Computing A-polynomials using Puiseux expansions

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The A-polynomial of a knot complement is a two variable polynomial $A(x, y)$ with integer coefficients that is notoriously difficult to compute in general. The computation requires the elimination of variables in a large set of polynomial equations. We present a method of computing this polynomial using Puiseux expansions about certain geometrically meaningful solutions of the equations and then using linear algebra. We give some examples of A-polynomials that were previously inaccessible by earlier methods.

Computing A-polynomials using Puiseux expansion

K knot in S^3 $\longrightarrow$ $A(x,y) \in \mathbb{Z}[x,y]$

A-polynomial (not Alexander)

Two almost equivalent definitions:

(1) In terms of representations
$$\pi_1(K) \longrightarrow SL(2,\mathbb{C})$$
(CCGLS '94)

(2) In terms of triangulations of $X = S^3 \setminus K$ into ideal tetrahedra in $\mathbb{H}^3$, hyperbolic 3-space

- related to reps
$$\pi_1(K) \to PSL(2,\mathbb{C}) = Isom^+(\mathbb{H}^3)$$

Computing $A(x,y)$ is usually difficult requiring elimination of variables in large systems of polynomial eqns
- resultants
- Gröbner bases

For (1), one starts with a presentation of π_1 in terms of g generators

– in practice this works only if

$g = 2$ (and relns not too long)

For (2), one starts with a triangulation of $S^3 \setminus K$ into t ideal tetrahedra and this works in practice only if

$t \leq 8$ (or so, depending on complexity of eqns)

Our new method which uses Puiseux expansions depends on some other quantities being small;

(i) d = degree of the Shape field (e.g. $d \leq 12$)

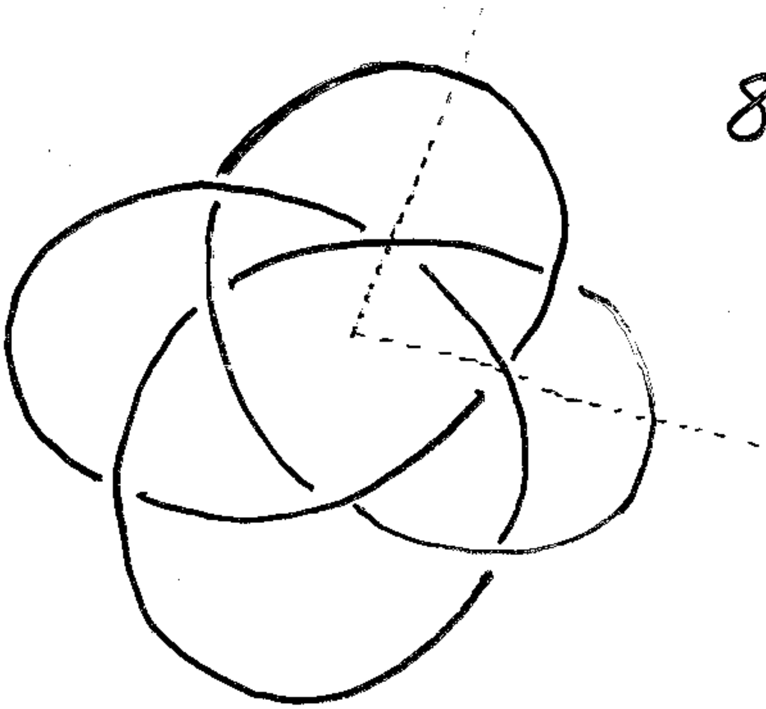
(ii) degree(A, x) and degree(A, y)

e.g. $A(x,y)$ 16×80 or so.

non 2-bridge 8-crossing knots

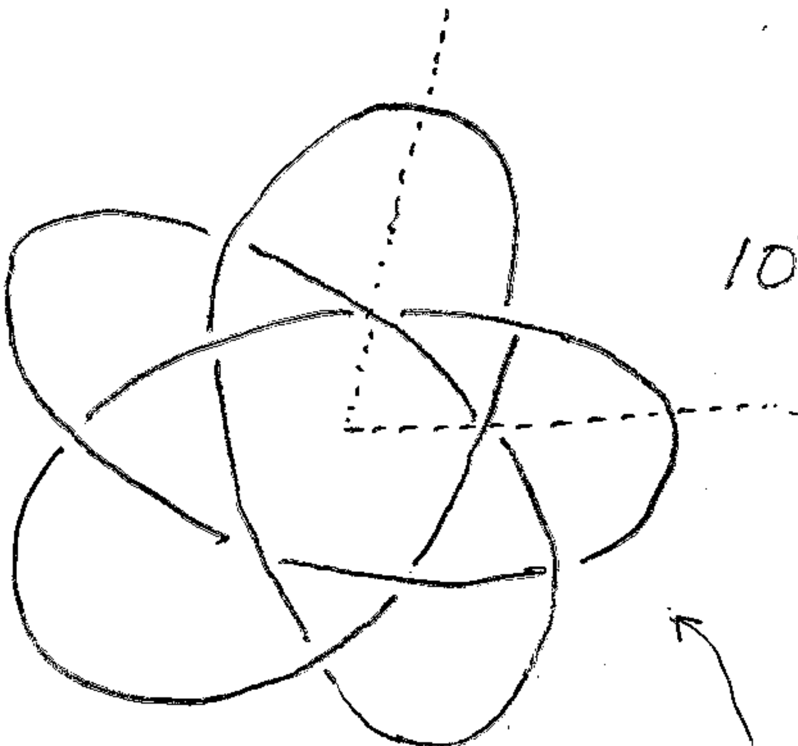
Rolfsen	Conway	#tet	#gen	deg	Sym
8_5	3,3,2	8	2	5	D_2
8_{10}	3,21,2	9	2	11	Z_2
8_{15}	21,21,2	11	2	7	D_2
8_{16}	.2.20	11	3	5	Z_2
8_{17}	.2.2	12	3	18	Z_2
8_{18}	8^*	13	3	4	D_8
8_{19}	3,3,2-	non-hyperbolic			
8_{20}	3,21,2-	5	2	5	Z_2
8_{21}	21,21,2-	7	2	4	D_2

Some Turk's Head knots



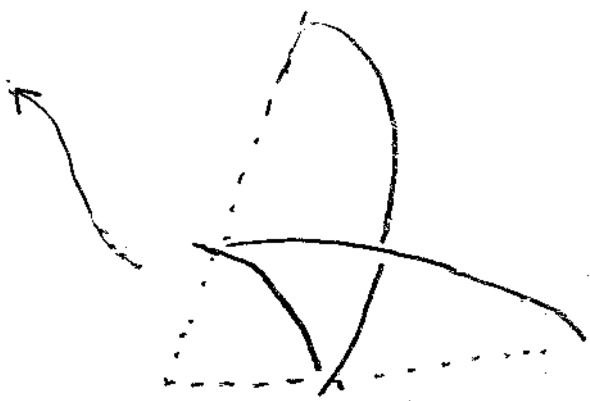
$8^* = 8_{18}$

Sym: D_8
amphicheiral



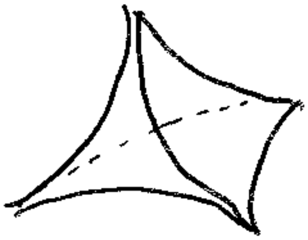
$10^* = 10_{123}$

Sym: D_{10}
amphicheiral



The gluing equations

If $X = S^3 \setminus K = \bigcup_{k=1}^{t} \Delta_k$, each

$\Delta_k =$  an ideal tetrahedron in $\mathbb{H}^3$ = hyperbolic 3-space.
$\cong \mathbb{C} \times (0,\infty)$

Up to isometry,

$\Delta_k \cong \Delta(z)$ with vertices at $0, 1, z, \infty$, $z \in \mathbb{C}$

$\mathrm{Vol}(\Delta_k) = D(z)$, the Bloch-Wigner dilog.

(& $\pi m(A) = \sum D(\alpha_j)$, certain $\alpha_j \in \overline{\mathbb{Q}}$)

The combinatorics of the triangulation lead to the gluing eqns

$$(G) \quad f_j = \prod_{j=1}^{t} z_j^{a_{ij}} (1-z_j)^{b_{ij}} = 1, \quad i = 1, \dots, t+2$$

where $f_j = 1$ ($j \le t$) are "edge" eqns &

$f_{t+1} = 1$, $f_{t+2} = 1$ are the "longitude" & "meridian" eqns.

We "deform" the last ~~two~~ eqns to

$$f_{t+1} = x^2, \quad f_{t+2} = y$$

and then eliminate $z_1, \dots, z_t$ from the resulting system to obtain $\quad A(x,y) \in \mathbb{Z}[x,y]$

The "Puiseux" method.

(1) Find the geometric soln of (G) (with $x=y=1$), say $z_j = \alpha_j \in \mathbb{C}$ with $\mathrm{Im}(\alpha_j) > 0$.

The α_j are algebraic and $Sh = \mathbb{Q}(\alpha_1, \ldots, \alpha_t)$ is the shape field.

– use SnapPea (Weeks)
or Snap (Goodman et al)

(2) Write $y = 1+s$ and then the soln of $A(x, 1+s) = 0$ is $x = 1 + \sum_{k=1}^{\infty} c_{0,k} s^k$ (no ramification, by Neumann – Zagier).

Indeed, using their proof, the shapes z_j also satisfy

$$z_j = \alpha_j + \sum_{k=1}^{\infty} c_{jk} s^k \qquad (j=1, \ldots, t).$$

(3) Compute the c_k, c_{jk} for $k = 1, 2, 3, \ldots, N$ by iteratively substituting in the eqns (G) and solving a system of linear eqns for

$$(c_{0k}, c_{1k}, \ldots, c_{tk})$$

The c_{jk} are all in Sh.

(4) ① Basic linear algebra method

Assuming $A(x,y) = \sum_{\substack{0\le i\le d_1 \\ 0\le j\le d_2}} a_{ij}\, x^i y^j$, $(a_{ij}\in\mathbb{Z})$

with $d\,N \ge (d_1+1)(d_2+1)$ (usually $=$)

$d = \deg(Cu_\infty)$ where $Cu_\infty = \mathbb{Q}(c_{01}, c_{02}, c_{03}, \dots) \subseteq Sh$.

If a solution is found with $A\neq 0$, check various conditions to see if is a plausible candidate for A.

If $A=0$ then N is too small or the guess for d_1, d_2 is wrong. — try again!

Variants:

② Not uncommonly, $d_1 = \deg(A, x) = d = \deg(Sh)$.

Then

$$A(x,y) = c_0(y)\prod_{j=1}^{d}(x - X^{(j)}(s)) + O(s^{N+1})$$

(Galois conjugates)

$$= \sum_{i=0}^{d} x^i c_i(y), \quad y = 1+s.$$

So e.g. $Tr(X(s)) = -\frac{c_1(y)}{c_0(y)} = -\frac{c_1(1+s)}{c_0(1+s)}$,

a rational fn of s obtainable by continued fractions if $N > d_2$.

Similarly for $Tr(X^k(s))$ & hence $c_i(1+s)$, by Newton's formulae.

Variant (3). ("hints")

From just a few terms of the Puiseux expansion $z_j = Z_j(a)$ of the shapes, one may surmise relation $z_i = z_j$ from $Z_i(a) = Z_j(a) + O(a^N)$ (e.g. $N=10$ is reasonably convincing).

These "hints" can simplify the eqns (G) enough so that they can be solved by elimination methods.

Variant (4). Use other solns besides the geometric soln, e.g. for $(n,0)$ surgery which has $y = \zeta_n$.

(a) e.g. if $\deg(Cu_\infty(n,0)) = d_1 > \deg(Cu_\infty)$, one can use Variant (2) but with the $(n,0)$-soln.

(b) Combine the solutions for various n to solve for a_{ij} as in (1) with smaller N. ($n=2$ is best since otherwise have to work over an extension of $\mathbb{Q}$).

Examples:

i.e. $d_1 = d$

$8^* = 8_{18}$ $t=13$, $g=3$, $d=4$ (method ②)

$\rightarrow A(x,y)$ 4×16 ht 106

12n706 $t=14$, in fact $d_j = 8[\zeta_4] + 6[\zeta_6]$

so $Sh = \mathbb{Q}(\sqrt{-1}, \sqrt{-3})$ but $C_u = \mathbb{Q}(\sqrt{-1})$

Using "hints" $\rightarrow A(x,y)$ 12×42 ht 60120

$(Sh = \mathbb{Q}(\zeta_5))$

$10^* = 10_{123}$ $t=18$, $g=3$, $d=4$

$\rightarrow A(x,y)$ 8×40 ht 18870

Using ④(a) since $\deg(Sh(2,0)) = 8 = \deg(A,x)$

$14^* = 14a19470$ $t=28$, $g=3$, $d=6$

$\rightarrow A(x,y)$ 12×84 ht $= 14859035072$

Here not enough time to find $A(x,y)$ but enough to find $P_j(z_j, y)$ for each shape $j = 1, \ldots, t$ and hence a 2nd soln of degree 6 at $x=-1, y=1$

Combine and use ④(b)

$16^* = 16a379778 \quad t=32, g=3, d=8$

$\longrightarrow A(x,y)$ 8×64 ht 1264620800

method ② since $d_1 = d$.

$20^* \qquad t=42, g=3, d=8$

$\longrightarrow A(x,y)$ 8×80 ht 32192632169005O

method ④(b) using
$\deg Sh = 8$ <u>and</u> $\deg Sh(2,0) = 8$.

<u>dodec</u>$_1$: $\quad t=46, g=4, d=4$

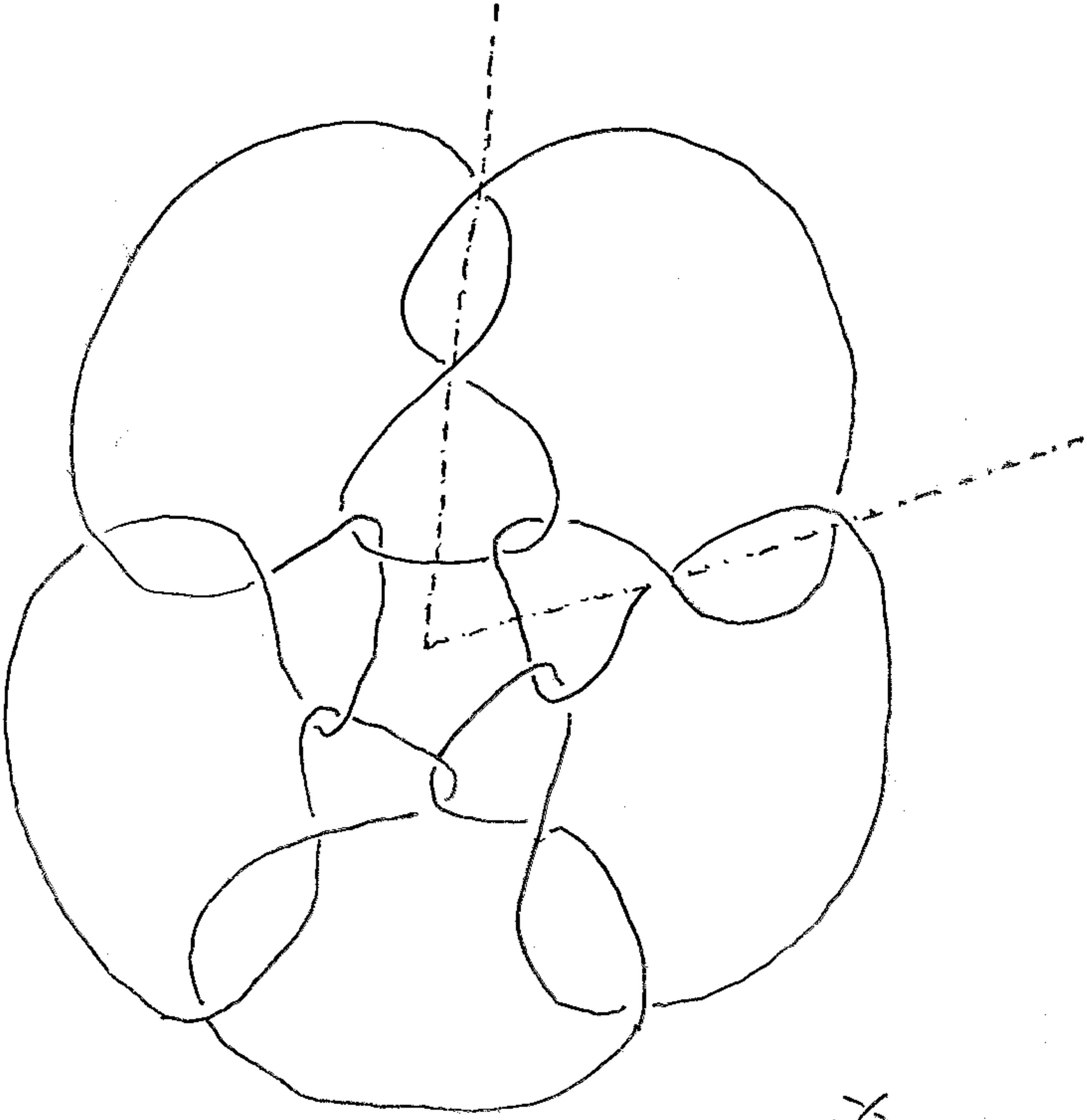
but can show $\deg(A,x) \geq 24$ so the Puiseux method runs out of steam

However - as demonstrated at Bromsgrove Conference

$A(x^5, y)$ = product of 10 polys in $\mathbb{Q}(\zeta_{10})[x,y]$
degree 16×16

$\rightarrow$ $A(x,y)$ 32×160 with

height = 40233375155685120881066569759384

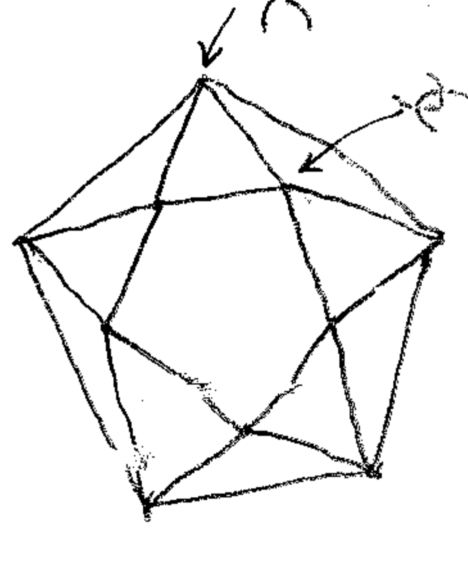


dodec$_1$

Sym: D_{10}
amphicheiral

$g = 4$, $t = 46$, $d = 4$

$\text{degree}(A, x) \geq 24$



8* Apoly

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & 54 & 0 & 0 \\ 0 & 0 & -112 & 0 & 0 \\ 0 & -2 & 109 & -2 & 0 \\ 0 & 12 & -64 & 12 & 0 \\ 0 & -14 & 74 & -14 & 0 \\ 0 & -28 & -100 & -28 & 0 \\ 1 & 68 & 106 & 68 & 1 \\ 0 & -28 & -100 & -28 & 0 \\ 0 & -14 & 74 & -14 & 0 \\ 0 & 12 & -64 & 12 & 0 \\ 0 & -2 & 109 & -2 & 0 \\ 0 & 0 & -112 & 0 & 0 \\ 0 & 0 & 54 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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10" A-polynomial

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -25 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 270 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1640 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6075 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & -13710 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & -75 & 16850 & -75 & 0 & 0 & 0 \\
0 & 0 & 0 & 585 & -5215 & 585 & 0 & 0 & 0 \\
0 & 0 & 0 & -2400 & -11290 & -2400 & 0 & 0 & 0 \\
0 & 0 & 0 & 5125 & 7275 & 5125 & 0 & 0 & 0 \\
0 & 0 & 6 & -3353 & 9720 & -3353 & 6 & 0 & 0 \\
0 & 0 & -75 & -8250 & -7400 & -8250 & -75 & 0 & 0 \\
0 & 0 & 360 & 17470 & -4740 & 17470 & 360 & 0 & 0 \\
0 & 0 & -755 & -4900 & 2515 & -4900 & -755 & 0 & 0 \\
0 & 0 & 275 & -11425 & -725 & -11425 & 275 & 0 & 0 \\
0 & 4 & 1099 & -578 & 6848 & -578 & 1099 & 4 & 0 \\
0 & -25 & 1475 & 17200 & -4950 & 17200 & 1475 & -25 & 0 \\
0 & 45 & -10910 & -7565 & -9320 & -7565 & -10910 & 45 & 0 \\
0 & 5 & 14080 & -5740 & 18870 & -5740 & 14080 & 5 & 0 \\
0 & -150 & 400 & 7600 & -1250 & 7600 & 400 & -150 & 0 \\
1 & 242 & -11914 & -7396 & -16312 & -7396 & -11914 & 242 & 1 \\
0 & -150 & 400 & 7600 & -1250 & 7600 & 400 & -150 & 0 \\
0 & 5 & 14080 & -5740 & 18870 & -5740 & 14080 & 5 & 0 \\
0 & 45 & -10910 & -7565 & -9320 & -7565 & -10910 & 45 & 0 \\
0 & -25 & 1475 & 17200 & -4950 & 17200 & 1475 & -25 & 0 \\
0 & 4 & 1099 & -578 & 6848 & -578 & 1099 & 4 & 0 \\
0 & 0 & 275 & -11425 & -725 & -11425 & 275 & 0 & 0 \\
0 & 0 & -755 & -4900 & 2515 & -4900 & -755 & 0 & 0 \\
0 & 0 & 360 & 17470 & -4740 & 17470 & 360 & 0 & 0 \\
0 & 0 & -75 & -8250 & -7400 & -8250 & -75 & 0 & 0 \\
0 & 0 & 6 & -3353 & 9720 & -3353 & 6 & 0 & 0 \\
0 & 0 & 0 & 5125 & 7275 & 5125 & 0 & 0 & 0 \\
0 & 0 & 0 & -2400 & -11290 & -2400 & 0 & 0 & 0 \\
0 & 0 & 0 & 585 & -5215 & 585 & 0 & 0 & 0 \\
0 & 0 & 0 & -75 & 16850 & -75 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & -13710 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6075 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1640 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 270 & 0 & 0 & 0 & 0
\end{bmatrix}
$$