Mathematics 539 – Exercises # 6

The prime number theorem

1. Let $M(x) = \sum_{n \le x} \mu(n)$. Suppose that $M(x) = O(x^\theta)$ for some $\theta < 1$ and all $x \ge 1$. Prove that

$$\frac{1}{\zeta(s)} = s \int_1^\infty \frac{M(x)}{x^{s+1}}\, dx$$

for $\mathrm{Re}(s) > \theta$, and hence that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > \theta$.

2. Show that the smallest prime p that does not divide the integer n satisfies $p \le \log n(1 + o(1))$.

3. Let $H_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$. Write $H_n = \dfrac{a_n}{b_n}$ where a_n, b_n are positive integers with $\gcd(a_n, b_n) = 1$.
Show that $\log a_n = n + o(n)$ and $\log b_n = n + o(n)$.