Mathematics 539 – Exercises # 1

Arithmetical Functions

1. Prove that
$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

2. Prove that
$$\sum_{d^2|n} \mu(d) = \mu^2(n)$$
and, more generally,
$$\sum_{d^k|n} \mu(d) = \begin{cases} 0 & \text{if } m^k|n \text{ for some } m > 1, \\ 1 & \text{otherwise.} \end{cases}$$

3. Suppose that $f(x)$ is defined for $0 \le x \le 1$ and we define
$$F(n) = \sum_{1 \le k \le n} f(k/n) \quad \text{and} \quad G(n) = \sum_{\substack{1 \le k \le n \\ (k,n)=1}} f(k/n).$$
(a) Prove that $G(n) = \sum_{d|n} \mu(d) F(n/d)$.
(b) Prove that $\mu(n)$ is the sum of the primitive nth roots of unity:
$$\mu(n) = \sum_{\substack{1 \le k \le n \\ (k,n)=1}} \exp(2\pi i k/n).$$

4. Let $f(n) = [\sqrt{n}] - [\sqrt{n-1}]$. Show that f is multiplicative but not completely multiplicative.

5. Prove or disprove the following: if $f(n)$ is multiplicative, and if $F(n) = \prod_{d|n} f(d)$, then $F(n)$ is multiplicative.

6. Prove or disprove the following:
(a) If $f(n)$ is a multiplicative function for which $f(p^m) \to 0$ as $m \to \infty$ for each prime p, then $f(n) \to 0$ as $n \to \infty$.
(b) If $f(n)$ is multiplicative and $f(n) \to 0$ as $n \to \infty$ through the sequence of all prime powers $\{p^m\} = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, \ldots\}$ then $f(n) \to 0$ as $n \to \infty$. More explicitly, the hypothesis of (b) means that if we arrange the prime powers in a sequence $q_1 < q_2 < \ldots$ then $\lim_{k\to\infty} f(q_k) = 0$.

7. Let $\lambda(n)$ denote Liouville's function.
(a) Prove that
$$\sum_{d|n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$
(b) Prove that
$$\lambda(n) = \sum_{m^2|n} \mu(n/m^2).$$