Mathematics 539 – Exercises # 4

Dirichlet's theorem on primes in arithmetic progressions

1. The following is an outline of a proof that $L(1,\chi) \neq 0$ for any real non-principal character χ. Fill in the details:
   (a) Let $A(n) = \sum_{d|n} \chi(d)$. Then $A(n) \geq 0$ and $A(n) \geq 1$ if n is a square.
   (b) Let $B(x) = \sum_{n \leq x} \frac{A(n)}{\sqrt{n}}$. Then $B(x) \to \infty$ as $x \to \infty$.
   (c) $B(x) = 2\sqrt{x}L(1,\chi) + O(1)$, as $x \to \infty$.

2. Show that Dirichlet's theorem is equivalent to the following statement: if $q > 1$ and $a > 0$ are any two integers with $(a,q) = 1$ then there is a prime p of the form $p = a + nq$.

3. Show that if a is a positive integer for which a is a square modulo p for all primes p, then a is the square of an integer.

4. For each $N \geq 1$, show that there are infinitely many primes p such that $\left(\frac{a}{p}\right) = 1$ for all a with $1 \leq a \leq N$.

5. Construct an infinite set S of primes with the property that if p and q are in S then
$$((p-1)/2, (q-1)/2) = (p, q-1) = (p-1, q) = 1.$$

6. Show that if a and q are positive integers then the arithmetic progression $a, a+q, a+2q, \ldots$ contains an arbitrary number of consecutive composite terms.

7. Show that if $0 < q < 2000$ then at most 10 consecutive terms of the arithmetic progression $a, a+q, a+2q, \ldots$ can be primes. (The proof need not use Dirichlet's theorem).