, $\sigma=1,\ldots,4$. The solution of the corresponding determining equations[1] yields eight sets of nontrivial multipliers given by

$$\Lambda_1 = C_1W + C_2A^1 + C_3A^2 + C_4A_t^0 + C_5,$$

$$\Lambda_2 = C_1A^2 + C_2A^0 + C_3W + C_4A_t^1 + C_6,$$

$$\Lambda_3 = -C_1A^1 - C_2W + C_3A^0 + C_4A_t^2 + C_7,$$

$$\Lambda_4 = C_1A^0 + C_2A^2 - C_3A^1 - C_4W_t + C_8,$$

where $C_1,\ldots,C_8$ are arbitrary constants. The constants $C_5,\ldots,C_8$ simply yield four divergence expressions (6.8), whereas the constants $C_1,\ldots,C_4$ yield conservation laws,

$$\frac{1}{2}\frac{\partial}{\partial t}(w^2+(a^0)^2+(a^1)^2+(a^2)^2) + \frac{\partial}{\partial x}(-a^0a^1-a^2w) + \frac{\partial}{\partial y}(a^1w-a^0a^2)=0,$$

$$\frac{\partial}{\partial t}(a^2w-a^0a^1) + \frac{1}{2}\frac{\partial}{\partial x}(-w^2+(a^0)^2+(a^1)^2-(a^2)^2) + \frac{\partial}{\partial y}(a^1a^2-a^0w)=0,$$

$$\frac{\partial}{\partial t}(a^1w+a^0a^2) + \frac{\partial}{\partial x}(-a^1a^2-a^0w) + \frac{1}{2}\frac{\partial}{\partial y}(-w^2-(a^0)^2+(a^1)^2+(a^2)^2)=0,$$

$$\frac{\partial}{\partial t}(wa_t^0-a^0w_t-a^1a_t^2+a^2a_t^1) + \frac{\partial}{\partial x}(a^1w_t-wa_t^1+a^0a_t^2-a^2a_t^0)$$
$$+\frac{\partial}{\partial y}(a^2w_t-wa_t^2+a^1a_t^0-a^0a_t^1)=0. \tag{6.12}$$

Since the fluxes in conservation laws (6.12) explicitly involve potential variables (and not the combinations of derivatives of potential variables which are identified with the given dependent variables b,e^1,e^2 through potential equations), conservation laws (6.12) yield four nonlocal conservation laws of Maxwell's equations (6.2).

(B) The potential system $\mathbf{MAW}\{t,x,y;a,w\}$ *with an algebraic gauge*. Now consider the potential system $\mathbf{MAW}\{t,x,y;a,w\}$,

$$w_t = a^2_x - a^1_y, \quad -w_x = a^0_y - a^2_t,$$

$$-w_y = a^1_t - a^0_x, \quad a^2 = 0, \tag{6.13}$$

which has the algebraic (spatial) gauge $a^2=0$ instead of the Lorentz gauge. One can show that choosing multipliers

$$\Lambda_1 = A^1, \quad \Lambda_2 = A^0, \quad \Lambda_3 = -W, \quad \Lambda_4 = 0,$$

one obtains an additional nonlocal conservation law of Maxwell's equations (6.2) given by

$$\frac{\partial}{\partial t}(a^1 w) + \frac{\partial}{\partial x}(a^0 w) + \frac{1}{2}\frac{\partial}{\partial y}(w^2 - (a^0)^2 + (a^1)^2) = 0. \tag{6.14}$$

Using respectively the algebraic gauges $a^0=0$ and $a^1=0$, one obtains two further nonlocal conservation laws of Maxwell's equations (6.2).

(C) The potential system $\mathbf{MAW}\{t,x,y;a,w\}$ *with the divergence gauge*. Now consider the potential system $\mathbf{MAW}\{t,x,y;a,w\}$ with the divergence gauge, given by

$$w_t = a^2_x - a^1_y, \quad -w_x = a^0_y - a^2_t,$$

$$-w_y = a^1_t - a^0_x, \quad a^0_t + a^1_x + a^2_y = 0. \tag{6.15}$$

We again seek conservation law multipliers depending only on potential variables and their derivatives: $\Lambda_\sigma(A,W,\partial A,\partial W)$, $\sigma=1,\ldots,4$. One can obtain an additional divergence-type conservation law

$$\frac{1}{2}\frac{\partial}{\partial t}(w^2 - (a^0)^2 + (a^1)^2 + (a^2)^2) + \frac{\partial}{\partial x}(-a^2 w - a^0 a^1) + \frac{\partial}{\partial y}(a^1 w - a^0 a^2) = 0 \tag{6.16}$$

following from the set of multipliers

$$\Lambda_1 = W, \quad \Lambda_2 = A^2, \quad \Lambda_3 = -A^1, \quad \Lambda_4 = A^0,$$

which yields a nonlocal conservation law of Maxwell's equations (6.2).

(D) Other gauges. One can directly show that for conservation law multipliers depending on potential variables and their first derivatives, no additional conservation laws arise for the potential system $\mathbf{MAW}\{t,x,y;a,w\}$ with Cronstrom or Poincaré gauges. Other gauges have not been examined.

VII. NONLOCAL SYMMETRIES AND NONLOCAL CONSERVATION LAWS OF MAXWELL'S EQUATIONS IN (3+1) DIMENSIONS

Now consider Maxwell's equations $\mathbf{M}\{t,x,y,z;e,b\}$ (6.1) in four-dimensional Minkowski space-time $(x^0,x^1,x^2,x^3)=(t,x,y,z)$.

As it is written, each of the eight equations in (6.1) is a divergence-type conservation law. As per Table II in Ref. 1 in $n=4$ dimensions, each divergence-type conservation law gives rise to $n(n-1)/2=6$ potential variables, i.e, if one directly uses all equations of (6.1) to introduce potentials, one obtains 48 scalar potential variables, and a highly underdetermined potential system.

Instead of using divergence-type conservation laws, lower-degree conservation laws can be effectively used, as follows.[12]

In four-dimensional Minkowski space-time, the 4×4 metric tensor is given by $\eta^{\mu\nu}=\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$; $\mu,\nu=0,1,2,3$. The electromagnetic field tensor $F_{\mu\nu}$ and its dual $*F_{\mu\nu}$ are given by the matrices

$$F_{\mu\nu}=\begin{pmatrix} 0 & -e^1 & -e^2 & -e^3 \\ e^1 & 0 & b^3 & -b^2 \\ e^2 & -b^3 & 0 & b^1 \\ e^3 & b^2 & -b^1 & 0 \end{pmatrix}, \quad *F_{\mu\nu}=\begin{pmatrix} 0 & b^1 & b^2 & b^3 \\ -b^1 & 0 & e^3 & -e^2 \\ -b^2 & -e^3 & 0 & e^1 \\ -b^3 & e^2 & -e^1 & 0 \end{pmatrix},$$

where the dual is defined by $*F_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta}$ and $\varepsilon_{\mu\nu\alpha\beta}$ is the four-dimensional Levi–Civita symbol. (In this section, Greek indices are assumed to take on the values 0,1,2,3, whereas Latin indices take on the values 1,2,3 and correspond to spatial coordinates.)

Through use of the differential 2-forms $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$, $*F=*F_{\mu\nu}dx^\mu\wedge dx^\nu$, Maxwell's equations (6.1) can be written as two conservation laws of degree 2,

$$\mathrm{d}F=0, \quad \mathrm{d}*F=0. \tag{7.1}$$

In particular, the equation $\mathrm{d}F=0$ is equivalent to the four scalar equations div $\mathbf{B}=0$, $\mathbf{B}_t=-$curl $\mathbf{E}$, and the equation $\mathrm{d}*F=0$ is equivalent to the remaining four equations of (6.1).

Using Poincaré's lemma, one introduces the magnetic potential a and the electric potential c,

$$F=\mathrm{d}a, \quad *F=\mathrm{d}c, \tag{7.2}$$

where a and c are four-component one-forms $a=a_\mu \mathrm{d}x^\mu$, $c=c_\mu \mathrm{d}x^\mu$ (a total of eight scalar potential variables).

The corresponding determined singlet potential system $\mathbf{MA}\{t,x,y,z;e,b,a\}$ is given by

$$\begin{gathered} \mathrm{div}\ \mathbf{E}=0, \quad \mathrm{E}_t=\mathrm{curl}\ \mathbf{B}, \\ e^1=a^0_x-a^1_t, \quad e^2=a^0_y-a^2_t, \\ e^3=a^0_z-a^3_t, \quad b^1=a^3_y-a^2_z, \\ b^2=a^1_z-a^3_x, \quad b^3=a^2_x-a^1_y, \end{gathered} \tag{7.3}$$

the determined singlet potential system $\mathbf{MC}\{t,x,y,z;e,b,c\}$ is given by

$$\begin{gathered} \mathrm{div}\ \mathbf{B}=0, \quad \mathbf{B}_t=-\mathrm{curl}\ \mathbf{E}, \\ e^1=c^3_y-c^2_z, \quad e^2=c^1_z-c^3_x, \\ e^3=c^2_x-c^1_y, \quad b^1=c^1_t-c^0_x, \\ b^2=c^2_t-c^0_y, \quad b^3=c^3_t-c^0_z, \end{gathered} \tag{7.4}$$

and the determined couplet potential system $\mathbf{AC}\{t,x,y,z;a,c\}$ is given by

$$a_y^3 - a_z^2 = c_t^1 - c_x^0, \quad a_z^1 - a_x^3 = c_t^2 - c_y^0,$$

$$a_x^2 - a_y^1 = c_t^3 - c_z^0, \quad a_x^0 - a_t^1 = c_y^3 - c_z^2$$

$$a_y^0 - a_t^2 = c_z^1 - c_x^3, \quad a_z^0 - a_t^3 = c_x^2 - c_y^1, \tag{7.5}$$

where electric and magnetic field components have been excluded through substitutions.

The above potential systems are underdetermined. In particular, both a and c are defined to within arbitrary four-dimensional gradients. It is natural to use Lorentz gauges for these potentials due to the Minkowski geometry, as well as the symmetry and linearity of Maxwell's equations (6.2). In Sec. VII B, we will show that other gauges are also useful for finding nonlocal conservation laws.

A. Nonlocal symmetries

Consider the determined potential system which consists of six PDEs (7.5) appended by Lorentz gauges,

$$a_t^0 - a_x^1 - a_y^2 - a_z^3 = 0, \quad c_t^0 - c_x^1 - c_y^2 - c_z^3 = 0. \tag{7.6}$$

This appended potential system has 23 point symmetries including four space-time translations, one dilation, six rotations/boosts, six internal rotations/boosts, one scaling, one duality-rotation, and four conformal symmetries. In particular, the conformal symmetries,

$$\begin{aligned}
\mathrm{X}_1 = &-(t^2+x^2+y^2+z^2)\frac{\partial}{\partial t} - 2tx\frac{\partial}{\partial x} - 2ty\frac{\partial}{\partial y} - 2tz\frac{\partial}{\partial z}\\
&+(3ta^0+xa^1+ya^2+za^3)\frac{\partial}{\partial a^0} + (xa^0+3ta^1+zc^2-yc^3)\frac{\partial}{\partial a^1}\\
&+(ya^0+3ta^2-zc^1+xc^3)\frac{\partial}{\partial a^2} + (za^0+3ta^3+yc^1-xc^2)\frac{\partial}{\partial a^3}\\
&+(3tc^0+xc^1+yc^2+zc^3)\frac{\partial}{\partial c^0} + (-za^2+ya^3+xc^0+3tc^1)\frac{\partial}{\partial c^1}\\
&+(za^1-xa^3+yc^0+3tc^2)\frac{\partial}{\partial c^2} + (-ya^1+xa^2+zc^0+3tc^3)\frac{\partial}{\partial c^3},
\end{aligned} \tag{7.7}$$

$$\begin{aligned}
\mathrm{X}_2 = &\,2tx\frac{\partial}{\partial t} + (t^2+x^2-y^2-z^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z}\\
&+(-3xa^0+ta^1+zc^2+yc^3)\frac{\partial}{\partial a^0} - (ta^0+3xa^1+ya^2+za^3)\frac{\partial}{\partial a^1}\\
&+(ya^1+3xa^2-zc^0-tc^3)\frac{\partial}{\partial a^2} + (za^1-3xa^3+yc^0+tc^2)\frac{\partial}{\partial a^3}\\
&+(za^2-ya^3-3xc^0-tc^1)\frac{\partial}{\partial c^0} - (tc^0+3xc^1+yc^2+zc^3)\frac{\partial}{\partial c^1}\\
&+(za^0+ta^3+yc^1-3xc^2)\frac{\partial}{\partial c^2} + (-ya^0-ta^2+zc^1-3xc^3)\frac{\partial}{\partial c^3},
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
X_3 = {} & 2ty\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (t^2 - x^2 + y^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} \\
& + (-3ya^0 - ta^2 + zc^1 - xc^3)\frac{\partial}{\partial a^0} + (-3ya^1 + xa^2 + zc^0 + tc^3)\frac{\partial}{\partial a^1} \\
& - (ta^0 + xa^1 + 3ta^2 + za^3)\frac{\partial}{\partial a^2} + (za^2 - 3ya^3 - xc^0 - tc^1)\frac{\partial}{\partial a^3} \\
& + (-za^1 + xa^3 - 3yc^0 - tc^2)\frac{\partial}{\partial c^0} + (-za^0 - ta^3 - 3yc^1 + xc^2)\frac{\partial}{\partial c^1} \\
& - (tc^0 + xc^1 + 3yc^2 + zc^3)\frac{\partial}{\partial c^2} + (xa^0 + ta^1 + zc^2 - 3yc^3)\frac{\partial}{\partial c^3},
\end{aligned} \tag{7.9}$$

$$\begin{aligned}
X_4 = {} & 2tz\frac{\partial}{\partial t} + 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (t^2 - x^2 - y^2 + z^2)\frac{\partial}{\partial z} \\
& + (-3za^0 + ta^3 + yc^1 + xc^2)\frac{\partial}{\partial a^0} + (-3za^1 + xa^3 - yc^0 - tc^2)\frac{\partial}{\partial a^1} \\
& + (-3za^2 + ya^3 + xc^0 + tc^1)\frac{\partial}{\partial a^2} - (ta^0 + xa^1 + ya^2 + 3za^3)\frac{\partial}{\partial a^3} \\
& + (ya^1 - xa^2 - 3zc^0 - tc^3)\frac{\partial}{\partial c^0} + (ya^0 + ta^2 - 3zc^1 + xc^3)\frac{\partial}{\partial c^1} \\
& + (-xa^0 - ta^1 - 3zc^2 + yc^3)\frac{\partial}{\partial c^2} - (tc^0 + xc^1 + yc^2 + 3zc^3)\frac{\partial}{\partial c^3},
\end{aligned} \tag{7.10}$$

can be shown to correspond to four nonlocal symmetries of Maxwell system $\mathbf{M}\{t,x,y,z;e,b\}$ (6.1). In particular, one can show that the symmetry components corresponding to the electric and magnetic fields e,b essentially depend on symmetric combinations of derivatives of the potential variables and are not expressible through local variables via potential equations (7.3) and (7.4).

Additional nonlocal symmetries of Maxwell's equations (6.1) in four-dimensional space-time were obtained in Ref. 12 which arise as local (first-order) symmetries of determined potential system $\mathbf{AC}\{t,x,y,z;a,c\}$ (7.5) appended by Lorentz gauges.

B. Nonlocal divergence-type conservation laws

Consider system $\mathbf{AC}\{t,x,y,z;a,c\}$ (7.5). We seek nonlocal conservation laws of Maxwell's equations (6.1) arising as local conservation laws of its potential system (7.5), using the direct method, with multipliers depending only on potential variables and their derivatives: $\Lambda_\sigma(A,C,\partial A,\partial C)$. (In each subsequent case, only first derivatives that are not dependent through the equations of the system are included in the dependence of the multipliers, in order to exclude trivial conservation laws.)

(A) Gauge-invariant nonlocal conservation laws. First, consider conservation laws arising from underdetermined potential system (7.5). It follows that such conservation laws will hold for any gauge. Following the direct method, one obtains 2090 linear PDEs for the six unknown multipliers. Its complete solution yields seven independent sets of multipliers. Six of these sets correspond to conservation laws that are PDEs (7.5) themselves. The other set is given by

$$\Lambda_1 = C_y^3 - C_z^2 = E^1, \quad \Lambda_2 = C_z^1 - C_x^3 = E^2, \quad \Lambda_3 = C_x^2 - C_y^1 = E^3,$$

$$\Lambda_4 = A^3_y - A^2_z = B^1, \quad \Lambda_2 = A^1_z - A^3_x = B^2, \quad \Lambda_6 = A^2_x - A^1_y = B^3. \tag{7.11}$$

The corresponding conservation law given by the divergence expression (exterior derivative),

$$d\Psi = 0, \quad \Psi^1 = a \wedge F + c \wedge *F, \tag{7.12}$$

was found in Ref. 12 and is a gauge-invariant nonlocal conservation law of Maxwell's equations (6.1).

(B) Nonlocal conservation laws arising from algebraic gauges. As a specific example, consider potential system (7.5) with the algebraic gauge $a^0 = c^0 = 0$. Here we seek local conservation laws arising from multipliers of the form

$$\Lambda_\sigma = \Lambda_\sigma(\tilde{A}, \tilde{C}, \partial\tilde{A}, \partial\tilde{C}), \quad \sigma = 1, \ldots, 6, \quad \tilde{A} \equiv (A^1, A^1, A^3), \quad \tilde{C} \equiv (C^1, C^1, C^3).$$

The complete solution of the corresponding determining equations yields seven sets of multipliers. Six of these sets of multipliers correspond to PDEs (7.5) as before, and the other set of multipliers given by

$$\Lambda_i = C^i, \quad \Lambda_{i+3} = A^i, \quad i = 1,2,3, \tag{7.13}$$

yields the conservation law

$$\frac{1}{2}\frac{\partial}{\partial t}(a_m a_m + c_m c_m) - \frac{\partial}{\partial x^k}\varepsilon^{kij}(a^i c^j) = 0, \tag{7.14}$$

which is a nonlocal conservation law of Maxwell's equations (6.1).

(C) Nonlocal conservation laws for the divergence gauge. Now, consider potential system (7.5) appended with two divergence gauges

$$a^0_t + a^1_x + a^2_y + a^3_z = 0, \quad c^0_t + c^1_x + c^2_y + c^3_z = 0. \tag{7.15}$$

We seek local conservation laws of the resulting determined potential system arising from multipliers of the form

$$\Lambda_\sigma(A, C, \partial A, \partial C), \quad \sigma = 1, \ldots, 8.$$

The solution of the determining equations yields 11 sets of multipliers, corresponding to

- the eight obvious conservation laws [PDEs (7.5) and (7.15)];
- gauge-invariant conservation law (7.12);
- two additional sets of multipliers,

$$\Lambda_i = C^i, \quad \Lambda_{i+3} = A^i, \quad i = 1,2,3, \quad \Lambda_7 = A^0, \quad \Lambda_8 = C^0, \tag{7.16}$$

$$\Lambda_i = A^i_t, \quad \Lambda_{i+3} = -C^i_t, \quad i = 1,2,3, \quad \Lambda_7 = -C^0_t, \quad \Lambda_8 = A^0_t. \tag{7.17}$$

The additional conservation law corresponding to multipliers (7.16) is given by

$$\frac{1}{2}\frac{\partial}{\partial t}(\eta_{\mu\nu} a^\mu a^\nu) - \frac{\partial}{\partial x^k}(a^0 a^k + c^0 c^k + \varepsilon^{kij} a^i c^j) = 0. \tag{7.18}$$

The conservation law corresponding to multipliers (7.17) is given by

$$\frac{\partial}{\partial t}\eta^{\mu\nu}\left(a^\mu \frac{\partial}{\partial x^\nu}c^0 - c^\mu \frac{\partial}{\partial x^\nu}a^0\right) - \frac{\partial}{\partial x^k}\varepsilon^{kij}\left(a^i \frac{\partial}{\partial x^j}a^0 + c^i \frac{\partial}{\partial x^j}c^0\right) = 0. \tag{7.19}$$

Conservation laws (7.18) and (7.19) are nonlocal conservation laws of Maxwell's equations (6.1).

(D) Nonlocal conservation laws for the Lorentz gauge. As a last example, for determined potential system (7.5) and (7.6) with the Lorentz gauges, we obtain all local conservation laws arising from zeroth-order multipliers,

$$\Lambda_\sigma = \Lambda_\sigma(A,C), \quad \sigma = 1, \ldots, 8.$$

[First-order conservation laws of potential system (7.5) and (7.6) are given in Ref. 12.]

The solution of the multiplier determining equations yields 12 sets of multipliers. Eight of them correspond to eight equations (7.5) and (7.6) which are divergence expressions as they stand.

The additional four sets of multipliers,

$$\Lambda_1 = -p_0C^1 - p_1C^0 + p_2A^3 - p_3A^2, \quad \Lambda_2 = -p_0C^2 - p_1A^3 - p_2C^0 + p_3A^1,$$

$$\Lambda_3 = -p_0C^3 + p_1A^2 - p_2A^1 - p_3C^0, \quad \Lambda_4 = -p_0A^1 - p_1A^0 - p_2C^3 + p_3C^2,$$

$$\Lambda_5 = -p_0A^2 + p_1C^3 - p_2A^0 - p_3C^1, \quad \Lambda_6 = -p_0A^3 - p_1C^2 + p_2C^1 - p_3A^0,$$

$$\Lambda_7 = p_0A^0 + p_1A^1 + p_2A^2 + p_3A^3, \quad \Lambda_8 = p_0C^0 + p_1C^1 + p_2C^2 + p_3C^3, \tag{7.20}$$

involving arbitrary constants $p_1, \ldots, p_4$, respectively, yield four conservation laws which are nonlocal conservation laws of Maxwell's equations (6.1). In particular, the conservation law corresponding to $p_1=1$, $p_2=p_3=p_4=0$, is given by

$$\frac{1}{2}\frac{\partial}{\partial t}(|a|^2 + |c|^2) - \frac{\partial}{\partial x^k}(a^0a^k + c^0c^k + \varepsilon^{kij}a^ic^j) = 0, \tag{7.21}$$

and the other three are obtained from corresponding permutations of the indices.

VIII. NONLOCAL SYMMETRIES AND EXACT SOLUTIONS OF THE THREE-DIMENSIONAL MHD EQUILIBRIUM EQUATIONS

Consider the PDE system of ideal MHD equilibrium equations in three space dimensions given by

$$\mathrm{div}(\rho\mathbf{v}) = 0, \quad \mathrm{div}\ \mathbf{b} = 0, \tag{8.1a}$$

$$\rho\mathbf{v} \times \mathrm{curl}\ \mathbf{v} - \mathbf{b} \times \mathrm{curl}\ \mathbf{b} - \mathrm{grad}\ p - \frac{1}{2}\rho\ \mathrm{grad}|\mathbf{v}|^2 = 0, \tag{8.1b}$$

$$\mathrm{curl}\ \mathbf{v} \times \mathbf{b} = 0. \tag{8.1c}$$

In (8.1), the dependent variables are the plasma density ρ, the plasma velocity $\mathbf{v}=(v^1,v^2,v^3)$, the pressure p, and the magnetic field $\mathbf{b}=(b^1,b^2,b^3)$; the independent variables are the spatial coordinates (x,y,z). For closure, one must add an appropriate equation of state that relates pressure and density to MHD equations (8.1).

It has been shown[16,17,20] that an infinite number of nonlocal symmetries exist for MHD equations (8.1) for two different equations of state. These symmetries have been used in the literature for the construction of physical plasma equilibrium solutions. Moreover, the symmetries for incompressible equilibria (Sec. VIII A) preserve solution stability, i.e., map stable magnetohydrodynamic equilibria into stable magnetohydrodynamic equilibria. For additional details and examples, see Refs. 16, 19, 21, and 22.

A. Nonlocal symmetries for incompressible MHD equilibria

As a first simplified example, consider the incompressible MHD equilibrium system $\mathbf{I}\{x,y,z;\mathbf{b},\mathbf{v},p\}$ with constant density (without loss of generality, $\rho=1$), given by

$$\operatorname{div}\ \mathbf{v}=0,\quad \operatorname{div}\ \mathbf{b}=0, \tag{8.2a}$$

$$\mathbf{v}\times\operatorname{curl}\ \mathbf{v}-\mathbf{b}\times\operatorname{curl}\ \mathbf{b}-\operatorname{grad}\ p-\frac{1}{2}\operatorname{grad}|\mathbf{v}|^2=0, \tag{8.2b}$$

$$\operatorname{curl}(\mathbf{v}\times\mathbf{b})=0. \tag{8.2c}$$

Using lower-degree conservation law (8.2c), one introduces a potential variable ψ,

$$\mathbf{v}\times\mathbf{b}=\operatorname{grad}\ \psi. \tag{8.3}$$

(Note that ψ has the direct physical meaning of a function enumerating *magnetic surfaces*, i.e., two-dimensional surfaces to which streamlines and magnetic field lines are tangent. In general, every three-dimensional plasma domain is spanned by such surfaces.)

The resulting determined potential system $\mathbf{I\Psi}\{x,y,z;\mathbf{b},\mathbf{v},p,\psi\}$ is given by

$$\operatorname{div}\ \mathbf{v}=0,\quad \operatorname{div}\ \mathbf{b}=0,\quad \mathbf{v}\times\mathbf{b}=\operatorname{grad}\ \psi, \tag{8.4a}$$

$$\mathbf{v}\times\operatorname{curl}\ \mathbf{v}-\mathbf{b}\times\operatorname{curl}\ \mathbf{b}-\operatorname{grad}\ p-\frac{1}{2}\operatorname{grad}|\mathbf{v}|^2=0. \tag{8.4b}$$

Now a comparison is made between the point symmetries of PDE systems (8.2) and (8.4). Incompressible MHD equilibrium system (8.2) has ten point symmetries: translations in pressure and two scalings, given, respectively, by

$$\mathrm{X_p}=\frac{\partial}{\partial p},\quad \mathrm{X_D}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},\quad \mathrm{X_S}=b^i\frac{\partial}{\partial b^i}+v^i\frac{\partial}{\partial v^i}+2p\frac{\partial}{\partial p},$$

the interchange symmetry given by

$$\mathrm{X_I}=v^i\frac{\partial}{\partial b^i}+b^i\frac{\partial}{\partial v^i}-(\mathbf{b}\cdot\mathbf{v})\frac{\partial}{\partial p},$$

and the Euclidean group (three space translations and three rotations) given by

$$\mathrm{X_E}=\boldsymbol{\zeta}\lrcorner\frac{\partial}{\partial\mathbf{x}}+(\mathbf{b}\cdot\operatorname{grad})\boldsymbol{\zeta}\lrcorner\frac{\partial}{\partial\mathbf{b}},$$

where the hook symbol denotes summation over vector components, $\mathbf{x}=(x,y,z)$, $\boldsymbol{\zeta}=\mathbf{a}+(\mathbf{b}\times\mathbf{x})$, and $\mathbf{a}$, $\mathbf{b}$ are arbitrary constant vectors in $\mathbb{R}^3$.

The first nine symmetries of MHD system (8.2) directly yield point symmetries of potential system (8.4). In addition, potential system (8.4) has the obvious potential shift symmetry given by

$$\mathrm{X}_\psi=\frac{\partial}{\partial\psi},$$

as well as an infinite number of point symmetries given by

$$\mathrm{X}_\infty = M(\psi)\left(v^i\frac{\partial}{\partial b^i} + b^i\frac{\partial}{\partial v^i} - (\mathbf{b}\cdot\mathbf{v})\frac{\partial}{\partial p}\right), \tag{8.5}$$

where $M(\psi)$ is an arbitrary smooth function of its argument. Point symmetries (8.5) yield nonlocal symmetries of incompressible MHD equilibrium system (8.2). One can show that globally symmetries (8.5) transform a given solution $(\mathbf{b},\mathbf{v},p)$ to a family of solutions $(\mathbf{b}',\mathbf{v}',p')$ given by[16,23]

$$x' = x, \quad y' = y, \quad z' = z,$$

$$\mathbf{b}' = \mathbf{b}\cosh M(\psi) + \mathbf{v}\sinh M(\psi),$$

$$\mathbf{v}' = \mathbf{v}\cosh M(\psi) + \mathbf{b}\sinh M(\psi),$$

$$p' = p + (|\mathbf{b}|^2 - |\mathbf{b}'|^2)/2. \tag{8.6}$$

Since transformations (8.6) depend on an arbitrary function $M(\psi)$, that is, constant on magnetic surfaces, they can be used to obtain families of physically interesting solutions from a known MHD equilibrium solution. Transformations (8.6) preserve magnetic surfaces: $\mathbf{b}'\times\mathbf{v}'$ is parallel to $\mathbf{b}\times\mathbf{v}$.

As a simple example, consider the well-known simple "transverse flow" solution of MHD equilibrium system (8.2) given by

$$\mathbf{b} = H(r)\mathbf{e}_z, \quad \mathbf{v} = \omega(r)(-y\mathbf{e}_x + x\mathbf{e}_y),$$

$$p(r) = F(r) - H^2(r)/2, \quad F(r) = \int_0^r q\omega^2(q)dq \tag{8.7}$$

depending on two arbitrary functions $H(r)$, $\omega(r)$. This solution describes the differential rotation of a constant-density ideal gas plasma around the z-axis, for the vertical magnetic field; $r=\sqrt{x^2+y^2}$ is a cylindrical radius. The magnetic surfaces $\psi=\text{const}$ are cylinders $r=\text{const}$ around the z-axis. In Fig. 1(a), field lines of solution (8.7) tangent to the cylinder $r=1$ are shown for $H(r)=e^{-r}$, $\omega(r)=2e^{-2r}$. Using transformations (8.6) with an arbitrary function $M(\psi)=f(r)$, one obtains an infinite family of solutions (8.7) for a noncollinear magnetic field and velocity given by

$$\mathbf{b} = H(r)\cosh(f(r))\mathbf{e}_z + \mathbf{v}\sinh(f(r)),$$

$$\mathbf{v} = \cosh(f(r))\mathbf{v} + H(r)\sinh(f(r))\mathbf{e}_z. \tag{8.8}$$

Here the magnetic field lines and plasma streamlines are helices that are tangent to cylindrical magnetic surfaces $r=\text{const}$, with slopes depending on r. For $f(r)=e^{-r^2}$, original and transformed magnetic field lines and streamlines tangent to the cylinder $r=1$ are shown in Fig. 1.

One can show[16,20] that for incompressible plasma equilibria with nonconstant plasma density, there exist infinite sets of transformations that generalize (8.6), as follows. If the density ρ is constant on magnetic surfaces, i.e.,

$$\operatorname{grad}\ \rho\cdot\mathbf{B} = \operatorname{grad}\ \rho\cdot\mathbf{V} = 0,$$

then the infinite set of transformations,

$$x' = x, \quad y' = y, \quad z' = z,$$

$$\mathbf{B}' = b(\Psi)\mathbf{B} + c\sqrt{\rho}\mathbf{V}, \quad \mathbf{V}' = \frac{c(\Psi)}{a(\Psi)\sqrt{\rho}}\mathbf{B} + \frac{b(\Psi)}{a(\Psi)}\mathbf{V},$$

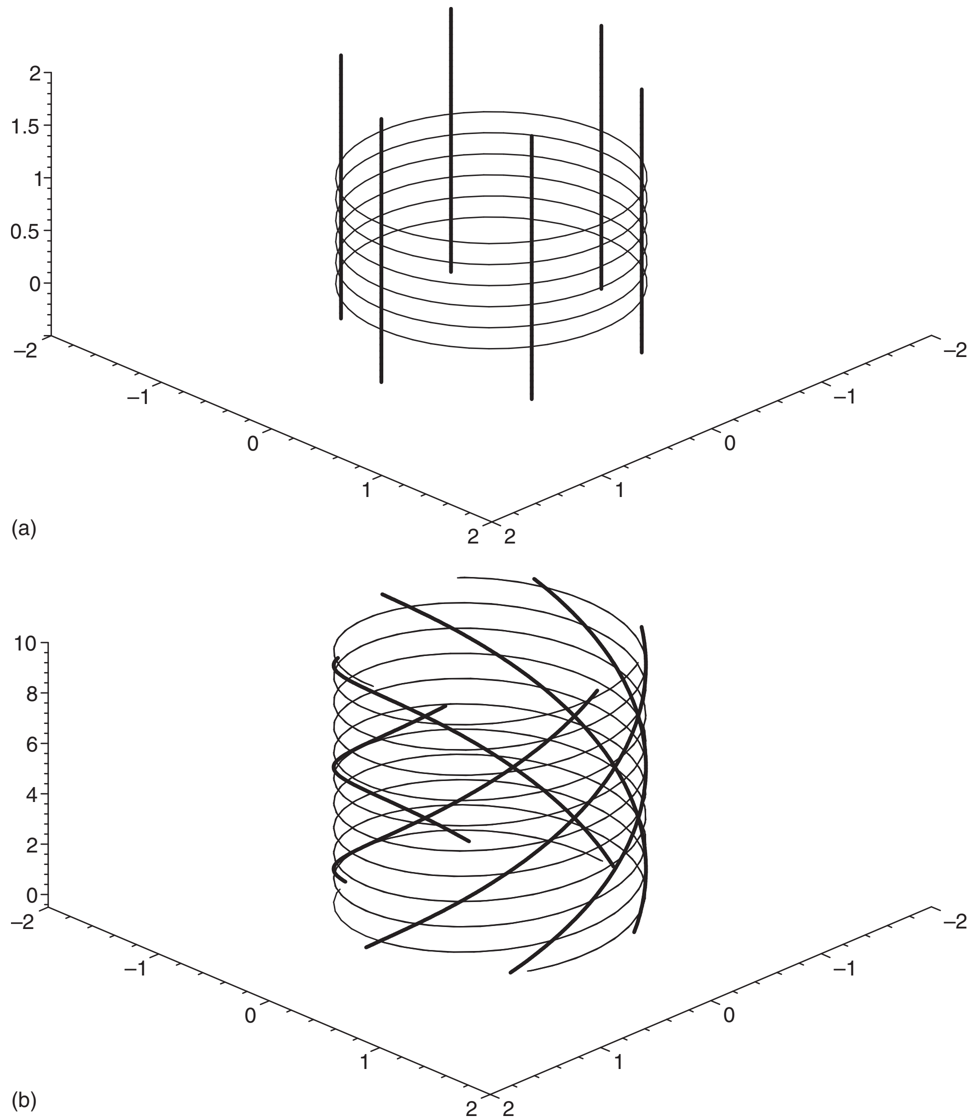

FIG. 1. Magnetic field lines and streamlines of "transverse flow" MHD equilibrium solution (8.7) (a) and its transformed version (8.8) (b). Magnetic field lines are shown with thick lines and plasma streamlines with thin lines.

$$\rho' = a^2(\Psi)\rho, \quad P' = CP + \frac{1}{2}(C|\mathbf{B}|^2 - |\mathbf{B}'|^2), \tag{8.9}$$

maps a given solution $(\mathbf{B},\mathbf{V},P,\rho)$ of PDE system (8.1) into a family of solutions $(\mathbf{B}',\mathbf{V}',P',\rho')$ with the same set of magnetic field lines. In (8.9), $a(\Psi)$ and $b(\Psi)$ are arbitrary functions constant on magnetic surfaces $\Psi=\text{const}$ and $b^2(\Psi)-c^2(\Psi)=C=\text{const}$.

B. Nonlocal symmetries for compressible adiabatic MHD equilibria

Now consider the system of compressible MHD equilibrium equations $\mathbf{C}\{x,y,z;\mathbf{b},\mathbf{v},p,\rho\}$ given by

$$\operatorname{div}(\rho\mathbf{v}) = 0, \quad \operatorname{div}\ \mathbf{b} = 0, \tag{8.10a}$$

$$\mathbf{v}\cdot \operatorname{grad}\ p + \gamma p\ \operatorname{div}\ \mathbf{v} = 0, \tag{8.10b}$$

$$\rho \mathbf{v} \times \text{curl } \mathbf{v} - \mathbf{b} \times \text{curl } \mathbf{b} - \text{grad } p - \frac{1}{2}\rho \text{ grad}|\mathbf{v}|^2 = 0, \tag{8.10c}$$

$$\text{curl}(\mathbf{v} \times \mathbf{b}) = 0. \tag{8.10d}$$

PDE system (8.10) describes plasmas corresponding to the ideal gas equation of state and undergoing an adiabatic process. Here the entropy $S=p/\rho^{\gamma}$ is constant throughout the plasma domain.

A determined potential system $\mathbf{C\Psi}\{x,y,z;\mathbf{b},\mathbf{v},p,\rho,\psi\}$ is obtained, as before, through replacing conservation law (8.10d) by potential equations (8.3). The resulting potential system $\mathbf{C\Psi}\{x,y,z;\mathbf{b},\mathbf{v},p,\rho,\psi\}$ has an infinite number of point symmetries given by the infinitesimal generator,

$$\mathrm{X}_C = N(\psi)\left(v^i \frac{\partial}{\partial v^i} - 2\rho \frac{\partial}{\partial \rho}\right) + \left(\int N(\psi) d\psi\right)\frac{\partial}{\partial \psi}, \tag{8.11}$$

where $N(\psi)$ is an arbitrary smooth function.[17] Point symmetries (8.11) yield nonlocal symmetries of compressible MHD equilibrium system (8.10). The finite form of the transformations of physical variables is readily found to be given by

$$x' = x, \quad y' = y, \quad z' = z, \quad \mathbf{b}' = \mathbf{b}, \quad p' = p,$$

$$\mathbf{v}' = f(\psi)\mathbf{v}, \quad \rho' = \rho/f^2(\psi). \tag{8.12}$$

Some generalizations of symmetry transformations (8.12) are considered in Refs. 16 and 20.

IX. DISCUSSION AND OPEN PROBLEMS

In this paper, the systematic framework for obtaining nonlocally related PDE systems in multidimensions ($n \geq 3$ independent variables), including procedures for obtaining determined nonlocally related PDE systems, as presented in Ref. 1 has been illustrated with examples. Nonlocal symmetries and nonlocal conservation laws have been used as a measure of "usefulness" of nonlocally related PDE systems due to their straightforward computation and, often, transparent physical meaning. In particular, new examples of nonlocal symmetries and nonlocal conservation laws have been found for the following situations.

- A nonlocal symmetry arising from a nonlocally related subsystem in (2+1) dimensions (Sec. II).
- Nonlocal symmetries and nonlocal conservation laws of a nonlinear "generalized plasma equilibrium" PDE system in three space dimensions. [These nonlocal symmetries and nonlocal conservation laws arise as local symmetries and conservation laws of a potential system following from a lower-degree (curl-type) conservation law (Sec. III).]
- Nonlocal symmetries of dynamic Euler equations of incompressible fluid dynamics arising from reduced systems for axial as well as helical symmetries (Sec. V).
- Nonlocal conservation laws of Maxwell's equations in (2+1)-dimensional Minkowski space, arising from a potential system appended with algebraic and divergence gauges (Sec. VI).
- Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations in (3+1)-dimensional Minkowski space, arising from a potential system of degree 2, appended with algebraic and divergence gauges (Sec. VII).

Moreover, known examples from the existing literature were discussed and synthesized within the framework presented in Ref. 1.

The well-known Geroch group[24,25] of nonlocal (potential) symmetries of Einstein's equations with a metric that admits a Killing vector has not been considered in this paper. This example is a natural generalization of ideas discussed above on the calculus on manifolds. It uses a conservation law of degree 1 to introduce a scalar potential variable. The symmetries are used in Refs. 24 and 25 to generate new exact solutions of Einstein's equations. The Geroch group example reinforces the understanding that a given PDE system has to have an internal geometric structure (in this case, a Killing vector) in order to have lower-degree conservation laws.

Although this paper has substantially synthesized and extended known results for multidimensional nonlocally related PDE systems, many more examples are needed to arrive at a better understanding of interconnections between nonlocally related PDE systems with $n \geq 3$ independent variables. The principal difficulty in performing computations lies in the complexity in solving determining equations for symmetries and conservation laws in multidimensions. Open problems include the following.

(1) Find examples of *nonlinear* PDE systems with $n \geq 3$ independent variables, for which nonlocal symmetries arise as local symmetries of a potential system following from a divergence-type conservation law(s), appended with some gauge constraint(s).
(2) Find efficient procedures to obtain "useful" gauge constraints (e.g., yielding nonlocal symmetries and/or nonlocal conservation laws) for potential systems arising from divergence-type conservation laws (as well as for underdetermined potential systems arising from lower-degree conservation laws). In particular, do there exist further refinements of Theorems 6.1 and 6.3 of Ref. 1 that can rule out consideration of specific families of gauges for particular classes of potential systems?
(3) Find further examples of lower-degree conservation laws for PDE systems of physical importance. [Conservation laws of degree one (curl-type in $\mathbb{R}^3$) would be of particular interest, since corresponding potential systems are determined.] Examples suggest that lower-degree conservation laws are rather rare and are only expected to exist when a given PDE system has a special geometrical structure. On the other hand, divergence-type conservation laws are rather common.

ACKNOWLEDGMENTS

The authors are grateful to NSERC for research support.

[1] A. F. Cheviakov and G. W. Bluman, J. Math. Phys. **51**, 103521 (2010).
[2] G. W. Bluman and A. F. Cheviakov, J. Math. Phys. **46**, 123506 (2005).
[3] G. W. Bluman, A. F. Cheviakov, and N. M. Ivanova, J. Math. Phys. **47**, 113505 (2006).
[4] G. W. Bluman and S. Kumei, J. Math. Phys. **28**, 307 (1987).
[5] G. W. Bluman and P. Doran-Wu, Acta Appl. Math. **2**, 79 (1995).
[6] A. Ma, "Extended group analysis of the wave equation," M.Sc. thesis, University of British Columbia, 1991.
[7] I. S. Akhatov, R. K. Gazizov, and N. H. Ibragimov, Sov. Math. Dokl. **35**, 384 (1987).
[8] A. Sjöberg and F. M. Mahomed, Appl. Math. Comput. **150**, 379 (2004).
[9] G. W. Bluman, A. F. Cheviakov, and J.-F. Ganghoffer, J. Eng. Math. **62**, 203 (2008).
[10] G. W. Bluman and A. F. Cheviakov, J. Math. Anal. Appl. **333**, 93 (2007).
[11] S. C. Anco and G. W. Bluman, J. Math. Phys. **38**, 3508 (1997).
[12] S. C. Anco and D. The, Acta Appl. Math. **89**, 1 (2005).
[13] A. F. Cheviakov and S. C. Anco, Phys. Lett. A **372**, 1363 (2008).
[14] A. F. Cheviakov, Comput. Phys. Commun. **176**, 48 (2007).
[15] G. W. Bluman, A. F. Cheviakov, and S. C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Applied Mathematical Sciences Vol. 168 (Springer, New York, 2010).
[16] O. I. Bogoyavlenskij, Phys. Lett. A **291**, 256 (2001).
[17] F. Galas, Physica D **63**, 87 (1993).
[18] O. Kelbin, A. F. Cheviakov, and M. Oberlack (unpublished).
[19] O. I. Bogoyavlenskij, Phys. Rev. E **62**, 8616 (2000).
[20] O. I. Bogoyavlenskij, Phys. Rev. E **66**, 056410 (2002).
[21] A. F. Cheviakov, Phys. Rev. Lett. **94**, 165001 (2005).
[22] K. I. Ilin and V. A. Vladimirov, Phys. Plasmas **11**, 3586 (2004).
[23] A. F. Cheviakov, Phys. Lett. A **321**, 34 (2004).
[24] R. Geroch, J. Math. Phys. **12**, 918 (1971).
[25] R. Geroch, J. Math. Phys. **13**, 394 (1972).
[26] Note that system (5.3) contains only first-order PDEs. Normally, in the point symmetry analysis procedure, if all differential equations are of the same order, no differential consequences are used in symmetry determining equations. However, in PDE system (5.3), an important differential consequence of PDEs (5.4c) is div $\boldsymbol{\omega}=0$. Without explicitly using this constraint, one misses infinite symmetries Y_4 in (5.9) and Y_3 in (5.14).