Mathematics 539 – Exercises # 5

Estimates of sums over primes

1. Prove that the following two inequalities are equivalent:

   (a) $$\pi(x) = \frac{x}{\log x} + O\Big(\frac{x}{\log^2 x}\Big),$$

   (b) $$\theta(x) = x + O\Big(\frac{x}{\log x}\Big).$$

2. Let $\Lambda(n)$ be von Mangoldt's function. Show that

$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

3. Let $\psi(x) = \sum_{n\le x} \Lambda(n)$. Prove for $\sigma = \mathrm{Re}(s) > 1$ that

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx.$$

4. Use the formula of the previous problem to deduce Chebyshev's theorem that if $\lim_{x\to\infty} \psi(x)/x = A$ exists, then $A = 1$. (Don't use the prime number theorem!). As a hint, you only need to use s real and the fact that $\lim_{s\to 1}(s-1)\zeta'(s)/\zeta(s) = -1$.

5. Let $m > 1$ and $(a, m) = 1$. Show that there is a constant $A = A(a, m)$ such that, if $x \ge 2$, then

$$\sum_{\substack{p\le x\\ p\equiv a \pmod m}} \frac{1}{p} = \frac{1}{\phi(m)}\log\log x + A + O(\frac{1}{\log x}).$$

6. Show that if p and q denote primes in the following sum, then

$$\sum_{pq\le x} \frac{1}{pq} = (\log\log x)^2 + O(\log\log x).$$

7. Here is the outline of a proof that $\int_0^\infty e^{-u}\log u\,du = -\gamma$ which uses only the definition of γ, i.e. that $\sum_{n\le x} 1/n = \log x + \gamma + \epsilon(x)$, with $\epsilon(x) = O(1/x)$ as $x \to \infty$.

   (a) Let $H(x) = \sum_{n\le x} 1/n$. Show for $\delta > 0$ that

$$\int_0^\infty \delta e^{-\delta x} H(x)\,dx = \sum_{n=1}^\infty \frac{e^{-\delta n}}{n} = -\log\delta + o(1), \text{ as } \delta \to 0+.$$

   (b) Show that

$$\int_0^\infty \delta e^{-\delta x}(\log x + \gamma + \epsilon(x))\,dx = \int_0^\infty e^{-u}\log u\,du - \log\delta + \gamma + o(1), \text{ as } \delta \to 0+.$$

   (c) Conclude from (a) and (b) that $\displaystyle\int_0^\infty e^{-u}\log u\,du = -\gamma$.