Mathematics 539 – Exercises # 3

Estimation of Arithmetical functions – Dirichlet Characters

1. Let $f(x)$ be defined for $x > 0$. Define
$$g(x) = \sum_{k \le x} f(\frac{x}{k}).$$
Show that
$$f(x) = \sum_{k \le x} \mu(k) g(\frac{x}{k}),$$
where $\mu(n)$ is the Möbius function.

2. Use the Euler-Maclaurin formula to prove that, for $x \ge 2$:
$$\text{(a)} \quad \sum_{n \le x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + A + O(\frac{\log x}{x}),$$
$$\text{(b)} \quad \sum_{2 \le n \le x} \frac{1}{n \log n} = \log\log x + B + O(\frac{1}{x \log x}),$$
where A and B are constants.

3. Show that
$$\sum_{n \le x} \frac{1}{\phi(n)} = O(\log x).$$

4. Prove that
$$\sum_{n \le x} \frac{\phi(n)}{n^2} = \frac{1}{\zeta(2)} \log x + \frac{\gamma}{\zeta(2)} - A + O(\frac{\log x}{x}),$$
as $x \to \infty$, where γ is Euler's constant and $A = \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^2}$.

5. For $\mathrm{Re}(s) > 0$ and integer $m \ge 1$, find an asymptotic formula for the partial sums
$$\sum_{\substack{n \le x \\ (n,m)=1}} \frac{1}{n^s},$$
with an error term that tends to 0 as $x \to \infty$.

6. For n an odd integer, write $\chi_2(n) = (-1)^{(n-1)/2}$, $\chi_3(n) = (\frac{2}{n})$ (Jacobi symbol), and for n even $\chi_2(n) = \chi_3(n) = 0$. Write $\chi_4(n) = \chi_2(n)\chi_3(n)$. Show that χ_2, χ_3 and χ_4 are the non-principal characters modulo 8.

7. (a) Using the fact that $\int_0^1 t^{n-1} dt = 1/n$, show that, if χ is a non-principal character mod m, then $L(1, \chi)$ can be written as the integral over the interval $[0, 1]$ of a rational function.
(b) Use this to evaluate $L(1, \chi_k)$ for each of the characters χ_k of the previous problem.