Mathematics 539 – Exercises # 2

Dirichlet series and Euler products

1. Let $\mu(n)$ be the Möbius function. Show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}.$$

2. Let $d(n)$ be the number of divisors of n. Prove that the following identity holds for $\sigma > 1$ (where as usual $s = \sigma + it$):

$$\sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)}.$$

3. Let $\nu(n)$ denote the number of distinct prime divisors of n so $\nu(1) = 0$ and $\nu(p_1^{a_1} \cdots p_k^{a_k}) = k$. Show that the following holds for $\sigma > 1$:

$$\sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \zeta(s) \sum_{p} \frac{1}{p^s}.$$

4. (a) Prove that the series $\sum_{n=1}^{\infty} n^{-1-it}$ has bounded partial sums if $t \neq 0$, but unbounded partial sums if $t = 0$.
   (b) Prove that the series $\sum_{n=1}^{\infty} n^{-1-it}$ diverges for all real t.

5. Prove that

$$\sum_{\substack{m \geq 1 \\ n \geq 1 \\ (m,n)=1}} \frac{1}{m^2 n^2} = \frac{\zeta^2(2)}{\zeta(4)} = \frac{5}{2}.$$

   (You may assume as known the explicit values for $\zeta(2)$ and $\zeta(4)$.)

6. Let ϕ be the Euler phi function and let $F(s) = \sum_{m=1}^{\infty} \phi(m)^{-s}$.
   (a) Show that the series converges uniformly in compact subsets of the half plane $\sigma > 1$.
   (b) Show that

$$F(s) = \prod_{p} \big(1 + (p-1)^{-s}(1 - p^{-s})^{-1}\big) = \zeta(s) Q(s),$$

   where

$$Q(s) = \prod_{p} \big(1 - p^{-s} + (p-1)^{-s}\big),$$

   and that this product converges absolutely and uniformly in compact subsets of the half-plane $\sigma > 0$.
   (c) Define $a(n) = \#\{m : \phi(m) = n\}$ so that $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$. Is $a(n)$ multiplicative? Is the product in (b) an Euler product for $F(s)$?