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Nonclassical analysis of the nonlinear Kompaneets equation

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Abstract The dimensional nonlinear Kompaneets (NLK) equation $u_t = x^{-2}\left[x^4(\alpha u_x + \beta u + \gamma u^2)\right]_x$ describes the spectra of photons interacting with a rarefied electron gas. Recently, Ibragimov obtained some time-dependent exact solutions for several approximations of this equation. In this paper, we use the nonclassical method to construct time-dependent exact solutions for the NLK equation $u_t = x^{-2}\left[x^4(\alpha u_x + \gamma u^2)\right]_x$ for arbitrary constants $\alpha > 0$, $\gamma > 0$. Solutions arising from "nonclassical symmetries" are shown to yield wider classes of time-dependent exact solutions for the NLK equation $u_t = x^{-2}\left[x^4(\alpha u_x + \gamma u^2)\right]_x$ beyond those obtained by Ibragimov. In particular, for five classes of initial conditions, each involving two parameters, previously unknown explicit time-dependent solutions are obtained. Interestingly, each of these solutions is expressed in terms of elementary functions. Three of the classes exhibit quiescent behavior, i.e., $\lim_{t\to\infty} u(x,t) = 0$, and the other two classes exhibit blow-up behavior in finite time. As a consequence, it is shown that the corresponding nontrivial stationary solutions are unstable.

Keywords Blow-up behavior · Invariant solution · Nonlinear Kompaneets equation · Nonclassical method · Stationary solution · Stability

1 Introduction

Group-theoretic methods are useful for finding symmetry reductions and corresponding group-invariant solutions of a partial differential equation (PDE) system [1–3]. Classical symmetry reductions due to Lie [4] have been generalized to the nonclassical method [5,6], in which one seeks local symmetries of an augmented PDE system consisting of the given PDE system, the invariant surface condition, and their differential consequences. In contrast to classical symmetries, "nonclassical symmetries" leave only submanifolds of solutions invariant. As a consequence, the nonclassical method is useful for finding further specific solutions in addition to those obtained by the classical Lie method.

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In this paper, we investigate the dimensional nonlinear Kompaneets (NLK) equation [7]

$$\frac{\partial u}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\alpha u_x + \beta u + \gamma u^2\right)\right], \tag{1.1}$$

where $\alpha > 0$, $\beta \geq 0$, and $\gamma > 0$ are arbitrary constants. This equation, also known as the photon diffusion equation, was first presented by Kompaneets [7], and in dimensionless form, after appropriate scalings of x, t, and u, can be written as either

$$\frac{\partial u}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(u_x + u + u^2\right)\right] \tag{1.2a}$$

when $\beta \neq 0$ or

$$\frac{\partial u}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(u_x + u^2\right)\right] \tag{1.2b}$$

in the case with dominant induced scattering $\beta = 0$ (i.e., $u^2 \gg u$).

As stated in Ibragimov [8], the NLK equation (1.1) describes an interaction of free electrons and electromagnetic radiation, specifically, the interaction of a low-energy homogeneous photon gas with a rarefied electron gas through Compton scattering. In Eq. (1.1), u is the density of the photon gas (photon number density), t is a dimensionless time, and $x = \frac{\hbar\nu}{kT}$, where $\hbar$ is Planck's constant and ν is the photon frequency. Then $\hbar\nu$ denotes the photon energy. T is the electron temperature and k is Boltzmann's constant. The terms u and u^2 in Eq. (1.1) correspond to spontaneous scattering (Compton effect) and induced scattering, respectively (see, e.g., [9]). Many numerical and analytical solutions have been found for the NLK equation (1.1) (see, e.g., [10–15]).

Recently, time-dependent exact solutions of the NLK equation (1.2b) were obtained by Ibragimov [8]. The NLK equation (1.2b) has two point symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{1.3}$$

Using these two point symmetries, Ibragimov obtained two sets of invariant solutions given by

$$u(x,t) = \frac{1}{x(1 - Ce^{2t})}, \tag{1.4}$$

where C is an arbitrary constant, and

$$u(x,t) = \frac{\phi(\lambda)}{x} \quad \text{with } \lambda = xe^{\rho t}, \tag{1.5}$$

where ρ is an arbitrary constant and $\phi(\lambda)$ satisfies the ordinary differential equation (ODE)

$$\lambda\phi'' + (2\phi + 2 - \rho)\phi' + \frac{2}{\lambda}\left(\phi^2 - \phi\right) = 0.$$

The purpose of this paper is to use the nonclassical method to seek further exact solutions of the NLK equation (1.1). By construction, the solutions obtained by the nonclassical method of course include the solutions obtained by Ibragimov [8]. More importantly, we obtain new explicit time-dependent solutions of the NLK equation (1.2b), which are useful in practice. These new solutions cannot be obtained by classical symmetry reductions.

This paper is organized as follows. In Sect. 2, we review the nonclassical method and apply it to the NLK equations (1.2a) and (1.2b) to obtain their "nonclassical symmetries." In Sect. 3, using the "nonclassical symmetries" obtained in Sect. 2, we construct corresponding families of explicit solutions for the NLK equation (1.2b), which cannot be found as invariant solutions for any of its local symmetries. Correspondingly, these new solutions yield five families of solutions with initial conditions of physical interest. It is shown that three of these families of solutions exhibit quiescent behavior, i.e., $\lim_{t\to\infty} u(x,t) = 0$, and that the other two families of solutions exhibit blow-up behavior, i.e., $\lim_{t\to t^*} u(x,t^*) = \infty$ for some finite t^* depending on a constant in their initial conditions. In Sect. 4, we consider nontrivial stationary solutions of (1.2b). We exhibit four families of stationary solutions not presented explicitly in [14] for the NLK equation (1.1) for the case $\alpha = \gamma = 1$ and $\beta = 0$. We show that two of these families of stationary solutions are unstable using the results presented in Sect. 3.

2 Nonclassical analysis of the NLK equation

2.1 The nonclassical method

A symmetry of a PDE system leaves invariant the solution manifold of the PDE system. Invariant solutions of a given PDE system arise from symmetries of the PDE system. These solutions are found by solving a corresponding reduced PDE system with fewer independent variables. In [5], Lie's method [4] was generalized to the nonclassical method. In the nonclassical method, one seeks local symmetries that leave only a submanifold of the solution manifold invariant. Such a "nonclassical symmetry" maps solution surfaces not in the submanifold to surfaces that are not solutions of the PDE system.

Consider a PDE system $\mathbf{R}\{x; u\}$ of N PDEs of order k with n independent variables $x = \left(x^1, \ldots, x^n\right)$ and m dependent variables $u(x) = \left(u^1(x), \ldots, u^m(x)\right)$, given by

$$R^\sigma[u] \equiv R^\sigma\left(x, u, \partial u, \ldots, \partial^k u\right) = 0, \quad \sigma = 1, \ldots, N. \tag{2.1}$$

In the nonclassical method, instead of seeking local symmetries of the given PDE system $\mathbf{R}\{x; u\}$ (2.1), one seeks local symmetries that leave invariant a submanifold of the solution manifold of the PDE system $\mathbf{R}\{x; u\}$ (2.1). In particular, one seeks functions $\xi^i(x, u)$, $\eta^\mu(x, u)$, $i = 1, \ldots, n$, $\mu = 1, \ldots, m$, so that

$$\mathbf{X} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\mu(x, u)\frac{\partial}{\partial u^\mu} \tag{2.2}$$

is a "symmetry" ("nonclassical symmetry") of the submanifold, which is the augmented PDE system $\mathbf{A}\{x; u\}$ consisting of the given PDE system $\mathbf{R}\{x; u\}$ (2.1), the invariant surface condition equations

$$\mathbf{I}^\nu(x, u, \partial u) \equiv \eta^\nu(x, u) - \xi^j(x, u)\frac{\partial u^\nu}{\partial x^j} = 0, \quad \nu = 1, \ldots, m, \tag{2.3}$$

and their differential consequences. Consequently, one obtains an overdetermined set of nonlinear determining equations for the unknown functions $\xi^i(x, u)$, $\eta^\mu(x, u)$, $i = 1, \ldots, n$, $\mu = 1, \ldots, m$.

Indeed, for any functions $\xi^i(x, u)$, $\eta^\mu(x, u)$, $i = 1, \ldots, n$, $\mu = 1, \ldots, m$, one has

$$\mathbf{X}^{(1)}\mathbf{I}^\nu(x, u, \partial u) = \left(\frac{\partial \eta^\nu}{\partial u^\mu} - \frac{\partial \xi^j}{\partial u^\mu}\frac{\partial u^\nu}{\partial x^j}\right) \cdot \mathbf{I}^\mu, \quad \nu = 1, \ldots, m, \tag{2.4}$$

which vanish for $\mathbf{I}^\nu(x, u, \partial u) = 0$, $\nu = 1, \ldots, m$, where $\mathbf{X}^{(1)}$ is the first extension of the infinitesimal generator (2.2). Therefore, the nonclassical method includes Lie's classical method.

In the nonclassical method, invariance of the given PDE system $\mathbf{R}\{x; u\}$ (2.1) is replaced by invariance of the augmented PDE system $\mathbf{A}\{x; u\}$. Consequently, it is possible to find symmetries leaving invariant the augmented PDE system $\mathbf{A}\{x; u\}$ which are not symmetries of the given PDE system $\mathbf{R}\{x; u\}$ (2.1). In turn, this can lead to further exact solutions of the given PDE system $\mathbf{R}\{x; u\}$ (2.1).

Consider a scalar PDE with two independent variables. Let $x^1 = x$, $x^2 = t$, $\xi^1 = \xi(x, t, u)$, $\xi^2 = \tau(x, t, u)$. Without loss of generality, one need only consider two essential cases when solving the determining equations for $(\xi(x, t, u), \tau(x, t, u), \eta(x, t, u))$. If the infinitesimal generator $\mathbf{X} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u}$ generates a "nonclassical symmetry" of the PDE system $\mathbf{R}\{x; u\}$ (2.1), then so does $\mathbf{Y} = p(x, t, u)\mathbf{X}$, where $p(x, t, u)$ is any smooth function. It follows that, if $\tau \neq 0$, one can set $\tau \equiv 1$, so that only two cases need to be considered: $\tau \equiv 1$ and $\tau \equiv 0$, $\xi \equiv 1$.

2.2 Nonclassical analysis of the NLK equation

The nonclassical method is now applied to the NLK equations (1.2a) and (1.2b), respectively. Here the invariant surface condition equation becomes

$$\xi(x, t, u)u_x + \tau(x, t, u)u_t = \eta(x, t, u). \tag{2.5}$$

2.2.1 Nonclassical symmetries of the NLK equation (1.2a)

Case I $\tau \equiv 1$.

The nonclassical method applied to (1.2a) yields the following determining equation system for the infinitesimals $(\xi(x,t,u), \eta(x,t,u))$:

$$2\xi_x\eta - 8xu\eta - \frac{2\xi\eta}{x} - 4x\eta + 4u^2\xi + 4u\xi + \eta_t - x^2\eta_{xx} - 4x\eta_x + 4xu^2\eta_u - 2x^2u\eta_x - x^2\eta_x - 8xu^2\xi_x + 4xu\eta_u - 8xu\xi_x = 0, \tag{2.6a}$$

$$4\xi - 2x^2\eta + \frac{2\xi^2}{x} - \xi_t - 2x^2u\xi_x - 2x^2\eta_{xu} + 2\xi_u\eta - 12xu^2\xi_u - 12xu\xi_u - 4x\xi_x + x^2\xi_{xx} - x^2\xi_x - 2\xi\xi_x = 0, \tag{2.6b}$$

$$2x^2\xi_{xu} - 2x^2\xi_u - 4x^2u\xi_u - 8x\xi_u - x^2\eta_{uu} - 2\xi_u\xi = 0, \tag{2.6c}$$

$$x^2\xi_{uu} = 0. \tag{2.6d}$$

The solution of the determining equation system (2.6a–d) is given by

$$\begin{cases} \xi(x,t,u) = 0, \\ \eta(x,t,u) = 0. \end{cases} \tag{2.7}$$

Hence the corresponding "nonclassical symmetry" is $Y_1 = \frac{\partial}{\partial t}$, which directly results from the point symmetry X_1.

Case II $\tau \equiv 0$, $\xi \equiv 1$.

In this case, the determining equation for $\eta(x,t,u)$ is

$$\begin{aligned} 4xu\eta_u &- x^2\eta^2\eta_{uu} - 2x^2\eta\eta_{xu} - 4u - 4u^2 - 4\eta + \eta_t - x^2\eta_{xx} - 6x\eta_x - 6x\eta - 2x^2\eta^2 \\ &- x^2\eta_x + 4xu^2\eta_u - 2x^2u\eta_x - 12xu\eta - 2x\eta\eta_u = 0. \end{aligned} \tag{2.8}$$

One is unable to find the general solution of (2.8). Hence one must consider ansatzes to obtain particular solutions of (2.8). If one considers an ansatz of the form $\eta = f(x,t) + g(x,t)u + h(x,t)u^2$, one obtains

$$\eta(x,t,u) = \frac{a_1}{x^4} - u - u^2, \tag{2.9}$$

where a_1 is an arbitrary constant. The corresponding "nonclassical symmetry" is $Y_2 = \frac{\partial}{\partial x} + \left(\frac{a_1}{x^4} - u - u^2\right)\frac{\partial}{\partial u}$.

2.2.2 Nonclassical symmetries of the NLK equation (1.2b)

Case I $\tau \equiv 1$.

The nonclassical method applied to (1.2b) yields the following determining equation system for the infinitesimals $(\xi(x,t,u), \eta(x,t,u))$:

$$4u^2\xi - 8xu^2\xi_x - \frac{2\xi\eta}{x} - 2x^2u\eta_x - 8xu\eta - x^2\eta_{xx} + \eta_t + 4xu^2\eta_u + 2\xi_x\eta - 4x\eta_x = 0, \tag{2.10a}$$

$$x^2\xi_{xx} - 12xu^2\xi_u - 2x^2u\xi_x - 2\xi_x\xi - \xi_t - 2x^2\eta + 4\xi + \frac{2\xi^2}{x} + 2\xi_u\eta - 2x^2\eta_{xu} - 4x\xi_x = 0, \tag{2.10b}$$

$$2x^2\xi_{xu} - 4x^2u\xi_u - 8x\xi_u - 2\xi\xi_u - x^2\eta_{uu} = 0, \tag{2.10c}$$

$$x^2\xi_{uu} = 0. \tag{2.10d}$$

The solutions of the determining equation system (2.10a–d) are given by

$$\begin{cases} \xi(x,t,u) = a_2 x, \\ \eta(x,t,u) = -a_2 u, \end{cases} \tag{2.11}$$

where a_2 is an arbitrary constant, and

$$\begin{cases} \xi(x,t,u) = -2x^2 u, \\ \eta(x,t,u) = 4xu^2 - 2u. \end{cases} \tag{2.12}$$

The solution (2.11) yields the "nonclassical symmetry" $Y_3 = a_2 x \frac{\partial}{\partial x} + \frac{\partial}{\partial t} - a_2 u \frac{\partial}{\partial u}$, which directly results from the point symmetry $X_1 + a_2 X_2$. The solution (2.12) yields the "nonclassical symmetry" $Y_4 = -2x^2 u \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \left(4xu^2 - 2u\right) \frac{\partial}{\partial u}$, which does not result from any point symmetry of (1.2b).

Case II $\tau \equiv 0, \xi \equiv 1$.

In this case, the determining equation for $\eta(x,t,u)$ is

$$-4\eta - 4u^2 - 6x\eta_x - 2x\eta\eta_u - 12xu\eta - 2x^2\eta^2 - 2x^2 u\eta_x + \eta_t + 4xu^2\eta_u - x^2\eta_{xx} - 2x^2\eta\eta_{xu} - x^2\eta^2\eta_{uu} = 0. \tag{2.13}$$

If one considers an ansatz of the form $\eta = f(x,t) + g(x,t)u + h(x,t)u^2$, the equation (2.13) has solutions

$$\eta(x,t,u) = \frac{c_2 \mathrm{e}^{-2t}}{x^2\left(c_1 + c_2\mathrm{e}^{-2t} - x\right)} + \frac{\left(x - 2c_1 - 2c_2\mathrm{e}^{-2t}\right)u}{x\left(c_1 + c_2\mathrm{e}^{-2t} - x\right)}, \tag{2.14}$$

$$\eta(x,t,u) = \frac{1}{x^2\left(1 + c_3\mathrm{e}^{2t}\right)} - \frac{2u}{x}, \tag{2.15}$$

$$\eta(x,t,u) = \frac{c_4}{x^4} - u^2, \tag{2.16}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

Hence, the corresponding "nonclassical symmetries" are $Y_5 = \frac{\partial}{\partial x} + \left[\frac{c_2\mathrm{e}^{-2t}}{x^2(c_1+c_2\mathrm{e}^{-2t}-x)} + \frac{(x-2c_1-2c_2\mathrm{e}^{-2t})u}{x(c_1+c_2\mathrm{e}^{-2t}-x)}\right]\frac{\partial}{\partial u}$, $Y_6 = \frac{\partial}{\partial x} + \left[\frac{1}{x^2(1+c_3\mathrm{e}^{2t})} - \frac{2u}{x}\right]\frac{\partial}{\partial u}$, and $Y_7 = \frac{\partial}{\partial x} + \left(\frac{c_4}{x^4} - u^2\right)\frac{\partial}{\partial u}$, respectively.

3 New exact solutions of the NLK equation

It is obvious that the invariant solutions arising from Y_1 and Y_3 are those obtained by Ibragimov [8], given by solutions (1.4) and (1.5). Moreover, the invariant solution corresponding to Y_2 is the stationary solution obtained by Dubinov [14] for the NLK equation (1.2a).

Consider the "nonclassical symmetry" Y_4 of the NLK equation (1.2b). Using the direct substitution method, one seeks solutions of the PDE system

$$\begin{cases} u_t = 4x\left(u_x + u^2\right) + x^2(u_{xx} + 2uu_x), & (3.1) \\ u_t = 2x^2 u u_x + \left(4xu^2 - 2u\right). & (3.2) \end{cases}$$

After equating the right-hand sides of (3.1) and (3.2), one obtains

$$4xu_x + x^2 u_{xx} + 2u = 0. \tag{3.3}$$

The solution of (3.3) is given by

$$u(x,t) = \frac{A(t) + B(t)x}{x^2}, \tag{3.4}$$

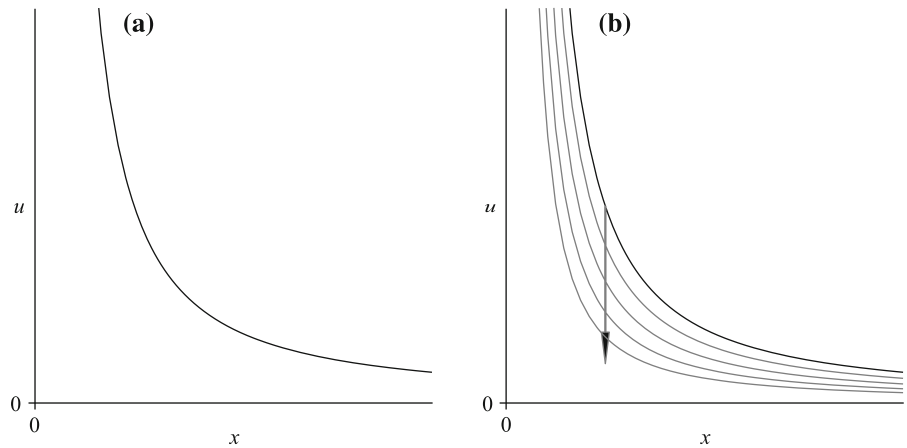

Fig. 1 (**a**) $u(x,0) = U(x) = \frac{b(x-c)}{x^2}$, $0 < b < 1$, $c \le 0$, $x > 0$. In (**b**), $u(x,t)$ is given by Eq. (3.11) for $x > 0$, $t > 0$, with the arrow pointing in the direction of increasing t

where $A(t)$ and $B(t)$ are arbitrary functions. Substituting (3.4) into (3.1), one obtains an ordinary differential equation (ODE) system for $A(t)$ and $B(t)$:

$$\begin{cases} A'(t) + 2A(t) - 2A(t)B(t) = 0, & (3.5)\\ B'(t) + 2B(t) - 2B(t)^2 = 0. & (3.6)\end{cases}$$

From (3.6), one obtains $B(t) \equiv 0$ or $B(t) = \frac{1}{1+De^{2t}}$, where D is an arbitrary constant. In particular, there are three families of solutions when $B(t) \not\equiv 0$. In terms of an arbitrary constant t_0, $-\infty < t_0 < \infty$, these solutions are given by

$$B(t) = \frac{1}{2}\left[1 - \tanh(t + t_0)\right], \quad \text{where } 0 < B(t) < 1;$$

$$B(t) = \frac{1}{2}\left[1 - \coth(t + t_0)\right], \quad \text{where } \begin{cases} B(t) < 0 & \text{if } t > -t_0,\\ B(t) > 1 & \text{if } t < -t_0;\end{cases}$$

$$B(t) \equiv 1.$$

If $B(t) \not\equiv 0$, one has $A(t) = -cB(t)$, where c is an arbitrary constant. If $B(t) \equiv 0$, one has $A(t) = Ee^{-2t}$, where E is an arbitrary constant. Therefore, there are four families of solutions of (1.2b):

$$\mathcal{F}_1: \ u(x,t) = \frac{x-c}{2x^2}\left[1 - \tanh(t + t_0)\right]; \tag{3.7}$$

$$\mathcal{F}_2: \ u(x,t) = \frac{x-c}{2x^2}\left[1 - \coth(t + t_0)\right]; \tag{3.8}$$

$$\mathcal{F}_3: \ u(x,t) = \frac{x-c}{x^2}; \tag{3.9}$$

$$\mathcal{F}_4: \ u(x,t) = \frac{E}{x^2 e^{2t}}. \tag{3.10}$$

The first two families of solutions $\mathcal{F}_1$ and $\mathcal{F}_2$ are new and cannot be obtained through classical symmetry reductions.

The corresponding initial conditions $u(x,0) = U(x)$ are given below.

For $\mathcal{F}_1$:

Case I $U(x) = \frac{b(x-c)}{x^2}$ with $0 < b < 1$, $c \le 0$, on the domain $0 < x < \infty$. Such a $U(x)$ is illustrated in Fig. 1a. The corresponding solutions of (1.2b) are given by

$$u(x,t) = \frac{x-c}{2x^2}\left[1 - \tanh(t + t_0)\right] \tag{3.11}$$

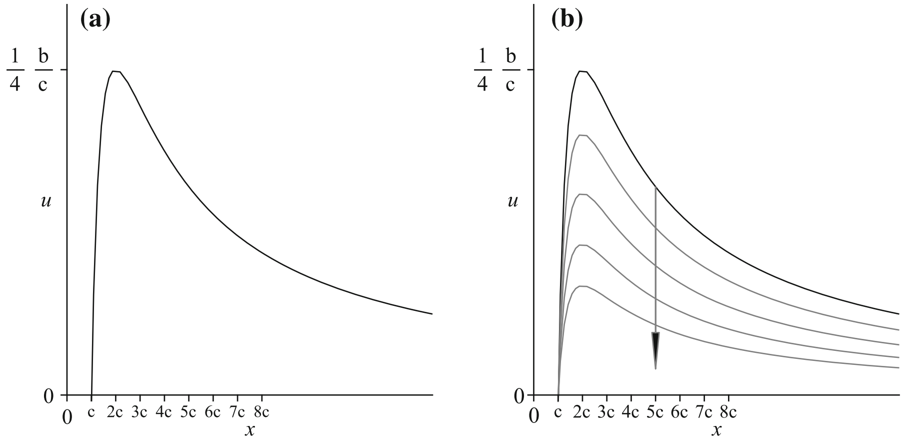

Fig. 2 (**a**) $u(x, 0) = U(x) = \frac{b(x-c)}{x^2}$, $0 < b < 1$, $c > 0$, $x \geq c$. In (**b**), $u(x, t)$ is given by Eq. (3.12) for $x \geq c$, $t > 0$, with the arrow pointing in the direction of increasing t

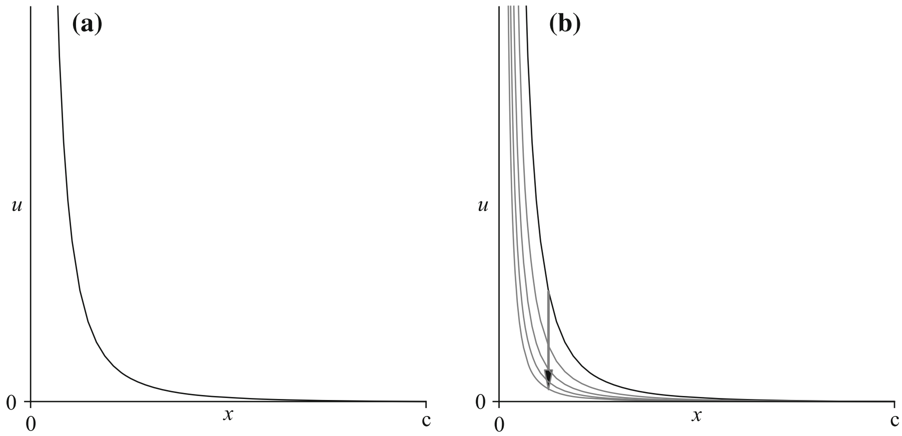

Fig. 3 (**a**) $u(x, 0) = U(x) = \frac{b(x-c)}{x^2}$, $b < 0$, $c > 0$, $0 < x \leq c$. In (**b**), $u(x, t)$ is given by Eq. (3.13) for $0 < x \leq c$, $t > 0$, with the arrow pointing in the direction of increasing t

with constants $t_0 = \frac{1}{2} \ln \left(\frac{1}{b} - 1\right)$, $0 < b < 1$, and $c \leq 0$, valid for $x > 0$, $t > 0$. For each value of x, the solution $u(x, t)$ is monotonically decreasing as a function of t. Moreover, $\lim_{t\to\infty} u(x, t) = 0$ for any $x > 0$. The evolution of a solution $u(x, t)$ is illustrated in Fig. 1b.

Case II $U(x) = \frac{b(x-c)}{x^2}$ with $0 < b < 1$, $c > 0$, on the domain $x \geq c$. Such a $U(x)$ is illustrated in Fig. 2a. The corresponding solutions of (1.2b) are given by

$$u(x, t) = \frac{x - c}{2x^2} [1 - \tanh(t + t_0)] \tag{3.12}$$

with constants $t_0 = \frac{1}{2} \ln \left(\frac{1}{b} - 1\right)$, $0 < b < 1$, and $c > 0$, valid for $x \geq c$, $t > 0$. For each value of x, the solution $u(x, t)$ is monotonically decreasing as a function of t. Moreover, $\lim_{t\to\infty} u(x, t) = 0$ for any $x \geq c$. The evolution of a solution $u(x, t)$ is illustrated in Fig. 2b.

For $\mathcal{F}_2$:

Case III $U(x) = \frac{b(x-c)}{x^2}$ with $b < 0$, $c > 0$, on the domain $0 < x \leq c$. Such a $U(x)$ is illustrated in Fig. 3a. The corresponding solutions of (1.2b) are given by

$$u(x, t) = \frac{x - c}{2x^2} [1 - \coth(t + t_0)] \tag{3.13}$$

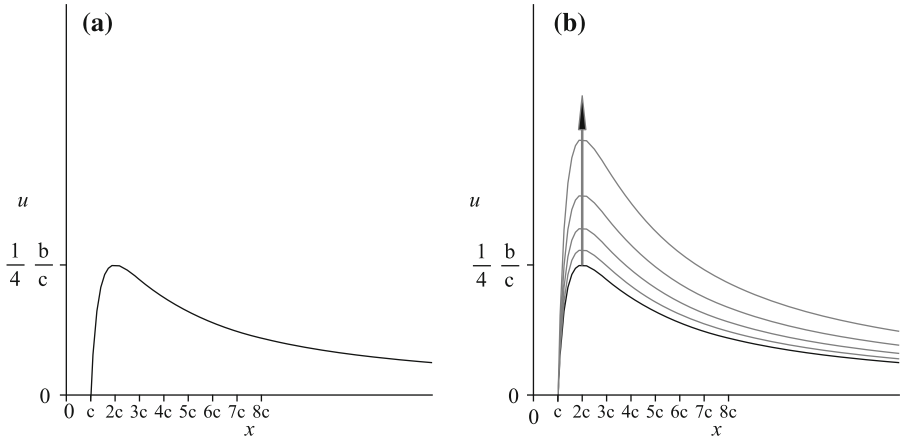

Fig. 4 (**a**) $u(x,0) = U(x) = \frac{b(x-c)}{x^2}$, $b > 1$, $c > 0$, $x \geq c$. In (**b**), $u(x,t)$ is given by Eq. (3.14) for $x \geq c$, $0 < t < -t_0$, with the arrow pointing in the direction of increasing t

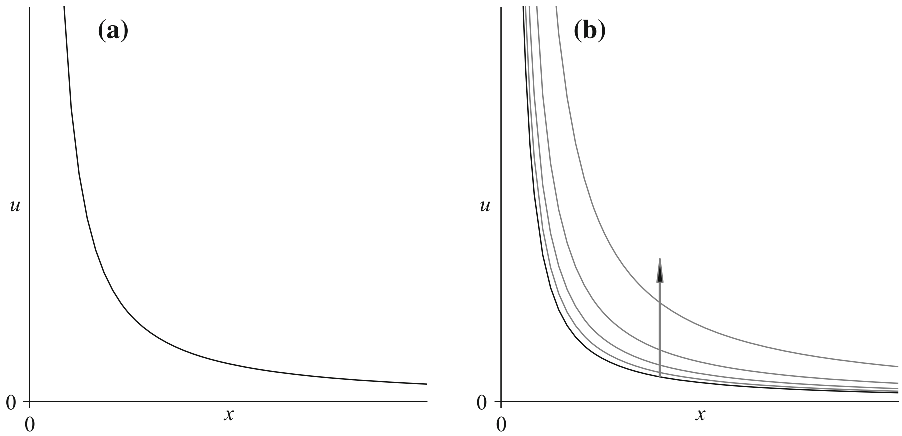

Fig. 5 (**a**) $u(x,0) = U(x) = \frac{b(x-c)}{x^2}$, $b > 1$, $c \leq 0$, $x > 0$. In (**b**), $u(x,t)$ is given by Eq. (3.15) for $x > 0$, $0 < t < -t_0$, with the arrow pointing in the direction of increasing t

with constants $t_0 = \frac{1}{2}\ln\left(1 - \frac{1}{b}\right)$, $b < 0$, and $c > 0$, valid for $0 < x \leq c$, $t > 0$. For each value of x, the solution $u(x,t)$ is monotonically decreasing as a function of t. Moreover, $\lim_{t\to\infty} u(x,t) = 0$ for $0 < x \leq c$. The evolution of a solution $u(x,t)$ is illustrated in Fig. 3b.

Case IV $U(x) = \frac{b(x-c)}{x^2}$ with $b > 1$, $c > 0$, on the domain $x \geq c$. Such a $U(x)$ is illustrated in Fig. 4a. The corresponding solutions of (1.2b) are given by

$$u(x,t) = \frac{x-c}{2x^2}\left[1 - \coth(t + t_0)\right] \tag{3.14}$$

with constants $t_0 = \frac{1}{2}\ln\left(1 - \frac{1}{b}\right)$, $b > 1$, and $c > 0$, valid for $x \geq c$, $0 < t < -t_0$. For each value of x, the solution $u(x,t)$ is monotonically increasing as a function of t. Moreover, $\lim_{t\to -\frac{1}{2}\ln\left(1-\frac{1}{b}\right)} u(x,t) = \infty$ for each value of $x \geq c$. The evolution of a solution $u(x,t)$ is illustrated in Fig. 4b.

Case V $U(x) = \frac{b(x-c)}{x^2}$ with $b > 1$, $c \leq 0$, on the domain $x > 0$. Such a $U(x)$ is illustrated in Fig. 5a. The corresponding solutions of (1.2b) are given by

$$u(x,t) = \frac{x-c}{2x^2}\left[1 - \coth(t + t_0)\right] \tag{3.15}$$

with constants $t_0 = \frac{1}{2}\ln\left(1-\frac{1}{b}\right)$, $b > 1$, and $c \le 0$, valid for $x > 0$, $0 < t < -t_0$. For each value of x, the solution $u(x,t)$ is monotonically increasing as a function of t. Moreover, $\lim_{t\to -\frac{1}{2}\ln\left(1-\frac{1}{b}\right)} u(x,t) = \infty$ for each value of $x > 0$. The evolution of a solution $u(x,t)$ is illustrated in Fig. 5b.

4 Stationary solutions

Stationary solutions of the NLK equation (1.1) were found in [14] in terms of the doubly degenerate Heun's function (HeunD) and its derivative (HeunD′). A stationary solution $u(x,t) \equiv V(x)$ of the NLK equation (1.2b) satisfies the ODE

$$V'(x) + V^2 = \frac{Q}{x^4} \tag{4.1}$$

for some constant Q which represents the photon flux in the frequency domain. One can show that a nontrivial stationary solution

$$V(x) = \frac{b(x-c)}{x^2} \tag{4.2}$$

satisfies Eq. (4.1) for some constant Q if and only if $b = 1$ and c is an arbitrary constant. Consequently, $Q = c^2$. Interestingly, the explicit family of solutions $V(x) = \frac{x-c}{x^2}$ is not exhibited in [14]. For $c > 0$, $V(x)$ is exhibited in Fig. 6; for $c \le 0$, $V(x)$ is exhibited in Fig. 7.

From the solutions obtained in Sect. 3, we see that all of these nontrivial stationary solutions are unstable, since a slight change in the initial condition will lead to a solution blowing up in finite time or decaying to the trivial stationary solution $u(x,t) \equiv 0$ as $t \to \infty$.

Moreover, if one applies the "nonclassical symmetry" Y_7 to the NLK equation (1.2b), one obtains two more families of explicit stationary solutions, $\boldsymbol{\mathcal{F}_5}$ and $\boldsymbol{\mathcal{F}_6}$.

The family of stationary solutions $\boldsymbol{\mathcal{F}_5}$, in terms of an arbitrary positive constant a, is given by

$$\boldsymbol{\mathcal{F}_5}\colon V(x) = \frac{x + a\tan\left(\frac{a}{x}\right)}{x^2}, \tag{4.3}$$

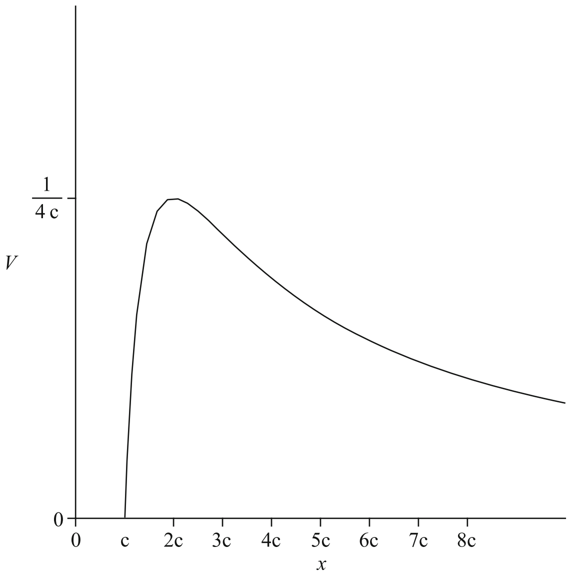

Fig. 6 The stationary solution $V(x) = \frac{x-c}{x^2}$, $c > 0$

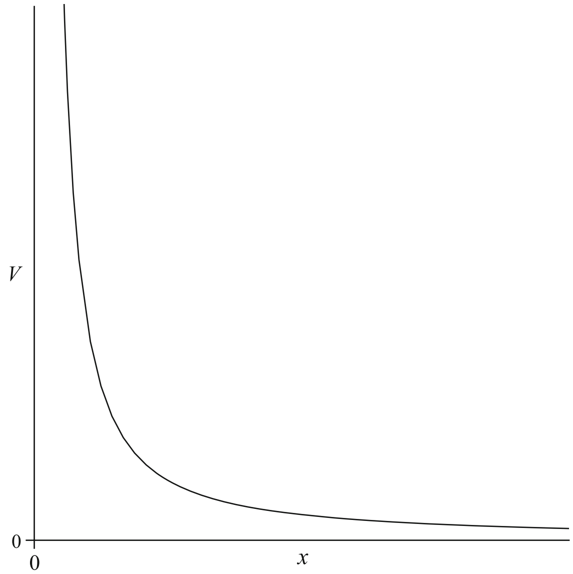

Fig. 7 The stationary solution $V(x) = \frac{x-c}{x^2}$, $c \le 0$

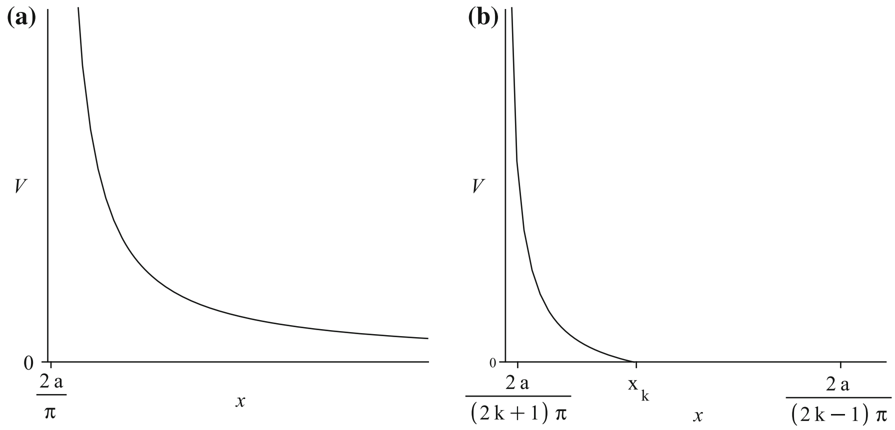

Fig. 8 (**a**) The stationary solution $V(x) = \frac{x+a\tan\left(\frac{a}{x}\right)}{x^2}$, $x > \frac{2a}{\pi}$. (**b**) The stationary solution $V(x) = \frac{x+a\tan\left(\frac{a}{x}\right)}{x^2}$, $\frac{2a}{(2k+1)\pi} < x \leq x_k$

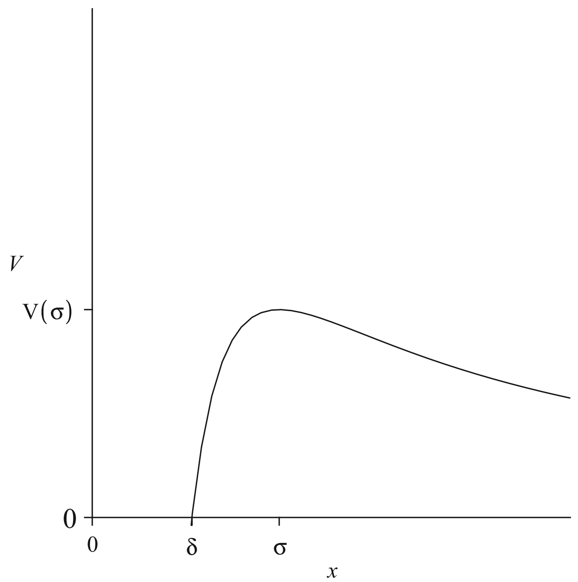

Fig. 9 The stationary solution $V(x) = \frac{x-a\tanh\left(\frac{a}{x}\right)}{x^2}$, $x \geq \delta$

valid on the domains:

(1) $x > \frac{2a}{\pi}$, illustrated in Fig. 8a;

(2) $\frac{2a}{(2k+1)\pi} < x \leq x_k$, where $x_k \in \left(\frac{2a}{(2k+1)\pi}, \frac{2a}{(2k-1)\pi}\right)$ satisfies $x_k + a\tan\left(\frac{a}{x_k}\right) = 0,\ k = 1, 2, \ldots$, illustrated in Fig. 8b.

The family of stationary solutions $\mathcal{F}_6$, in terms of an arbitrary positive constant a, is given by

$$\mathcal{F}_6: \ V(x) = \frac{x - a\tanh\left(\frac{a}{x}\right)}{x^2}, \tag{4.4}$$

valid on the domain $x \geq \delta$, where δ is the unique positive solution of the equation $\delta - a\tanh\left(\frac{a}{\delta}\right) = 0$. The maximum value of $V(x)$ occurs at $x = \sigma = \frac{2a}{1+\mathrm{LambertW}(\mathrm{e}^{-1})}$, in terms of the Lambert W function. Such a solution is illustrated in Fig. 9.

5 Conclusions

In this paper, we used the nonclassical method to obtain some previously unknown solutions of the NLK equation (1.2b). These solutions do not arise as invariant solutions of the NLK equation (1.2b) with respect to its local symmetries. It is interesting to note that the newly obtained exact solutions are explicit solutions of (1.2b) expressed in terms of elementary functions. It is further observed that these solutions exhibit both quiescent and blow-up behavior depending on their initial conditions. It is also shown that related stationary solutions are unstable.

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